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Critical exponents of self-avoiding walks on fractals with dimension 2-ε

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Résumé. — Nous étudions les exposants critiques de marches auto-évitantes sur une famille de fractals du type Sierpinski de connectivité finie. Les éléments de la famille sont caractérisés par un entier b, $2 \leqslant b < \infty$. Pour b grand, la dimension fractale du réseau tend vers 2 par valeurs inférieures. Nous montrons que, quand $b \to \infty$, l'exposant de la susceptibilité ne tend pas vers sa valeur en dimension 2 et nous déterminons la correction dominante autour de $b = \infty$.

Abstract. — We study critical exponents of self-avoiding walks on a family of finitely ramified Sierpinki-type fractals. The members of the family are characterized by an integer b, $2 \leqslant b < \infty$. For large b, the fractal dimension of the lattice tends to 2 from below. We use scaling theory to determine the critical exponents for large b. We show that as $b \to \infty$ the susceptibility exponent does not tend to its 2-dimensional value, and determine the leading correction to critical exponents for large but finite b.

1. Introduction.

In order to understand the influence of dimensionality on phase transitions, the critical behaviour of various Hamiltonians on fractal lattices has been extensively investigated in recent years. These have been studied using exact real-space renormalization group techniques [1-4], approximate renormalization schemes using bond-moving [5, 6], and more recently with Monte Carlo techniques [7-8]. These studies showed that the critical exponents on fractals do not depend on a single parameter such as the Hausdorff dimension. Several parameters such as the spectral dimension [2], lacunarity [3] etc. have been suggested as additional characterizations of the fractal that influence critical behaviour. In general, however, the extent to which universality of critical exponents applies to fractal lattices, and the relationship of critical exponents on fractals with those on integer-dimensional lattices has remained obscure [9-11].

In a recent paper Elezovic *et al.* [12, hereafter referred to as EKM] have studied the critical exponents of self-avoiding walks (SAW's) on a family of Sierpinski-type fractals introduced first by Given and Mandelbrot [13]. Different members of this family are characterized by an integer b, which runs from 2 to infinity. EKM found that as b is increased from 2 to 8, both the Hausdorff and the spectral dimensions of the fractal increase monotonically, and tend towards 2, but the exactly determined magnetic susceptibility critical exponent γ systematically increases and moves *away* from the exactly known 2-dimensional value $\gamma = 43/32$. EKM argued that this indicates a non-monotonic dependence of γ and b, and that for b much larger than 8, γ woulds decrease again and tends towards the 2-dimensional value for large b. However, a direct calculation of the critical exponents for much larger values of b appears infeasible as the computer time required to generate the exact renormalization equations by direct enumeration increases as $\exp(b^2)$.

In this paper, we use finite-size scaling theory to study the variation of critical exponents of SAW's with b, for large b. We show that the limiting value of γ for large b is *not* equal to its two dimensional value, but is determinable in terms of *other* exponents of the SAW in two dimensions. Thus, the relationship between the exponents on fractals and those on regular integer-dimensional lattices is quite subtle. For large but finite b, the difference of

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critical exponents from the $b = \infty$ value can be evaluated as a systematic asymptotic expansion. We calculate explicitly the first term in the expansion and show that it is proportional to $(2-\tilde{d})$ where $\tilde{d}$ is the spectral dimension of the lattice.

Our result may be described as an analogue of the ε-expansions for fractal lattices. The treatment may be generalized to other families of fractals, and to other Hamiltonians such as branched polymers [14-15], n-vector model for $n < 1$ etc., which exhibit a finite temperature phase transition, and similar expansions may be derived for these problems.

2. Preliminaries.

In the following, we shall discuss only the case of SAW's on a fractal family which is in the same universality class as the fractal family studied by EKM. The recursive construction of the fractal family is shown in figure 1. The basic geometrical unit of the construction is an equilateral triangle with 3 corner vertices. The graph of the first order triangle consists of 4 vertices. The 3 corner vertices are joined to the fourth central vertex by 3 bonds.

