ON COHOMOLOGY THEORY FOR TOPOLOGICAL GROUPS

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Abstract. We construct some new cohomology theories for topological groups and Lie groups and study some of its basic properties. For example, we introduce a cohomology theory based on measurable cochains which are continuous in a neighbourhood of identity. We show that if G and A are locally compact and second countable, then the second cohomology group based on locally continuous measurable cochains as above parametrizes the collection of locally split extensions of G by A.

1. Introduction

The cohomology theory of topological groups has been studied from different perspectives by W. T. van Est, Mostow, Moore, Wigner and recently Lichtenbaum amongst others. W. T. van Est developed a cohomology theory using continuous cochains in analogy with the cochain construction of cohomology theory of finite groups. However, this definition of cohomology groups has a drawback, in that it gives long exact sequences of cohomology groups only for those short exact sequences of modules that are topologically split.

Based on a theorem of Mackey (*cf.* [Di]), guaranteeing the existence of measurable cross sections for locally compact groups, Moore developed a cohomology theory of topological groups using measurable cochains in place of continuous cochains. This cohomology theory works for the category of Polish groups G and G-modules A which are again Polish. We recall, a topological group G is said to be Polish, if its topology is induced by a complete separable metric on G. This theory satisfies the nice properties expected from a cohomology theory (*cf.* [Mo76]) viz., there exists long exact sequences of cohomology groups of a Polish group G corresponding to a short exact sequence of Polish G-modules, the correct interpretation of the first measurable cohomology as the space of continuous crossed homomorphism, when G and A are locally compact; an interpretation of the second cohomology $H^2_m(G, A)$ in terms of topological extensions of G by A, etc. Here $H^*_m(G, A)$ denotes the Moore cohomology group of a topological group G and a topological G-module A. It further agrees with the van Est continuous cohomology groups, when G is profinite and the coefficient module A is discrete.

The cohomology theory developed by Moore has had numerous applications (for some recent applications and also for further results on Moore cohomology groups

1991 *Mathematics Subject Classification.* Primary 22D99; Secondary 22E99, 46M20, 57T10.

see [Aus]). The motivation for us to consider the cohomology theory of topological groups, is to explore the possibility of deploying such theories to the study of the non-abelian reciprocity laws as conjectured by Langlands, just as the continuous cohomology theory of Galois groups has proved to be immensely successful in class field theory. In this context, the analogues of the Galois group like the Weil group W_k attached to a number field k (or the conjectural Langlands group whose finite dimensional representations are supposed to parametrize automorphic representations) are no longer profinite but are locally compact. Indeed such a motivation led the second author to generalize a classical theorem of Tate on the vanishing of the Schur cohomology groups to the context of Weil groups ([Ra]), to show that $H^2_m(W_k, \mathbb{C}^*)$ vanishes, where we impose the trivial module structure on $\mathbb{C}^*$.

The immediate inspiration for us is the recent work of Lichtenbaum ([Li]), where he outlines deep conjectures explaining the special values of zeta functions of varieties in terms of Weil-étale cohomology. Here the cohomology of the generic fibre turns out to be the cohomology of the Weil group. Lichtenbaum studies the cohomology theory of topological groups from an abstract viewpoint based on the work of Grothendieck ([SGA4]), where he embeds the category of G-modules in a larger abelian category with sufficiently many injectives. The cohomology groups are then the right derived functors of the functor of invariants (we refer to the paper by Flach ([Fl]) for more details and applications to the cohomology of Weil groups). Lichtenbaum imposes a Grothendieck topology on the category of G-spaces, where the covers have local sections. The required abelian category is the category of sheaves with respect to this Grothendieck topology.

In this paper, we introduce a new cohomology theory of topological groups. We modify Moore's construction and impose a local regularity condition on the cochains in a neighbourhood of identity (like continuity or smoothness in the context of Lie groups) but assume the cochains to be measurable everywhere. The basic observation which makes this possible is the following: given a short exact sequence of Lie groups

$$1 \to G' \to G \to G'' \to 1,$$

there is a continuous section from a neighbourhood of identity in G'' to G. More generally, using the solution to Hilbert's fifth problem and the observation for Lie groups, Mostert (*cf.* [Mos]) showed that every short exact sequence of finite dimensional locally compact groups

$$1 \to G' \to G \xrightarrow{\pi} G'' \to 1$$

is locally split *i.e.*, there exists a continuous section from a neighbourhood of identity in G'' to G.

Define the group of 0-cochains $C^0(G, A)$ to be A. For $n \geq 1$, define the group $C^n_{lcm}(G, A)$ of locally continuous measurable cochains to be the space of all measurable functions $f \colon G^n \to A$ which are continuous in a neighbourhood of the identity in G^n. The coboundary map is given by the standard formula. Now we define our locally continuous cohomology theory $H^n_{lcm}(G, A)$ as the cohomology of this cochain complex. These cohomology groups interpolate the continous cohomology and the measurable cohomology theory of Moore: there are natural maps

$$H_c^n(G,A) \to H_{lcm}^n(G,A) \to H_m^n(G,A),$$

where $H_c^n(G,A)$ denotes the continuous cohomology groups of G with values in A.

For the category of Lie groups, we replace continuity with the property of being smooth around identity and we define the locally smooth measurable cohomology theory (denoted by $\{H_{lsm}^n(G,A)\}_{n\geq 0}$) of a Lie group G that acts smoothly on A. Similarly, we can define locally holomorphic measurable cohomology theory (denoted as $\{H_{lhm}^n(G,A)\}_{n\geq 0}$) in the holomorphic category, based on measurable cochains holomorphic in a neighbourhood of identity.

We remark out here that although the cohomology theories developed by Moore and Lichtenbaum seem sufficient for many purposes, the richness of applications of cohomology arises from the presence of different cohomology theories that can be compared to each other. The multiplicity of such theories allow the use of cohomological methods in a variety of contexts. In this regard, we expect that the principle of imposing local regularity on the cochains, will allow it's use in more geometric and arithmetical contexts. For example, it is tantalizing to explore the relationship of these theories to the measurable Steinberg 2-cocycle ([Mo68]), which is continuous on a dense open subset (but not at identity!).

These locally regular cohomology theories can be related to the underlying category theoretic properties of the group and it's modules. For example, suppose there is an extension

$$1 \to A \to E \to G \to 1$$

of G by A given by a measurable 2-cocycle. From the construction of this extension (as given by Moore, *cf.* [Mo76, page 30]), it seems difficult to relate the topology of E to that of G and A. If G and A are locally compact, it is a difficult theorem of Mackey that E is locally compact (*cf.* [Mac]). Another difficulty arises, when we work with a Lie group G and a smooth G-module A. It is not clear when an extension of G by A defined by a measurable 2-cocycle is a Lie group. Further, there does not seem to be any relationship between the Moore cohomology groups and the cohomology groups of the associated Lie Algebra and its module.

We now describe some of our results towards establishing the legitimacy of these theories. It can be seen that these locally regular cohomology theories are cohomological, in that there exists long exact sequence of cohomology groups associated to locally split short exact sequences of modules. Further, the zeroth cohomology group is the space A^G of G-invariant elements in A. There exists a natural map

$$H_{lcm}^n(G,A) \to H_m^n(G,A).$$

When G and A are Lie groups and the G-action is smooth, the following are natural maps between cohomology groups.

$$H_{lsm}^n(G,A) \to H_{lcm}^n(G,A) \to H_m^n(G,A).$$

For any topological group G and continuous G-module A, the first cohomology group

$$H^1_{lcm}(G, A) = \{c : G \to A \mid c(st) = c(s) + s \cdot c(t),\ c \text{ is continuous}\}.$$

When $G,\ A$ are locally compact, second countable topological groups, it follows by a theorem of Banach, that measurable crossed homomophisms from G to A are continuous. Thus we have

$$H^1_m(G, A) = H^1_{lcm}(G, A) = H^1_{cont}(G, A).$$

For a Lie group G and smooth G-module A (a smooth G-module A is an abelian Lie group such that the action $G \times A \to A$ is smooth) which is locally compact and second countable, the first cohomology group agrees with Moore cohomology group.

$$H^1_m(G, A) = H^1_{lcm}(G, A) = H^1_{lsm}(G, A).$$

which is the group of all smooth crossed homomorphisms from G to A. The advantage of working with locally regular cohomology can be seen in the holomorphic category.

Proposition 1. *Given a complex Lie group G and a holomorphic G-module A, $H^1_{lhm}(G, A)$ is the group of all holomorphic crossed homomorphisms from G to A.*

The main theorem of this paper is to show that the second cohomology group $H^2_{lcm}(G, A)$ for locally compact second countable G and A, parametrizes all the locally split extensions of G by A .

Theorem 1. *If $G,\ A$ are both locally compact, second countable topological groups, the second cohomology group, $H^2_{lcm}(G, A)$ parametrizes all the isomorphism classes of extensions E of G by A,*

$$1 \to A \xrightarrow{\imath} E \xrightarrow{\pi} G \to 1$$

which are locally split.

It is this theorem that confirms our expectation that these locally regular cohomology theories can be a good and potentially useful cohomology theory for topological groups. Our other attempts to constuct suitable cohomology theories failed to give a suitable interpretation for the second cohomology group.

The proof of this theorem is a bit delicate. Given a locally continous measurable 2-cocycle, we construct an abstract extension group E. We topologize E by first defining the product topology in a sufficiently small 'tubular' neighbourhood of identity in E, and by imposing the condition that left translations are continuous. To conclude that E is a topological group, we need to show that inner conjugation by any element of E is continuous at identity. For this, we follow the idea of proof of Banach's theorem that a measurable homomorphism of locally compact second countable groups is continuous. We find that the proof of Banach's theorem extends perfectly to prove that inner conjugations are continuous.

In the smooth category, we have the following analogue of Theorem 1.

Theorem 2. *Let G be a Lie group, A be a smooth G-module. The second cohomology groups, $H^2_{lsm}(G, A)$ parametrizes all the locally split smooth extensions of G by A.*

Further, as a consequence of the positive solution to Hilbert's fifth problem, we have a comparison theorem as follows:

Theorem 3. *Let G be a Lie group, A be a smooth G-module. Then the natural map,*

$$H^2_{lsm}(G,A) \to H^2_{lcm}(G,A)$$

is an isomorphism.

The locally smooth measurable cohomology groups can be related to Lie algebra cohomology. This is easily done via the cohomology theory $H_\Box(G,A)$ based on germs of smooth cochains defined in a neighbourhood of identity developed by Swierczkowski ([Sw, page 477]). We have a restriction map,

$$H^n_{lsm}(G,A) \to H^n_\Box(G,A),$$

given by restricting a locally smooth measurable cochain to a neighbourhood of identity in G^n where it is smooth.

Now suppose G acts on a finite dimensional real vector space V. Let L denotes the Lie algebra associated to the Lie group G and let $H(L,V)$ denotes the Lie algebra cohomology. It has been proved (*cf.* [Sw, Theorem 2]) that

$$H_\Box(G,V) \simeq H(L,V).$$

2. Continuous and measurable cohomology theories

We briefly recall the earlier constructions of Čech type cohomology theories of topological groups and their modules. Let G be a topological group and A be an abelian topological group. Assume that there is an action of G on A. We say A is a topological G-module if the group action,

$$G \times A \to A,$$

is continuous.

Suppose $A',\ A,\ A''$ are topological G-modules and form a short exact sequence of abelian groups:

$$0 \to A' \xrightarrow{\imath} A \xrightarrow{\jmath} A'' \to 0,$$

We say it is a short exact sequence of topological G-modules, if $\imath,\ \jmath$ are continuous, $\imath$ is closed, $\jmath$ is open and there is an isomorphism of topological groups,

$$A/\imath(A') \simeq A''.$$

2.1. Cohomology theory based on continuous cochains.

Definition 1. Define the continuous cohomology $H^*_{cont}(G,A)$ as the cohomology of the cochain complex $\{C^n_{cont}(G,A), d^n\}_{n\geq 0}$ given as:

(i) $C^0_{cont}(G,A) = A$
(ii) For $n \geq 0$, $C^n_{cont}(G,A)$ denotes the group of all continuous maps from the product G^n to A.

The coboundary formula is given by the standard definition. For any $f \in C^n_{cont}(G, A)$,

$$\begin{aligned} d^n f(s_1, s_2, \ldots, s_{n+1}) =& s_1 \cdot f(s_2, \cdots, s_{n+1}) + \sum_{i=1}^{n} (-1)^i f(s_1, \ldots, s_i s_{i+1}, \ldots, s_{n+1}) \\ &+ (-1)^{n+1} f(s_1, s_2, \ldots, s_n), \end{aligned}$$

This theory coincides with the abstract cohomology theory, when the group is finite. When G is profinite acting on a discrete abelian group, this cohomology theory is compatible with direct limits. This theory was introduced by van Est (*cf.* [vEst]).

Example 1. Suppose G is a connected topological group and A is a discrete G-module. Then it can be seen that G acts trivially on A. The continuous cochains are just constant maps, and from the definition of the coboundary map, it can be seen that all the higher cohomology groups vanish.

Consider the short exact sequence of trivial $G = S^1$-modules,

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\pi} S^1 \to 1.$$

From the above remark, and the fact that there are no continuous homomorphisms from S^1 to $\mathbb{R}$, we see that,

$$H^1_{cont}(S^1, \mathbb{Z}) = H^1_{cont}(S^1, \mathbb{R}) = H^2_{cont}(S^1, \mathbb{Z}) = 0,$$

whereas

$$H^1_{cont}(S^1, S^1) = \mathrm{Hom}_{cont}(S^1, S^1) \simeq \mathbb{Z}.$$

Hence this theory does not have long exact sequences of cohomology groups even for the exponential sequence as above.

Further, the second cohomology group $H^2_{cont}(S^1, \mathbb{Z}) = (0)$. Thus, the second continuous cohomology does not account for even the natural exponential exact sequence given above.

2.2. **Moore cohomology theory.** The problem with continuous cohomology theory arises from the fact that continuous cross sections do not exist in general. However, we have the following theorem guaranteeing the existence of measurable sections:

Theorem 4 (Mackey, Dixmier)**.** *Let G be a polish group and H be a closed subgroup of G. Then H is a polish group and the projection map $G \to G/H$ admits a measurable cross section.*

We recall that a topological group G is said to be polish if its topology admits a complete separable metric.

Define the group $C^n_m(G, A)$ of measurable n-cochains with values in a topological G-module A to be the space of all measurable maps from G^n to A. With the coboundary map as before, we obtain a cochain complex $\{C^n_m(G, A), d^n\}$. We define the measurable or the Moore cohomology groups $H^n_m(G, A)$ are as cohomology groups $\{C^n_m(G, A), d^n\}$.

The measurable cohomology groups have many of the nice properties required for a cohomology theory, viz., long exact sequences of cohomology groups for any short exact sequence of G-modules, correct interpretation of the low rank cohomology groups, comparison with continuous cohomology, a form of Shapiro's lemma, weak forms of Hochschild-Serre spectral sequences, etc.

One of the difficult aspects of Moore's theory, is that even if we restrict ourselves to the category of topological groups and continuous G-modules, to compute the cohomology, we need to work in the measurable category. The measurable cochain groups are large and it is difficult to compute them. For instance, the construction of the extension group corresponding to a measurable 2-cocycle is not direct, and uses the topology on induced modules.

2.3. **Induced modules and extension groups.** Suppose that $G,\ A$ are second countable, locally compact topological groups and A is a topological G-module. Denote by $I(A)$ the group of measurable maps from G to A.

Since A is Polish, there exists a metric ρ on A whose underlying topology is the same as the original topology on A. Further, we can assume that ρ is bounded. We take a finite measure $d\nu$ on G, which is equivalent to the Haar measure on G (*cf.* [Mo76]). Define a metric on $I(A)$ as follows

$$\bar{\rho}(f_1, f_2) = \int\limits_G \rho(f_1(x), f_2(x))d\nu(x)$$

This makes $I(A)$ a Polish group. We define a G-action on $I(A)$ by,

$$(s \cdot f)(t) = sf(s^{-1}t) \quad \forall\ s,\ t \in G,\ f \in U(G, A).$$

Via this action $I(A)$ becomes a topological G-module and A embeds as a G-submodule of $I(A)$ as submodule of the constant maps. It can be seen that the higher measurable cohomology groups of $I(A)$ are trivial (*cf.* [Mo76]). We have a short exact sequence,

$$0 \to A \to I(A) \to U(A) \to 0.$$

Moore showed that the second measurable cohomology group parametrizes the collection of extensions:

Proposition 2. *Suppose a Polish group G acts continuously on an abelian Polish group A. Then $H^2_m(G, A)$ parametrizes the isomorphism classes of topological extensions of G by A,*

$$1 \to A \to E \to G \to 1.$$

Proof. Given a topological extension E of G by A, the construction of the 2-cocycle corresponding to the extension is done (as before) using the existence of the measurable cross-section guarenteed by the theorem of Dixmier-Mackey.