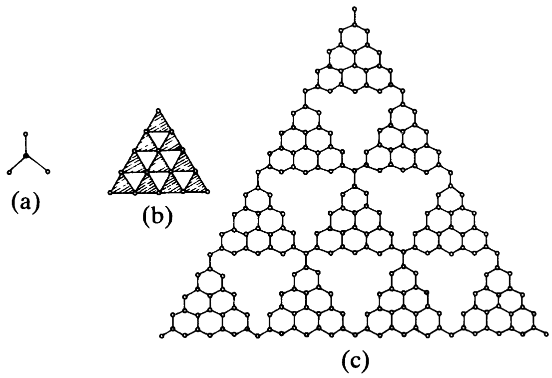

Fig. 1. — The recursive construction of the Given-Mandelbrot family of fractals. (a) The graph of the first order triangle. The corner-sites of the « triangle » are denoted by unfilled circles. (b) The recursive construction for $b = 4$. The graph of $(r+1)$-th order triangle. Only the corner verticles of the r-th order triangles (shaded) are shown. (c) The third order triangle for $b = 4$. All the internal vertices and bonds are depicted.

The recursive construction involves joining $b(b+1)/2$ small triangles (say of sides of length L each) into a larger triangle of length bL. The lattice resulting after a repeated application of the construction rule is a planar fractal lattice with finite ramification number 3. Each site of the lattice has either 2 or 3 neighbours, and the lattice can be obtained from the two dimensional hexagonal lattice by a selective deletion of sites. Its Hausdorff dimension is easily seen to be $\log [b(b+1)/2]/\log b$. The recursive construction, and hence all the critical properties of this lattice are the same as of the fractal family studied by EKM. The spectral dimension of the latter family has been calculated by Hilfer and Blumen [16], for b lying between 2 and 10, and is in principle calculable for any b. As b tends to infinity, the lattice resembles a two-dimensional plane more and more, and both the Hausdorff and spectral dimensions tend to 2.

Since the lattice is finitely ramified, the critical exponents for any fixed b can be determined exactly by the real-space renormalization group techniques. Following the treatment of Dhar [17] and EKM, we attach a weight x^L to a self-avoiding walk configuration having L steps, and introduce the restricted partition functions $A^{(r)}$, $B^{(r)}$ and $C^{(r)}$ (Fig. 2) denoting the sum over all possible SAW configurations within an r-th order triangle, consistent with the constraints shown in figure 2. The generating function for the number of L-stepped walks can be easily written down in terms of these restricted partition functions [12]. These functions satisfy simple recursion relations, which are of the form

$$B^{(r+1)} = f(B^{(r)}, b) \tag{2.1}$$

$$A^{(r+1)} = P_{11}(B^{(r)}, b)\, A^{(r)} + P_{12}(B^{(r)}, b)\, C^{(r)} \tag{2.2}$$

$$C^{(r+1)} = P_{21}(B^{(r)}, b)\, A^{(r)} + P_{22}(B^{(r)}, b)\, C^{(r)}. \tag{2.3}$$

Here $f(x, b)$, $P_{11}(x, b)$, $P_{12}(x, b)$, $P_{21}(x, b)$ and $P_{22}(x, b)$ are finite degree (dependent on b) polynomials of their argument x. These can be determined by explicit enumeration for any finite b. A listing of these polynomials for $b \leqslant 7$ (and $f(x, 8)$) may be found in EKM.

For the lattice studied here the starting values of these vertex weights are

$$A^{(1)} = x + 2x^2 \tag{2.4}$$

$$B^{(1)} = x^2 \tag{2.5}$$

$$C^{(1)} = 0\,. \tag{2.6}$$

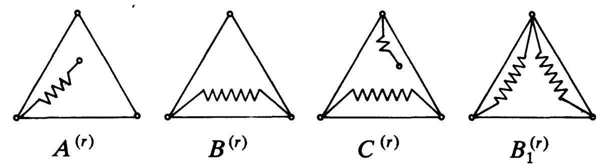

Fig. 2. — The restricted partition functions $A^{(r)}$, $B^{(r)}$, $C^{(r)}$ and $B_1^{(r)}$. $A^{(r)}$ is sum over all configurations of SAW's on an r-th-order triangle with one endpoint inside the triangle and one at a corner site. $B^{(r)}$ is sum over all internal configurations of walks (denoted by a wiggly line) that go from one corner site to a second corner site. $B_1^{(r)}$ is a sum over those walks that contribute to $B^{(r)}$, and also visit the third corner site of the triangle. $C^{(r)}$ is defined analogously.

The reason we choose to study this fractal family instead of the one studied by EKM is that for this family the recursion relations (2.1)-(2.3) are exact for all values of r, whereas for EKM the exact recursions involve an additional variable $B_1^{(r)}$ (Fig. 2). For large r, $B_1^{(r)}$ tends to zero near the fixed point of the recursions, and putting $B_1^{(r)} = 0$ we get back the recursion equations (2.1)-(2.3). In either case the critical exponents calculated are the same, as they only depend on the behaviour of recursion relations for large r.

3. Scaling limit of the recursion equations.

The configurations of walks which contribute to the polynomials f, P_{11}, P_{12}, P_{21} and P_{22} are, of course, exactly the configurations of SAW's on a finite equilateral triangular shaped subset of sites of a hexagonal lattice. This is obvious for the second order triangle, and the form of the recursion equations is independent of order of the triangle.

The critical growth constant μ_{hex} for the SAW's on the hexagonal lattice is known exactly [18]

$$\mu_{\text{hex}} = (2 + \sqrt{2})^{1/2} . \qquad (3.1)$$

For any fixed b, if $B^*(b)$ is the fixed point of the recursion equation (2.1), we have the growth constant of SAW's on the fractal

$$\mu(b) = [B^*(b)]^{-1/2} . \qquad (3.2)$$

As $b \to \infty$, the constraint that the walks are confined to a triangular shaped region does not affect the critical constant. Hence we have

$$\lim_{b \to \infty} B^*(b) = \mu_{\text{hex}}^{-2} = 1/(2 + \sqrt{2}) . \qquad (3.3)$$

For the lattice studied by EKM, the growth constant $\mu(b)$ is not simply related to $B^*(b)$, and $\mu(b = \infty)$ is equal to the growth constant on a triangular lattice, which is not known exactly.

For B close to $B^*(\infty)$, the SAW's on the hexagonal lattice are close to criticality. Define a change of variables from B to ε by

$$B^{(r)} = B^*(\infty) \exp(\varepsilon^{(r)}) . \qquad (3.4)$$

To simplify notation, we shall drop the superscript over ε in the remainder of the paper.

Now $f(B^{(r)}, b)$ in equation (2.1) may be interpreted as measuring the corner-spin corner-spin correlation function of the $n = 0$ vector model on the finite hexagonal lattice. For small ε and large b, the behaviour of this function can be deduced from scaling theory. In this limit, the function $f(B, b)$ which is a function of two variables ε and b reduces to a scaling form involving only one variable $\varepsilon b^{1/\nu}$, where $\nu = 3/4$ is the exactly known correlation length exponent for two dimensional SAW's. Then the recursion equation (2.1) may be approximated as

$$B^{(r+1)} \simeq \frac{K}{b^a} \exp[g(\varepsilon b^{1/\nu})] . \qquad (3.5)$$

Here a is a critical exponent which describes the power law decay of the corner-spin corner-spin correlation function at criticality on a finite equilateral triangle of sides of length b, $g(x)$ is a scaling function and K is some constant. The scaling function $g(x)$ is easily seen to have the following properties :

i) By a redefinition of K, we can set $g(0) = 0$.