For our purpose, we briefly recall the proof of the converse. From the short exact sequence, and the cohomological triviality of $I(A)$, we obtain an isomorphism,

$$H^2_m(G,A) \simeq H^1_m(G,U(A)).$$

Corresponding to a 2-cocycle $b \in Z^2_m(G,A)$, we obtain a measurable crossed homomorphism, i.e., a continuous homomorphism $T: \ G \rightarrow: U(A) \rtimes G$. Let $\mathcal{E}$ be the image of $T(G)$. We have a short exact sequence,

$$0 \rightarrow A \rightarrow I(A) \rtimes G \rightarrow U(A) \rtimes G \rightarrow 0.$$

The required extension group E is obtained as the inverse image of $\mathcal{E}$ in $I(A) \rtimes G$. We equip the group E with subspace topology of $I(A) \rtimes G$. It can be verified that E is a closed subgroup of $I(A) \rtimes G$, and hence it is a Polish group. □

Remark 1. From this construction, it does not seem possible to directly relate the topology of E to that of G and A; for example, if G and A are locally compact, will E be locally compact? This question was already answered in the affirmative by Mackey (*cf.* [Mac]), but the proof is neither easy nor direct.

A similar problem arises when we work with Lie groups and we want to relate the manifold structure on E to that of G and A. This provides us another motivation (apart from the work of Lichtenbaum) for the construction of a cohomology theory based on measurable cochains which are continuous (or more generally are regular in a suitable sense) in a neighbourhood of identity.

3. Locally continuous measurable cohomology theory

We now describe our construction of a cohomology theory for topological groups, based on measurable cochains which are continuous in a neighbourhood of identity. We first establish basic properties of this cohomology theory. Our main aim is to show that the second cohomology group describes the equivalence classes of topological extensions that are locally split. The construction of the extension allows us to directly relate the topology on the extension group to that of the subgroup and the quotient group.

3.1. **Definition and basic functorial properties.** We define the group $C^n_{lcm}(G,A)$ of locally continuous measurable n-cochains on G with values in A to be the space of measurable functions from G^n to A which are continuous in some neighbourhood of identity in G^n. The standard coboundary map involves the group action, group multiplication on G, and composition in A.

$$(s_1, s_2, \cdots s_{n+1}) = s_1 \cdot (s_2, s_3, \cdots s_{n+1}),$$

$$(s_1, \cdots s_i, s_{i+1}, \cdots s_{n+1}) \mapsto (s_1, \cdots s_i s_{i+1}, \cdots s_{n+1}).$$

All these maps are continuous, and hence d^n in Definition 1 takes $C^n_{lcm}(G,A)$ to $C^{n+1}_{lcm}(G,A)$. Therefore, $\{C^\bullet_{lcm}(G,A), d^\bullet\}_{n\geq 0}$ forms a cochain complex. For every $n \geq 0$, we denote the group of n-cochains by

$$Z^n_{lcm}(G,A) = \{f\colon G^n \rightarrow A\colon \quad d^n f(s_1, s_2, \cdots, s_{n+1}) = 0\}.$$

and we denote the group of n-coboundaries by

$$B^n_{lcm}(G,A) = \text{Image } (d^{n-1}) \text{ for } n \geq 1.$$

We define $B^0_{lcm}(G,A) = 0$. The locally continuous cohomology groups $H^n_{lcm}(G,A)$ for $n \geq 0$ are defined as the n-th cohomology group of the cochain complex $\{C^\bullet_{lcm}(G,A), d^\bullet\}$, i.e.,

$$H^n_{lcm}(G,A) = \frac{Z^n_{lcm}(G,A)}{B^n_{lcm}(G,A)}.$$

It is clear that $H^0_{lcm}(G,A) = A^G$ the space of G-invariants in A. These cohomology groups lie between the continuous cohomology and the measurable cohomology, i.e., there are natural maps,

$$H^n_{cont}(G,A) \to H^n_{lcm}(G,A) \to H^n_m(G,A).$$

We will see that this cohomology theory addresses the problems related to the cohomology theories discussed earlier. We first verify some of the basic properties satisfied by these cohomology groups.

3.2. **Change of groups.** Let A, A' be topological modules for G, G' respectively. Suppose there are continuous homomorphism $\phi\colon G' \to G$, $\psi\colon A \to A'$ satisfying the following compatibility condition:

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ \phi\big\uparrow \quad \big\downarrow\psi & & \big\downarrow\psi \\ G' \times A' & \longrightarrow & A' \end{array}$$

$$g' \cdot \psi(a) = \psi(\phi(g') \cdot a) \quad \forall a \in A,\ g' \in G'.$$

Then there is a map of cohomology groups,

$$H^n_{lcm}(G,A) \to H^n_{lcm}(G',A').$$

In particular, this gives functorial maps for $G = G'$,

$$H^n_{lcm}(G,A) \to H^n_{lcm}(G,A').$$

For $G' = H$, a subgroup of G, we have the restriction homomorphism

$$H^n_{lcm}(G,A) \to H^n_{lcm}(H,A).$$

3.3. **Locally split short exact sequences.**

Definition 2. A short exact sequence of topological groups

$$0 \to G' \xrightarrow{\imath} G \xrightarrow{\jmath} G'' \to 0$$

is an algebraically exact sequence of groups with the additional property that $\imath$ is a closed and $\jmath$ is open. It is said to be *locally split*, if the homomorphism $\jmath$ admits a local section, i.e., there exists an open neighbourhood U'' of identity in G'' and a continuous map $\sigma\colon U'' \to G$ such that $\jmath \circ \sigma = id_{U''}$.

We recall the definition of a finite dimensional topological space and topological group:

Definition 3. A topological space X has finite topological dimension k, if every covering $\mathcal{U}$ of X has a refinement $\mathcal{U}'$ in which every point of X occurs in at most $k+1$ sets in $\mathcal{U}'$, and k is the smallest such integer. Finite dimensional topological groups are topological groups that have finite dimension as a topological space.

The following theorem due to Mostert (*cf.* [Mos, page 647]) provides examples of locally split short exact sequences:

Theorem. Let G be a finite dimensional locally compact group and H be a closed normal subgroup of G, then G/H admits a local cross-section.

This theorem is obvious when G is a Lie group, and it follows from the fact that that any finite dimensional locally compact group is an inverse limit of Lie groups. For detailed proof, refer to (*cf.* [Mos]).

Lemma 1. *Consider a locally split short exact sequence of topological groups* $1 \to G' \xrightarrow{\imath} G \xrightarrow{\jmath} G'' \to 1$. *Then there exists a measurable section* $\sigma : G'' \to G$ *which is continuous in a neighbourhood of identity on* G''.

Proof. By the hypothesis and Zorn's lemma, we can assume that there is a maximal $Z \subset G''$ measurable set containing an open neighbourhood of identity, and a section $\sigma : Z \to G$ continuous in a neighbourhood of identity on G''. Suppose Z is not equal to G. By translation, given an element w in the complement of Z in G'' we can find a neighbourhood U_w of w in G'', and a continuous section σ_w to $\jmath$ on U_w. Patching the sections σ and $\sigma_w|_{U_w \cap (G''-\mathbb{Z})}$ we get a measurable extension of σ. By maximality this implies $Z = G''$. □

Remark 2. When the group G'' is Lindelöf (meaning every open cover of G'' admits a countable subcover), the above lemma can be proved easily without using Zorn's lemma.

3.3.1. *Long exact sequence.* As a corollary of Lemma 1, we associate to each *locally split* short exact sequence, a long exact sequence of locally continuous measurable cohomology groups.

Corollary 1. *Consider a locally split short exact sequence of topological* G*-modules.* $0 \to A' \to A \to A'' \to 0$. *Then there is a short exact sequence of cochain complexes,*

$$0 \to C^*_{lcm}(G, A') \xrightarrow{\tilde{\imath}} C^*_{lcm}(G, A) \xrightarrow{\tilde{\jmath}} C^*_{lcm}(G, A'') \to 0.$$

Hence, there is a long exact sequence of locally continuous measurable cohomology groups,

$$0 \to H^0_{lcm}(G, A') \to H^0_{lcm}(G, A) \to H^0_{lcm}(G, A'') \xrightarrow{\delta} H^1_{lcm}(G, A') \to \cdots$$

Proof. Since this is fundamental to our construction of cohomology theories, we briefly indicate the construction of the connecting homomorphism

$$\delta : H^n_{lcm}(G, A'') \to H^{n+1}_{lcm}(G, A').$$

We choose a locally continous measurable section $\sigma : A'' \to A$ as given by Lemma 1. The connecting homomorphism is defined as,

$$\delta(s) = d(\sigma \circ s), \quad s \in Z^n_{lcm}(G, A'').$$

This gives a well-defined cocycle with values in A', and the cohomology class defined by this cocycle is independent of the choice of the section σ. □

3.3.2. *The first cohomology group.*

Proposition 3. *Suppose G is a topological group and and A is a topological G-module. Then*

$$Z^1_{lcm}(G, A) = Z^1_{cont}(G, A)$$

is the space of continuous crossed homomorphisms from G to A.