ii) For fixed b, the function $f(B, b)$ is a finite polynomial in $\exp(\varepsilon)$ with positive coefficients. This implies that $g(x)$ is a monotonically increasing convex function of x for all real values of x. In particular, for x near zero, $g(x)$ is approximately equal to $\alpha x + \beta x^2$, where α and β are positive constants.

iii) If $\varepsilon < 0$, the SAW's are below criticality, and the correlation function $B^{(r+1)}$ must decrease exponentially with distance for large b. Since the correlation length diverges as $|\varepsilon|^{-\nu}$ for small ε we must have in this case $B^{(r+1)} \sim \exp(-|\varepsilon|^{\nu} b)$ for large b. This implies that $g(x)$ must vary as $-|x|^{\nu}$ for large negative x.

iv) If $\varepsilon > 0$, the dominant contribution to $B^{(r+1)}$ comes from walks whose length is proportional to b^2. The walks then fill the equilateral triangle with finite density. The properties of this « dense phase » of the SAW's have been discussed in detail by Duplantier and Saleur [19]. Their results imply that for large b, $B^{(r+1)}$ must vary as $\exp(b^2)$. This is possible only if

$$g(x) \approx K_0 x^{2\nu} , \quad \text{for} \quad x \to \infty , \qquad (3.6)$$

where K_0 is some constant.

The scaling assumption of equation (3.5) is a crucial step in the analysis of this paper, and it is desirable to test it against available numerical data. Figure 3 shows the graphs of $\log[f(B^*(\infty)\, e^{\varepsilon}, b)/f(B^*(\infty), b)]$ *versus* $\varepsilon b^{1/\nu}$ for $b = 2$ to 8 from the exact recursion equations of EKM, and using the exactly known value $\nu = 3/4$. We note that there are no adjustable parameters in this fit. The different curves are fairly close to each other, and for large b appear to converge to a single curve. This is in agreement with our scaling assumption. It should however be remarked that the data suggests that corrections to scaling are quite significant for small values of $b \leqslant 8$.

The value of the exponent a in equation (3.5) is related to the scaling power of the corner-spin variable in the SAW problem on a plane. These have been calculated exactly by Cardy and Redner [20]

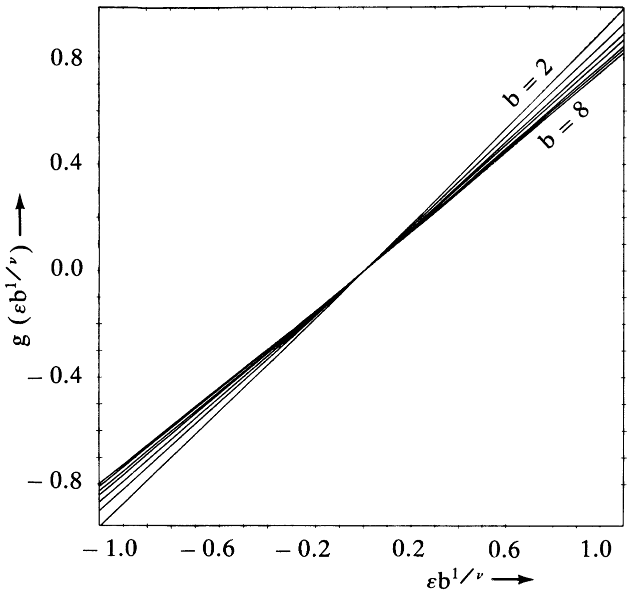

Fig. 3. — A test of scaling of the recursion equations : the function $g(\varepsilon, b)$ is plotted *versus* $\varepsilon b^{1/\nu}$ for $b = 2$ to 8. Convergence towards a limiting curve is clearly seen.

and Guttman and Torrie [21]. For the wedge angle $\pi/3$, their results imply that

$$a = 15/4 . \tag{3.7}$$

A direct numerical estimate of a can be obtained by studying the dependence of the exact functions $f(B^*(\infty), b)$ on b. For b varying from 2 to 8, these values from the EKM data are 0.110913, 0.050622, 0.026393, 0.015147, 0.009337, 0.006083 and 0.004141 respectively. Thus the effective value of the exponent a is approximately 1.93 between $b = 2$ and 3, and it increases to approximately 2.88 between $b = 7$ and 8. The discrepancy again indicates that the corrections to scaling are quite strong for small b.