Proof. A locally continuous measurable 1-cocycle is a measurable function $c : G \to A$ satisfying the cocycle condition

$$c(s_1 s_2) = s_1 \cdot c(s_2) + c(s_1) \quad \forall s_1, s_2 \in G.$$

Further, there exists an open set $U \subset G$ containing identity such that $c|_U$ is continuous. For any $x \in G$ arbitrary, the map $c|_{xU}$ satisfies following formula.

$$c(xs) = x \cdot c(s) + c(x) \text{ for all } s \in U.$$

Since the group action is continuous and the map of translation by $c(x)$ is continuous on A, we see that c is continuous on xU. □

Remark 3. This holds in greater generality in the context of the measurable cohomology groups constructed by Moore. Using Banach's theorem that any measurable homomorphism between two polish groups is continuous, it can be seen that if G and A are locally compact and G acts continuously on A, then the first measurable cohomology group is the group of all continuous crossed homomorphisms from G to A.

3.4. **Other constructions.** It is possible to construct other cohomology theories imposing different conditions on the nature of the cochains. For example, the proof of Lemma 1 can be modified to extend a continuous section to a dense, open subset of G''. We can then construct a cohomology theory based on continuous cochains defined on dense open subsets of products of G (or even measurable cochains which are continuous on dense open subsets of products of G). However, such a cohomology theory does not have restriction maps to subgroups in general. Further, it seems difficult to relate the second cohomology group (based on continuous cochains defined on dense open subsets of products of G) to extensions of G.

Another construction can be based on set theoretic cochains which are continuous in a neighbourhood of identity of G^n. But here again, the second cohomology group does not seem to correspond to extensions of G having local sections.

4. Locally split extensions and $H^2_{lcm}(G, A)$

Let G, A be locally compact second countable topological groups, and assume that A is a continuous G-module. Our aim out here is to establish a bijective correspondence between the second cohomology group $H^2_{lcm}(G, A)$ and the collection of all *locally split* extensions E of G by A, i.e., those extensions for which there exists a continuous section for the map $\pi\colon E \to G$ in some neighbourhood of identity.

Theorem 5. *Let G be a locally compact, second countable topological group acting continuously on a locally compact, second countable abelian group A. Then the second cohomology group $H^2_{lcm}(G, A)$ parametrizes isomorphism classes of locally split extensions of G by A.*

Consider a locally split extension E of G by A. We now associate a unique cohomology class in $H^2_{lcm}(G, A)$. By Lemma 1, choose a measurable section $\sigma\colon G \to E$ which is continuous in a neighbourhood of identity in G. Define $f_\sigma\colon G \times G \to A$ as $f_\sigma(s_1, s_2) = \sigma(s_1)\sigma(s_2)\sigma(s_1 s_2)^{-1}$. It can be verified that f_σ satisfies the cocycle condition and is continuous in a neighbourhood of identity in $G \times G$. If we choose some other section τ having properties as above, then

$$f_\tau(s_1, s_2) = f_\sigma(s_1, s_2) + d(\sigma\tau^{-1})(s_1, s_2).$$

Since $\sigma, \tau : G \to A$ are in $C^1_{lcm}(G, A)$, we see that $\sigma\tau^{-1} : G \to A$ is measurable. It is continuous around identity and hence it gives a 1-cochain in $C^1_{lcm}(G, A)$. Therefore, $d(\sigma\tau^{-1})$ is a locally continuous 2-coboundary and hence, both the sections give the same class in $H^2_{lcm}(G, A)$.

4.1. **Neighbourhoods of identity in E.** Consider a measurable 2-cocycle $F\colon G \times G \to A$ which is continuous on a neighbourhood $U_F \times U_F \subset G \times G$ of identity. Since it is an abstract 2-cocycle, we get an abstract extension

$$1 \to A \xrightarrow{\imath} E \xrightarrow{\pi} G \to 1.$$

In order to topologize E, we define a base $\mathcal{B}$ for the neighbourhoods of identity in E that consists of sets of the form $U_A \times U_G$, where U_A and U_G are open neighbourhoods of identity in A and G respectively, such that $F|_{U_G \times U_G}$ is continuous. It is easy to see that $\mathcal{B}$ is a filter base in the terminology of Bourbaki (*cf.* [Bou, Chapter 1, Section 6.3]). Let us call any subset of E, containing some member of $\mathcal{B}$, as a neighbourhood of identity in E.

Since the cocycle F is continuous in a neighbourhood of identity, it can be verified that the multiplication map $E \times E \to E$ (resp. inverse map $E \to E$) are continuous in a neighbourhood of identity. From Proposition 1 of Bourbaki (*cf.*[Bou, Chapter 3, Section 1.2, page 221]), for E to be a topological group with $\mathcal{B}$ as a base for the neighbourhoods at identity, it is necessary and sufficient that inner conjugation by any element $a \in E$ is continuous at identity: for $a \in \tilde{E}$ and any $V \in \tilde{\mathcal{B}}$, there exists $V' \in \tilde{\mathcal{B}}$ such that $V' \subset a \cdot V \cdot a^{-1}$. We single this out as a theorem:

Theorem 6. *Let E be an extension of the group G by A corresponding to the 2-cocycle $F : G \times G \to A$ and is provided with the neighbourhood topology $\mathcal{B}$ defined above. Then for any $x \in E$, the map of inner conjugation $\imath_x : E \to E$ is continuous at identity.*

Proof of this theorem will occupy next few sections. The proof of this theorem is modelled on the proof of Banach's theorem that measurable homomorphims of second coutntable locally compact topological groups are continuous. Heuristically, this can be considered as saying that the topology of a locally compact group can be recovered from the underlying measure theory. Our proof of the above theorem makes this heuristic precise.

In our situation, E can be equipped with a measure structure since the cocycle F is measurable. We topologize E with a neighbourhood base filter $\mathcal{B}$, imposing the condition that left translation is continuous. We show that there exists a left invariant measure on E. This will allow us to define convolution of measurable functions. We then use the fact that the multiplication and inverse maps are continuous in a neighbourhood of identity e, together with a global argument to prove the above theorem.

4.2. **Topology on extension group.** We topologize E by considering the left translates $x\mathcal{B}$ as a base for the open neighbourhoods of $x \in E$. With this topology left multiplication by any element $x \in E$ is a continuous map from E to E. It is easy to observe the following proposition listing some basic properties of the topological space E.

Proposition 4.
- (i) *The homomorphisms $\imath$, π are continuous and π is an open map.*
- (ii) *There exists an open neighbourhood $U_F \subset G$ of identity in G, and a continuous section $\sigma\colon U_F \to E$.*
- (iii) *The inclusion $\imath\colon A \to E$ is a homeomorphism onto its image and $\imath(A)$ is a closed subset of E.*
- (iv) *E is a locally compact, second countable, Hausdorff space.*
- (v) *The Borel algebra on E is generated by members of the filters $\underset{x\in E}{\cup} x\mathcal{B}$. Moreover, the measure structure on E is product of measure structures on G and A.*
- (vi) *The group law and the inverse map on E are Borel measurable functions. Hence, the map $\imath_x\colon E \to E$ of inner conjugation by any $x \in E$ is Borel measurable.*

Proof. Since E is second countable, its Borel measurable sets are generated by small open sets, namely the members of $\underset{x\in E}{\cup} x\mathcal{B}$. We observe the formulae for group law and the inverse map:

$$(a_1, s_1)(a_2, s_2) = (a_1 + s_1 \cdot a_2 + F(s_1, s_2),\ s_1 s_2),$$
$$(a, g)^{-1} = (s^{-1} \cdot (-a) + s^{-1}(-F(s, s^{-1})),\ e_G).$$

The cocycle, $F : G \times G \to A$ is measurable, and $G,\ A$ are topological groups. Therefore, the group law and inverse map are measurable on the product measure space $\mathcal{M}_A \times \mathcal{M}_G$ which is the Borel measure space $\mathcal{M}_E$ on E. □

4.3. **Construction of left invariant measure on E.** In this section we construct a left invariant Borel measure on E. By Riesz representation theorem, it is equivalent to construct an integral on E which is invariant under left translation. For the construction of the left invariant integral, we follow the method of [Fe, Chapter 3, Section 7]. We remark that in our setting, the only change is given by Lemma 4, analogous to the uniform continuity lemma given by [Fe, Chapter 2, Proposition 1.9].

Let $C_c(E)$ denote the space of real valued continuous functions with compact support on E, and $C_c^+(E) \subset C_c(E)$ the subspace of functions taking nonnegative real values. We denote by f, g, h the functions in $C_c(E)$. For a function f on E and $u \in E$, let $f_u(x) = f(ux),\ x \in E$ denote the left translation of f by u.