From the approximate recursion equation (3.5), the fixed point $\varepsilon^*(b)$ for a large but finite value of b is easily obtained. Taking the logarithm of both sides of equation (3.5) we get

$$g(\varepsilon^*(b)\, b^{1/\nu}) \approx a \log b \tag{3.8}$$

and using equation (3.6) we get

$$\varepsilon^*(b) \approx \left(\frac{a \log b}{K_0} \right)^{1/2\,\nu} b^{-1/\nu} . \tag{3.9}$$

Thus $\varepsilon^*(b)$ tends to zero as b tends to infinity in agreement with equation (3.3). Also, because the right hand side of equation (3.9) contains the logarithm factor, $\varepsilon^*(b)\, b^{1/\nu}$ increases slowly with b. This is also in agreement with the numerical data of EKM, from which the exact values of $\varepsilon^*(b)\, b^{1/\nu}$ for b lying between 2 and 8 are approximately 1.882, 2.735, 3.476, 4.124, 4.700, 5.215 and 5.680 respectively.

4. Calculation of critical exponents.

The critical exponent $\nu(b)$ is obtained by linearizing equation (3.5) around the fixed point $\varepsilon^*(b)$. If $\lambda_1(b)$ is the value of the derivative of f at the fixed point of equation (2.1), we get

$$\lambda_1(b) = b^{1/\nu} \frac{dg}{dx} \bigg|_{x = \varepsilon^*(b)\, b^{1/\nu}} . \tag{4.1}$$

Again using equation (3.6) we get

$$\lambda_1(b) \approx K_1\, b^{1/\nu} (\log b)^{\frac{2\nu - 1}{2\nu}} \tag{4.2}$$

where $K_1 = 2\,\nu a (K_0/a)^{1/2\,\nu}$ is a constant. The correlation length exponent $\nu(b)$ is given by $\log b / \log \lambda_1(b)$. Hence from equation (4.2) we get

$$\frac{1}{\nu(b)} = \frac{1}{\nu} + \frac{2\,\nu - 1}{2\,\nu} \frac{\log \log b}{\log b} + \text{ terms of order } \frac{1}{\log b} . \tag{4.3}$$

Since $\left(\frac{2\,\nu - 1}{2\,\nu} \right)$ is positive for $\nu = 3/4$, the first correction term in equation (4.3) is positive. This implies that $\nu(b)$ tends to ν as b tends to infinity *from below*. For $b \leqslant 8$, EKM found that $\nu(b) > \nu$. This implies that $\nu(b)$ is a *non monotonic* function of b.

That $\nu(b)$ for large b is less than ν is a reflection of the fact that dg/dx in equation (4.1) is greater than 1 for large enough x (and hence for large enough b). For x near zero, its value is less than 1 (Fig. 3). For b lying between 2 and 8, the exact values of this derivative at the fixed point are approximately 0.945, 0.923, 0.914, 0.9117, 0.9113, 0.9142 and 0.9163. Thus this factor is close to 1 and slowly increases with b (for $b > 6$). The initial *decrease* for small b is presumably due to corrections to scaling not included in our analysis. However, the crossover value of b beyond which this derivative will be larger than 1 is difficult to estimate, and could be quite large.