Lemma 2. *Let E be as in Proposition (6), and let $f,\ g \in C_c^+(E)$. Let $g \neq 0$ with nonnegative values. Then there exist finitely many positive real numbers $c_1, c_2, \dots, c_r$ and elements $u_1, u_2, \dots, u_r \in E$ such that*

$$f \leq c_1 g_{u_1} + c_2 g_{u_2} + \cdots + c_r g_{u_r} \tag{1}$$

where $g_{u_i} \colon E \to \mathbb{R}$ is defined as $g_{u_i}(s) = g(u_i s)$, for all $s \in E$.

The proof follows from the compactness of the support of f. This allows us to define the following:

Definition 4. Suppose $f, g \in C_c^+(E)$ are as above, we define the approximate integral of f relative to g as

$$(f; g) = \inf \left\{ \sum_{i=1}^{r} c_i \right\},$$

where the tuple $(c_1, c_2, \dots, c_r\}$ runs over all the finite sequences of non-negative numbers for which there exist group elements $u_1, u_2, \dots, u_r$ satisfying the proposition above. By linearity, we define $(f; g)$ for any $f \in C_c(E)$.

Definition 5. Fix a compactly supported function $g : E \to \mathbb{R}^+$. If $f, \phi \in C_c^+(E)$ and $\phi \neq 0$, define $I_\phi(f) = (g; \phi)^{-1}(f; \phi)$

It can be seen that the approximate integral $I_\phi(f)$ satisfies the following properties. The arguments are similar and follows from analogous properties satisfied by $(f; g)$ (see [Fe, Chapter 3, Lemma 7.4 and page 202]):

Lemma 3. *Let $f, f_1, f_2, \in C_c^+(E)$. Then:*

(i) *If* $f \neq 0$ *then* $(g;f)^{-1} \leq I_\phi(f) \leq (f;g)$*;*
(ii) $I_\phi(f_x) = I_\phi(f)$ for all $x \in G$*;*
(iii) $I_\phi(f_1 + f_2) \leq I_\phi(f_1) + I_\phi(f_2)$*;*
(iv) $I_\phi(cf) = cI_\phi(f)$ *for all* $c \in \mathbb{R}_{\geq 0}$*.*

We next show that, if ϕ has small compact support, I_ϕ is "nearly additive". For this purpose, we require a lemma on uniform continuity, the analogue of [Fe, Corollary 1.10, page 167], whose proof we give since we do not yet have an uniform structure on E.

Lemma 4. *Let f be a real valued continuous function on E and $\epsilon > 0$. Suppose C is a compact subset of E. Then, there is a neighbourhood V of e such that*

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } x,\ y \in C,\ x^{-1}y \in V.$$

Proof. Suppose the lemma is not true. Then there exists $\epsilon > 0$, a sequence V_i of neighbourhoods of identity e in E with $\cap V_i = \{e\}$, elements $x_i,\ y_i \in C$ with $x_i^{-1}y_i \in V_i$, such that

$$|f(x_i) - f(y_i)| \geq \epsilon.$$

We can assume that $V_{i+1}V_{i+1} \subset V_i$. Since $x_i,\ y_i$ are in a compact set C, we can assume by passing to a subsequence, that the sequence x_i (resp. y_i) converges to x_0 (resp. y_0). Since f is continuous,

$$|f(x_0) - f(y_0)| \geq \epsilon.$$

Since, $\{x_i\}$ converges to x_0, choose $N_k \geq k+1$ such that $x_i \in x_0V_{k+1}$ for $i \geq N_k \geq k+1$. Now,

$$y_i \in x_iV_i \subset (x_0V_{k+1}) \cdot V_i.$$

Since $i \geq k+1$,

$$(x_0V_{k+1}) \cdot V_i \subset (x_0V_{k+1}) \cdot V_{k+1} \subset x_0V_k.$$

Therefore, $y_i \in x_0V_k$ whenever $i \geq N_k$. Hence the sequence $\{y_i\}$ converges to x_0. Since f is continuous, this gives a contradiction.

□

We now prove that I_ϕ is nearly additive.

Lemma 5. *Given $f_1, f_2 \in C_c^+(E)$ and $\epsilon > 0$, we can find a neighbourhood V of e such that, if $\phi \in C_c^+(E)$ is non-zero and supp$(\phi) \subset V$, then*

$$|I_\phi(f_1) + I_\phi(f_2) - I_\phi(f_1 + f_2)| \leq \epsilon \tag{2}$$

The proof is similar to the proof of Lemma in [Fe, Chapter 3, 7.7], only we use Lemma 4 in place of the lemma on uniform continuity available for a locally compact topological group.

Proof. Fix a non-zero function $f' \in C_c^+(E)$ which is strictly positive on the $supp(f_1 + f_2)$. We can find this function because E is locally compact and Hausdorff. Choose δ to be a small positive number such that

$$(f'; g)\delta(1 + 2\delta) + 2\delta(f_1 + f_2; g) < \epsilon. \tag{3}$$

Now put $f = f_1 + f_2 + \delta f'$, and define

$$h_i(x) = \begin{cases} \frac{f_i(x)}{f(x)} & \text{if } f(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The functions $h_1,\ h_2$ have compact support. By the left uniform continuity (*cf.* Lemma 4) applied to $h_i,\ i = 1, 2$, we can choose a neighbourhood V of identity so that

$$|h_i(x) - h_i(y)| < \delta \text{ for } i = 1, 2 \quad \text{and } x^{-1}y \in V. \tag{4}$$

We can further assume by restricting to a smaller neighbourhood of identity $e \in E$ that multiplication and inverse maps are continuous on V.

Now let ϕ be any non-zero function in $C_c^+(E)$ with support contained inside V. We shall prove the lemma for ϕ. By Lemma 2, we can find $0 \leq c_j \in \mathbb{R}$ and $u_j \in G$ such that

$$f \leq \sum_{j=1}^{r} c_j \phi_{u_j}.$$

If $\phi(u_j x) \neq 0$, we have $u_j x \in V$. Therefore by Lemma 4,

$$|h_i(u_j^{-1}) - h_i(x)| < \delta.$$

Hence for $i = 1, 2$ and for every $x \in E$, we have

$$f_i(x) = h_i(x) f(x) \leq \sum_{j=1}^{r} c_j \phi(u_j x) h_i(x) \leq \sum_{j=1}^{r} c_j \phi(u_j x)(h_i(u_j^{-1}) + \delta).$$

This implies,

$$(f_i; \phi) \leq \sum_{j=1}^{r} c_j (h_i(u_j^{-1}) + \delta), \quad i = 1, 2.$$

But $h_1 + h_2 \leq 1$ implies that

$$(f_1; \phi) + (f_2; \phi) \leq \sum_{j=1}^{r} c_j (1 + 2\delta).$$

Taking infimum over $\{\sum c_j\}$, we obtain

$$(f_1; \phi) + (f_2; \phi) \leq (1 + 2\delta)(f; \phi)$$

Multiplying by $(g; \phi)^{-1}$, we get the relation of relative integrals,

$$I_\phi(f_1) + I_\phi(f_2) \leq (1 + 2\delta) I_\phi(f).$$

By properties of I_ϕ, and the choice of f', we see that

$$I_\phi(f_1) + I_\phi(f_2) \leq \delta I_\phi(f').$$

Therefore we have,

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + 2\delta I_\phi(f_1 + f_2) + \delta(1 + 2\delta)I_\phi(f') < I_\phi(f_1 + f_2) + \epsilon.$$

This proves the lemma. □

This completes the proof that I_ϕ is nearly additive, when ϕ has sufficiently small compact support. From Lemma 3 we also have

$$(g; f)^{-1} \leq I_\phi(f) \leq (f; g)$$

whenever $g \neq 0$ and f, ϕ, g are compactly supported real valued functions on E.

Proposition 5. *There exists a non-zero left invariant integral on E.*

Proof. For each $f \in C_c^+(E),\ f \neq 0$, let $S_f = [(g; f)^{-1}, (f; g)]$, a compact interval. Consider the set $S = \prod_{0 \neq f \in C_c^+(E)} S_f$ (with the cartesian product topology). By Tychonoff's theorem S is compact. Let $\{\phi_i\}$ be a net of non-zero elements of $C_c^+(E)$ such that, for each neighbourhood V of identity, $supp(\phi_i) \subset V$ for sufficiently large i. By the properties of I_{ϕ_i}, we know that $I_{\phi_i} \in S$ for each i. Since S is compact, we can replace $\{\phi_i\}$ by a subnet, and assume that $I_{\phi_i} \to I$ in S. Putting $I(0) = 0$, and passing to the limit over i, we get from the properties of I_{ϕ_i} the following:

(i) $(g; f)^{-1} \leq I(f) \leq (f; g)$, if $0 \neq f \in C_c^+(E)$.
(ii) $I(f_x) = I(f)$, for all $f \in C_c^+(E)$; for all $x \in E$.
(iii) $I(cf) = cI(f)$for all $c \in \mathbb{R}_+$; and for all $f \in C_c^+(E)$.
(iv) $I(f_1 + f_2) = I(f_1) + I(f_2)$ for all $f_1, f_2 \in C_c^+(E)$.

Now any continuous function $f : E \to \mathbb{C}$ with compact support, can be written as $f = (f_1 - f_2) + i(f_3 - f_4)$, with each $f_i \in C_c^+(E)$. Define

$$I(f) = I(f_1) - I(f_2) + i(I(f_3) - I(f_4)) \tag{5}$$

□

Remark 4. This integral defines a left invariant measure on E, Notice that the integral is positive, i.e., $I(f) > 0$, whenever $0 \neq f \in C_c^+(E)$. Let μ be the Borel measure on E, corresponding to the left invariant integral I on E. If $x \in G$, we see that

$$\mu(W) = \mu(xW)$$

where W is a Borel subset of E whose closure is compact. Further, if K is any compact subset of E then $\mu(K) < \infty$.

4.4. **Global argument.** We now derive a consequence of the existence of a non-trivial left invariant integral on E. We start with a general observation:

Lemma 6. *Let G_1, G_2 be two groups, and $f : G_1 \to G_2$ be a group homomorphism. Suppose that G_1, G_2 are topological spaces, and G_2 satisfies Lindelöf property. Assume further that there exist non-zero left invariant measures μ_1 (resp. μ_2), on the Borel subalgebra of G_1 (resp. of G_2).*

Let f be measurable and $W \subset G_2$ be an arbitrary open subset. Then

- *The measure $\mu_2(W) > 0$.*
- *If f is surjective, the measure of the preimage $\mu_1(f^{-1}(W)) > 0$.*

Proof. Since W is open in G_2 and G_2 is Lindelöf, there exist countably many left translates $\{t_i W\}_{i\in\mathbb{N}}$ which cover G_2. Since the measure is left invariant and non-zero, it follows that the measure of W is positive.

Since f is surjective, there exist elements $s_i \in G_1$, such that $f(s_i) = t_i$. Since f is measurable, the inverse image $f^{-1}(W)$ is a measurable subset of G_1. Since $\{t_i W\}_{i\in\mathbb{N}}$ cover G_2, the collection $\{s_i f^{-1}(W)\}_{i\in\mathbb{N}}$ covers G_1. Since the measure μ_1 is non-zero on G_1, it follows that $\mu_1(f^{-1}(W)) > 0$.

□

We apply this global argument when $E = G_1 = G_2$ with $f = \imath_x$ inner conjugation by an element $x \in E$:

Corollary 2. *Let E be as above and μ denote the left invariant measure constructed in the foregoing subsection. Let W be an open subset of E and x an element of E. Then*

$$\mu(i_x^{-1}(W)) > 0.$$

4.5. **Convolution.** The proof of Banach's theorem for locally compact groups proceeds by first showing that convolution of measurable functions satisfying suitable properties is continuous. In our context, we can carry out such an argument for measurable functions supported in a sufficiently small neighbourhood of identity in E. However, here we establish directly a statement that suffices for proving Theorem 6. The proof makes more use of symmetric subsets, has the advantage of simplifying the required arguments in our context by reducing the requirement of uniform continuity to Lemma 4. The key proposition is the following:

Proposition 6. *Let M be a measurable, symmetric (i.e. $M = M^{-1}$) subset of E. Suppose that $M \subset \pi^{-1}(U_2)$, for U_2 a symmetric relatively compact open neighbourhood of identity in G such that the product of the closures $\overline{U}_2\overline{U}_2 \subset U_F$. Assume that identity $e \in M$ and measure $\mu(M)$ is positive and finite. Then the set*

$$MM = \{xy \colon x \in M, y \in M\}$$

contains an open neighbourhood of identity in E.

Granting this proposition, we now prove Theorem 6.

Proof of Theorem 6. We need to show that for any sufficiently small neighbourhood V of identity in E the set $\imath_x^{-1}(V)$ contains an open neighbourhood of identity $e \in E$. Let W be a symmetric open neighbourhood of e in E satisfying following

(i) W is symmetric (i.e., $W = W^{-1}$)
(ii) $W \subset \pi^{-1}(xU_2x^{-1})$
(iii) $WW \subset V$

Let $M' = \imath_x^{-1}(W)$. By Corollary 2, $\mu(M') > 0$. Since $W \subset \pi^{-1}(xU_2x^{-1})$, we have $\imath_x^{-1}(W) \subset \pi^{-1}(U_2)$. Intersecting wih a symmetric compact set K containing identity $e \in E$, we can assume that $M = \imath_x^{-1}(W) \cap K$ has finite, positive measure, and is contained inside $\pi^{-1}(U_2)$. By Proposition 6, we see that the product set MM contains an open neighbourhood V_x' of e. Now

$$\imath_x^{-1}(V) \supseteq \imath_x^{-1}(WW) \supset MM \supset V_x'.$$

This proves Theorem 6. □

We now proceed to the proof of Proposition 6. For $x \in E$, define the function

$$u(x) = \mu(M \cap xM).$$

The proof of the proposition reduces to the following lemma:

Lemma 7. *Under the hypothesis of Proposition 6, u is a continuous function.*

Assuming Lemma 7 we now prove Proposition 6.

Proof of Proposition 6. If $u(x) \neq 0$. them $M \cap xM \neq \emptyset$. Hence $x \in MM^{-1} = MM$ as M is assumed to be symmetric. Further, $u(e) = \mu(M) > 0$. Since u is continuous, this proves Proposition 6. □

We now proceed to the proof of Lemma 7.

Proof of Lemma 7. Let χ_M denote the characteristic function of M. Then $u(x)$ can be defined by the following integral.

$$\begin{aligned} u(x) &= \mu(M \cap xM) \\ &= \int_{M \cap xM} d\mu(y) \\ &= \int_{M} \chi_M(x^{-1}y) d\mu(y) \\ &= \int_{E} \chi_M(y)\chi_M(x^{-1}y) d\mu(y). \end{aligned}$$

We observe that support of u is contained inside $MM \subseteq U_2U_2$. Since $M \subset \pi^{-1}(U_2)$, by Lusin's theorem choose a function $f \in C_C(E)$ with support contained inside $\pi^{-1}(\overline{U}_2)$

such that

$$\int_E |\chi_M(y) - f(y)| \; d\mu(y) < \epsilon_1,$$

for some sufficiently small $\epsilon_1 > 0$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to x_0 in $\pi^{-1}(U_2U_2)$. To show the continuity of u restricted to $\pi^{-1}(U_2U_2)$, it is enough to show that the sequence $\{u(x_n)\}$ converges to $u(x_0)$. We have

$$|u(x_n) - u(x_0)| = \left| \int_E \chi_M(y)(\chi_M(x_n^{-1}y) - \chi_M(x_0^{-1}y))d\mu(y) \right|.$$

Since M is symmetric, we also have $\chi_M(y^{-1}) = \chi_M(y)$. Therefore,

$$|u(x_n) - u(x_0)| \leq \int_E \chi_M(y)|\chi_M(y^{-1}x_n) - \chi_M(y^{-1}x_0)|d\mu(y).$$

We have

$$\begin{aligned} |u(x_n) - u(x_0)| \leq & \int_E \chi_M(y)|\chi_M(y^{-1}x_n) - \tilde{f}(y^{-1}x_n)|d\mu(y) \\ & + \int_E \chi_M(y)|\tilde{f}(y^{-1}x_n) - \tilde{f}(y^{-1}x_0)|d\mu(y) \\ & + \int_E \chi_M(y)|\tilde{f}(y^{-1}x_0) - \chi_M(y^{-1}x_0)|d\mu(y), \end{aligned}$$

where $\tilde{f}(z) = f(z^{-1})$ for $z \in E$. Since the integral is left invariant, by replacing y by x_ny (resp. by x_0y) in the first (resp. third) term on the right, we see that

$$\begin{aligned} \int_E \chi_M(y)|\chi_M(y^{-1}x_n) - \tilde{f}(y^{-1}x_n)|d\mu(y) &= \int_E \chi_M(x_ny)|\chi_M(y^{-1}) - \tilde{f}(y^{-1})|d\mu(y) \\ &\leq \int_E |\chi_M(y^{-1}) - \tilde{f}(y^{-1})|d\mu(y) \\ &= \int_E |\chi_M(y) - f(y)|d\mu(y) \\ &< \epsilon_1. \end{aligned}$$

Here we have used the fact that M is symmetric and definition of $\tilde{f}$. Similarly, we obtain

$$\int_E \chi_M(y)|\tilde{f}(y^{-1}x_0) - \chi_M(y^{-1}x_0)|d\mu(y) < \epsilon_1.$$

Now we estimate the middle term. Since inverse map is continuous in $\pi^{-1}(U_F)$ and support of f is contained inside $\pi^{-1}(U_F)$, the function $\tilde{f}(y) = f(y^{-1})$ is continuous.