We now determine the limiting value of the susceptibility exponent $\gamma(b)$. In two dimensions, it is known that a three leg vertex has a smaller scaling power than a one leg vertex [22, 23]. This implies that for large b, and close to criticality $C^{(r)}$ is smaller than $A^{(r)}$ by a power of b, and can be ignored in the recursion equations in the lowest order calculation. Then the recursion equation (2.2) becomes

$$A^{(r+1)} \simeq A^{(r)} P_{11}(B^{(r)}, b) . \tag{4.4}$$

This implies that the magnetic field eigenvalue $\lambda_2(b)$ (in the notation of EKM) is

$$\lambda_2(b) \simeq P_{11}(B^*(b), b) . \tag{4.5}$$

We can now write a scaling form for P_{11} in analogy with equation (3.5) as

$$P_{11}(B^*(\infty)\exp(\varepsilon), b) \approx K_3\, b^c \exp[h(\varepsilon b^{1/\nu})], \quad (4.6)$$

where K_3 is a constant, c is a critical exponent and $h(x)$ is a scaling function. That this scaling form is reasonable may be seen as follows: for fixed ε, P_{11} is a monotonically increasing function of b. If $\varepsilon = 0$, it is expected to increase as a power of b, say as b^c. For fixed b, P_{11} is polynomial in e^ε with positive coefficients. Hence, as in the case of $g(x)$, $h(x)$ is a monotonically increasing convex function of x, which may be chosen to be zero when $x = 0$.

For $\varepsilon < 0$, and large b it is known that P_{11} varies as $\varepsilon^{-1/64}$ [20, 21] and is independent of b. This is consistent with the scaling form (4.6) only if

$$h(x) \approx -\frac{1}{64}\log|x|, \quad \text{for} \quad x \to -\infty; \quad (4.7)$$

and

$$c = 1/48. \quad (4.8)$$

From the numerical data of EKM, the values of P_{11} at $\varepsilon = 0$ for $b = 2$ to 7 are 1.7574, 2.3733, 2.8912, 3.3374, 3.7291 and 4.0781 respectively. Thus the effective exponent c decreases from about 0.75 between $b = 2$ and 3 to a value about 0.58 for b between 6 and 7. These values again show slow convergence, but are quite consistent with a small positive value of c.

Using equation (3.5), equation (4.6) may be rewritten as

$$P_{11}(B^*(\infty)\exp(\varepsilon), b) \approx K_3'\, b^{c+a} \times \exp[h(\varepsilon b^{1/\nu}) - g(\varepsilon b^{1/\nu})]\, B^{(r+1)} \quad (4.9)$$

where K_3' is some constant.

For large positive values of x, both $h(x)$ and $g(x)$ increase as $K_0\, x^{2\nu}$, but these divergences due to finite free energy density in the low temperature phase of the $n = 0$ vector model cancel exactly in equation (4.9). Thus $[h(x) - g(x)]$ in equation (4.9) must vary for large x as a smaller power of x than $x^{2\nu}$. This implies using equation (3.9) that at the fixed point $\varepsilon = \varepsilon^*(b)$, the argument of the exponential in equation (4.9) varies at most as a sublinear power of $\log b$. Hence we get

$$\lim_{b \to \infty} \frac{\log \lambda_2(b)}{\log(b)} = c + a. \quad (4.10)$$

The magnetic field exponent $\gamma(b)$ is given by [12]

$$\gamma(b) = \log[2\lambda_2^2(b)/b(b+1)]/\log \lambda_1(b) \quad (4.11)$$

and thus using equation (4.10) we get

$$\gamma(b = \infty) = 2(c + a - 1)\,\nu = 133/32. \quad (4.12)$$

We can determine the functional form of the leading correction due to finite b to this value by the following heuristic argument: for a fixed value of $\varepsilon > 0$ and large b, we expect that $P_{11}/B^{(r+1)}$ will vary as $K(\varepsilon)\, b^\omega$ where $K(\varepsilon)$ is an ε-dependent constant, and ω is an exponent describing the endpoint correlations in the low temperature phase of the dense SAW's. This behaviour is consistent with equations (3.5) and (4.6) only if

$$h(x) - g(x) \approx (\omega - c - a)\,\nu \log x, \quad \text{for } x \to \infty. \quad (4.13)$$

Substituting this form in equation (4.9), we see that the first correction term to $\gamma(b)$ is of the form

$$\gamma(b) = \frac{133}{32} - K_2 \frac{\log\log b}{\log b} + \text{terms of order } (1/\log b), \quad (4.14)$$

where K_2 can be determined explicitly if the exponent ω appearing in equation (4.13) is known.