Given $\epsilon_2 > 0$, by Lemma 4, there exists a symmetric neighbourhood W_2 contained inside $\pi^{-1}(U_F)$ (here again we are using fact that inverse map is continuous on $\pi^{-1}(U_F)$) such that

$$|\tilde{f}(z_1) - \tilde{f}(z_2)| < \epsilon_2 \quad \text{for} \quad z_1^{-1}z_2 \in W_2.$$

Since x_n converges to x_0, there exists a natural number N such that for $n \geq N$,

$$x_n \in x_0 W_2. \quad i.e., \quad x_0^{-1}x_n \in W_2.$$

Since W_2 is symmetric, this condition can be rewritten as

$$x_n^{-1}x_0 \in W_2.$$

Hence,

$$(y^{-1}x_n)^{-1}(y^{-1}x_0) = x_n^{-1}x_0 \in W_2.$$

By applying Lemma 4 to the continuous function $\tilde{f}$, we obtain

$$|\tilde{f}(y^{-1}x_n) - \tilde{f}(y^{-1}x_0)| \leq \epsilon_2, \text{ for } n \geq N.$$

Hence for $n \geq N$, the middle term can be estimated as

$$\int_E \chi_M(y)|\tilde{f}(y^{-1}x_n) - \tilde{f}(y^{-1}x_0)|d\mu(y) < \epsilon_2 \int_E \chi_M(y) < \epsilon_2\mu(M).$$

Combining the above estimates, we obtain

$$|u(x_n) - u(x_0)| < 2\epsilon_1 + \epsilon_2\mu(M) \text{ for all } n \geq N.$$

This establishes continuity of u and hence proves Lemma 7. □

4.6. **Comparison with other cohomology theories.** Suppose G is a locally compact group acting on a locally compact group A. Then we have a natural map

$$H^2_{lcm}(G, A) \to H^2_m(G, A).$$

As a corollary to Banach's theorem we show that the above map is injective:

Corollary 3. *Let G, A be locally compact, second countable groups. Then the natural map*

$$H^2_{lcm}(G, A) \to H^2_m(G, A)$$

is injective.

Proof. Suppose a 2-cohomology class $\underline{c}$ in $H^2_{lcm}(G, A)$ is trivial in $H^2_m(G, A)$. Construct the corresponding extension E of $\underline{c}$. Then $\underline{c} = 0$ in $H^2_m(G, A)$, implies that there exists a measurable section $\sigma : G \to E$ which is a group homomorphism. By Theorem 6, we know that E is locally compact. It can be seen that E is also second countable. Hence by Banach's theorem, σ is a continuous group homorphism, and this implies that the extension $E \simeq A \rtimes G$. □

Corollary 4. *Suppose that either of the following conditions hold:*

(i) *G is a profinite group and A is a discrete G-module.*
(ii) *G is a Lie group and A is a finite dimensional vector space.*

Then the natural maps,

$$H^2_{cont}(G, A) \to H^2_{lcm}(G, A) \to H^2_m(G, A),$$

are isomorphisms.

Proof. This follows from the previous corollary and the isomorphism

$$H^2_{cont}(G, A) \to H^2_m(G, A)$$

for the given cases (*cf.* [Mo76]). □

5. Cohomology theory for Lie groups

In this chapter, we work in the smooth category in the context of Lie groups G, A with smooth actions $G \times A \to A$. Here we can define an analogous cohomology theory $H^n_{lsm}(G, A)$ where we impose the condition that the measurable cochains are locally smooth and study some of its basic properties. We show in the context of Lie groups that the second locally smooth measurable cohomology group $H^2_{lsm}(G, A)$ parametrizes the collection of locally split extensions of G by A. Further we observe as a corollary to the solution of Hilbert's Fifth problem and Theorem 6, an isomorphism of $H^2_{lcm}(G, A)$ with $H^2_{lsm}(G, A)$.

Let G be a Lie group and A be a smooth G-module. Analogous to the construction in the continuous case, we form a cochain complex $\{C^n_{lsm}(G, A), d^n\}_{n\geq 0}$. This starts with $C^0_{lsm}(G, A) = A$, and for higher n, $C^n_{lsm}(G, A)$ is the group of all measurable functions $f\colon G^n \to A$, which are smooth around identity. It is easily checked that the standard coboundary operator restricts to define a cochain complex.

Definition 6. The locally smooth cohomology theory $\{H^n_{lsm}(G, A)\}_{n\geq 0}$ is defined as the cohomology groups of the cochain complex $\{C^n_{lsm}(G, A); d^n\}_{n\geq 0}$.

It is clear that there exists a map $H^n_{lsm}(G, A'') \to H^n_{lcm}(G, A'')$. It is easy to establish the following properties of $H^n_{lsm}(G, A)$.

Proposition 7. (i) $H^0_{lsm}(G, A) = A^G$.

(ii) *The first cohomology group* $H^1_{lsm}(G, A)$ *is the group of all smooth crossed homomorphisms from* G *to* A.

(iii) *Given a short exact sequence of* G*-modules*

$$0 \to A \xrightarrow{\imath} A \xrightarrow{\jmath} A \to 0,$$

there is a long exact sequence of cohomology groups

$$\cdots \to H^i_{lsm}(G, A') \to H^i_{lsm}(G, A) \to H^i_{lsm}(G, A'') \xrightarrow{\delta} H^{i+1}_{lsm}(G, A) \to \cdots .$$

Proof. The proof of (ii) follows from the smoothness at identity and the cocycle condition. For (iii), given a short exact sequence of G-modules, it follows from the property that the sequence is locally split (i.e. $\jmath$ admits a smooth local section). We extend the section to a locally smooth measurable section $\sigma : A'' \to A$. Arguing as before, we obtain the long exact sequence of cohomology groups. □

Remark 5. We can introduce an analogous cohomology theory in the holomorphic context based on measurable cochains which are holomorphic in a neighbourhood of identity. In this context, we observe that the first cohomology group $H^1_{lhm}(G, A)$ is the space of all holomorphic crossed homomorphisms from G to A (see Proposition 1). Since the closed graph theorem is not applicable in the holomorphic context, we cannot obtain this result from the smooth version by an application of arguments as above.

5.1. **Extensions of Lie groups.** We describe now the second cohomology group.

Theorem 7. *Let G be a Lie group and A be a smooth G-module. Then the second cohomology group $H^2_{lsm}(G, A)$ parametrizes equivalence classes of extensions E of G by A,*

$$1 \to A \xrightarrow{\imath} E \xrightarrow{\pi} G \to 0,$$

where E is a Lie group with a measurable cross section $\sigma : G \to E$ such that σ is smooth around identity in G.

Proof. Given an extension E of G by A with a locally smooth measurable cross section $\sigma : G \to E$, we assign to it the 2-cohomology class of $F_\sigma : G \times G \to A \in Z^2_{lcm}(G, A)$ that takes $(s_1, s_2) \in G \times G$ to $F_\sigma(s_1)F_\sigma(s_2)F_\sigma(s_1s_2)^{-1}$.

For proving converse, we take an arbitrary cohomology class $\bar{F} \in H^2_{lsm}(G, A)$. Choose a representative F and construct an abtract group extension E

$$1 \to A \xrightarrow{\imath} E \xrightarrow{\pi} G \to 1.$$

Suppose G^0 is the connected component of identity in G, we claim that the subgroup $E^0 := \pi^{-1}(G^0) \prec E$ is a Lie group. Since G^0 is a normal subgroup of G and π is surjective, the subgroup E^0 is normal in E. We shall first verify that E^0 is a Lie group (we remark here that we can carry out a similar argument in the continuous case to show directly that E^0 is a topological group, instead of using Theorem 6). We then use the measurability condition to show that the extension E is a Lie group.

Since the cocycle F is smooth in a neighbourhood of identity, we assume that U_G is sufficiently small so that the following holds:

- The product map
$$(x, y, z) \mapsto xyz$$
is smooth from $U_G \times U_G \times U_G$ to U_F. This can be ensured by assuming that the following functions are smooth on $U_G \times U_G \times U_G$:

$$(s_1, s_2, s_3) \mapsto F(s_1s_2, s_3) \quad \text{and} \quad (s_1, s_2, s_3) \mapsto F(s_1, s_2s_3). \tag{6}$$

- The map $s \mapsto s^{-1}$ is smooth from $\pi^{-1}(U_G)$ to $\pi^{-1}(U_G)$ (here we have assumed U_G is symmetric).

We define an atlas on E^0 by imposing that left translations are diffeomorphisms and imposing the product of smooth structure on $\pi^{-1}(U_G) \simeq A \times U_G$, i.e., the atlas consists of $(xU, \phi \circ L_{x^{-1}})$, where $x \in E^0$ and U is an open subset of $\pi^{-1}(U_G)$. Here (U, ϕ) is a part of the atlas for the product smooth structure on $\pi^{-1}(U_G)$.

We first claim that this gives us an atlas: suppose there exists elements $x,\ y \in E^0$ and open sets $U,\ V$ contained inside $\pi^{-1}(U_G)$ such that $xU \cap yV \neq \emptyset$. By taking the union of U and V, we can assume that $U = V$. We have the charts,

$$xU \xrightarrow{L_{x^{-1}}} U \xrightarrow{\phi} W$$

$$yU \xrightarrow{L_{y^{-1}}} U \xrightarrow{\phi} W$$

where W is an open subset in some Euclidean space. Let $V = U \cap x^{-1}yU$ be the image of $L_{x^{-1}}(xU \cap yU)$. We need to show that the map,

$$\phi \circ L_{y^{-1}} \circ L_x \circ \phi^{-1} : \phi(V) \to W \text{ is smooth.}$$

For this it is enough to show that

$$L_{y^{-1}x} : V \to U \text{ is smooth.}$$

The hypothesis implies that there exists elements $z,\ z'$ in U such that $xz = yz'$, i.e., $y^{-1}x = z'z^{-1}$. This implies that

$$y^{-1}x \in \pi^{-1}(U_G) \times \pi^{-1}(U_G).$$

Hence the required smoothness follows from the assumption that the triple product is smooth from $\pi^{-1}(U_G) \times \pi^{-1}(U_G) \times \pi^{-1}(U_G)$ to $\pi^{-1}(U_F)$. This concludes the proof that E^0 with the above atlas is a smooth manifold. We remark that the manifold structure is such that left translations are diffeomorphisms.

We now have to show that E^0 is a Lie group. For this we first observe that inner conjugation by any element $x \in E^0$ is smooth at identity. Since G^0 is a connected Lie group, the neighbourhood U_G generates G as a group. It follows that the group E^0 is generated by $\pi^{-1}(U_G)$. Hence any element $x \in E^0$ can be written as

$$x = x_1 \cdots x_r, \text{ where each } x_i \in \pi^{-1}(U_G).$$

By our choice of U_G, inner conjugation by any $x_i \in \pi^{-1}(U_G)$ is smooth at identity. Since the inner conjugation by x is a composite of inner conjugations by the elements x_i, it follows that inner conjugation by any element of $x \in E^0$ is smooth at identity.

We now show that the multiplication map $E^0 \times E^0 \to E^0$ is smooth. Suppose $x,\ y \in E^0$. Let U be a sufficiently small neighbourhood of identity in E such that the conjugation map $z \mapsto y^{-1}zy$ is smooth where $z \in U$. Now the multiplication map $xU \times yU$ can be written as,

$$(xz)(yz') = (xy)(y^{-1}zy)z' \quad z,\ z' \in U.$$

We can assume that $U,\ y^{-1}Uy \subset \pi^{-1}(U_G)$. Since left multiplication by xy is smooth, and multiplication is smooth on $\pi^{-1}(U_G) \times \pi^{-1}(U_G)$, we conclude that multiplication is a smooth map from $E^0 \times E^0$ to E^0.

Similarly, to show that the inverse map is smooth on E^0, say around $x \in E^0$, we take U to be a sufficiently small neighbourhood of identity in E^0 such that $z \mapsto$

$xz^{-1}x^{-1}$, $z \in U$ is smooth on U. (we use the fact that inverse map is smooth on $\pi^{-1}(U_G)$ and assume that $U \subset \pi^{-1}(U_G)$). Now,

$$(xz)^{-1} = x^{-1}(xz^{-1}x^{-1}), \quad z \in U.$$

As left translations are smooth, it follows that the inverse map is smooth on E^0. This concludes the proof that E^0 is a Lie group.

Remark 6. We remark again out here, that the above arguments did not require to start with that E or E^0 is a topological group. The above arguments, carried out in the continuous category, will directly yield that E^0 is a topological group. We have only used the fact that any neighbourhood of identity in G generates G as a group and that cocyles are locally regular (locally regular means locally continuous or locally smooth depending on the setting).

Now we want to conclude that E is a Lie group. For this, we first show that E is a topological group. Since the cocycle is measurable, we see that inner conjugation i_x by any element $x \in E$ is a measurable automorphism of E^0. By Banach's theorem, it follows that i_x is continuous on E^0. (in particular it follows that E is a topological group).

Since $\imath_x$ is continuous on E^0, the graph of $\imath_x$ is closed in $E^0 \times E^0$. Therefore, the graph of $\imath_x$ is a closed subgroup of the Lie group $E^0 \times E^0$. Therefore,the graph of $\imath_x$ is a Lie group of $E^0 \times E^0$. Therefore, that i_x is a smooth diffeomorphism of E^0. We now argue as above to conclude that E is a Lie group. Therefore we get the following short exact sequences of topological groups,

$$1 \to A \xrightarrow{\imath} E \xrightarrow{\pi} G \to 1.$$

Since E and G are Lie groups with a continuous group homomorphism $\pi : E \to G$, we see that graph of π is a closed subgroup of $E \times G$ which is a Lie group. Therefore graph of π is a Lie subgroup. This implies that π is smooth. By implicit function theorem, π admits a smooth cross section in a neighbourhood of identity. We use arguments similar to those used in proving Lemma 1 and extend this to a locally smooth measurable cross section σ from G to E. This concludes proof of Theorem 7. □

5.2. **A comparison theorem.** In this section, as a corollary of positive solution to Hilbert's fifth problem, we show the following:

Theorem 8. *Let G be a Lie group and A be a smooth G-module. Then the natural map,*

$$H^2_{lsm}(G, A) \to H^2_{lcm}(G, A),$$

is an isomorphism.

Proof. Let $F : G \times G \to A$ be a locally continuous measurable 2-cocycle on G with values in A. We shall show that F is cohomologous to a locally smooth measurable 2-cocycle b. By Theorem 6, we obtain a locally split (topological) extension E of G

by A:

$$1 \to A \xrightarrow{\imath} E \xrightarrow{\pi} G \to 1.$$

Denote by E^c the connected component of E containing identity. Since the extension is locally split and G and A are Lie groups, it follows that E^c is locally Euclidean. Hence by positive solution to Hilbert fifth problem (*cf.* [Y],[Gl],[Mo-Zi]), we conclude that E^c is a Lie group.

Now the map $\pi|_{E^c} \to G$ is a continuous homomorphism. Hence the graph of $\pi|_{E^c}$ is a closed subgroup of the Lie group $E^c \times G$. Therefore, it is a Lie subgroup and this shows that the projection map $\pi|_{E^c}$ is smooth. Applying the implicit function theorem, we can find a smooth cross section of π in a neighbourhood of identity on G to E^c. By arguments similar to Lemma 1, we extend this to a measurable section σ from G to E.

The section σ gives raise to a 2-cocycle $b_\sigma : G \times G \to A$ in $Z^2_{lsm}(G, A)$ given by the formula $b_\sigma(s_1, s_2) = \sigma(s_1)\sigma(s_2)\sigma(s_1 s_2)^{-1}$. One can observe that b_σ is cohomologous to F in $Z^2_{lcm}(G, A)$. This yields a surjective map

$$H^2_{lsm}(G, A) \to H^2_{lcm}(G, A).$$

We, next claim this map to be injective. Suppose a class $\underline{\text{b}} \in H^2_{lsm}(G, A)$ is trivial in $H^2_{lcm}(G, A)$. Corresponding to $\underline{\text{b}} \in H^2_{lsm}(G, A)$, by Theorem 7 we obtain a Lie group E which is an extension of G by A Since $\underline{\text{b}} = 0$ in $H^2_{lcm}(G, A)$, there exists a locally continuous measurable section $\sigma : G \to E$ which is a group homomorphism. Since it is continuous at identity, it is continuous everywhere. Hence we obtain a continuous isomorphism between the Lie groups E and $A \rtimes G$. By an application of the closed graph theorem, this isomorphism is smooth. Therefore, the cohomology class $\underline{\text{b}}$ is trivial in $H^2_{lsm}(G, A)$. Hence it follows that

$$H^2_{lsm}(G, A) \to H^2_{lcm}(G, A)$$

is an isomorphism. $\square$

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