Duplantier and Saleur [19] have proposed formulas (conjectured to be exact) for the scaling dimensions of the surface and bulk operators in the dense polymer problem. For the 1-leg operator in bulk, and at a corner with wedge angle $\pi/3$, their results are $-\frac{3}{16}$ and $-\frac{3}{8}$ respectively. A straightforward calculation of ω in terms of these exponents give $\omega = 2 + \frac{3}{16} - \frac{3}{8} = \frac{29}{16}$. Substituting this in equations (4.9) and (4.13) gives

$$K_2 = \frac{321}{128}. \quad (4.15)$$

For $b = 2$ to 8, EKM found the values of $\gamma(b)$ to be 1.3752, 1.4407, 1.4832, 1.5171, 1.5467, 1.5738 and 1.5991 respectively. Thus $\gamma(b)$ increases with b but even for $b = 8$ its value is much less than the limiting $b = \infty$ value. This suggests that equation (4.14) is likely to be a good approximation to $\gamma(b)$ only for b's much larger than those reachable numerically. It would be interesting to design an approximation scheme which works well for intermediate values of b also (say $b \gtrsim 100$).

Corrections to critical exponents which vanish only as inverses of logarithms of scale factor (Eqs. (4.3) and (4.14)) are also encountered in some other real-space renormalization calculations — such as in the large-cell renormalization.

5. Discussion.

The fact that even for $b \to \infty$, when both the spectral and Hausdorff dimension of the lattice tend to 2, the critical exponent $\gamma(b = \infty)$ is not equal to the two-dimensional value $\gamma = 43/32$ is somewhat unexpected. In particular, this disagrees with a recent suggestion [7, 8] that if the spectral and

Hausdorff dimensions are equal, critical exponents on fractals are analytical continuation of their values on regular integer-dimensional lattices.

It should be stressed that this difference between $\gamma(b=\infty)$ and the two-dimensional value is not due to an inappropriate definition of γ on fractals, or some peculiarity of the $n=0$ vector model. The definition of γ assumed here is natural and equivalent to defining it in terms of the divergence of the susceptibility in the high temperature phase of the $n=0$ vector model on the fractal.

In fact, it is quite straightforward to see that not all exponents on the fractal in the limit $b \to \infty$ can equal their two dimensional value. In fact, for the $n=1$ vector model (Ising case), this family of fractals does not even undergo a finite temperature phase transition for any finite b. But the critical exponents do exist for the two dimensional planar problem. For nonintegral values of n lying between 0 and 1, a nontrivial fixed point of the recursion equations exists for any finite b, but the limiting $b \to \infty$ value of the critical coupling on fractals does not correspond to criticality in the bulk. A simple relation between the exponents on fractals and those in bulk thus appears unlikely.

How do we reconcile the fact that for $b \to \infty$ the lattice « looks like » a two dimensional hexagonal lattice ? This has to be understood in terms of crossover effects. For large b, chains of length $\leq b^{1/\nu}$ essentially see the hexagonal lattice, and their properties (say end to end length) are like those in the Euclidean plane. But chains much larger than $b^{1/\nu}$ do meet the constrictions (corners of high-order triangles) and their behaviour is governed by constrictions. As $b \to \infty$, the effective 2-dimensional behaviour is valid over a wider and wider regime, and a crossover to the true exponents $\nu(b)$ and $\gamma(b)$ occurs only in the narrow temperature $|\varepsilon| \lesssim b^{-1/\nu}$.

Finally, we note that, for large b, the spectral dimension $\tilde{d}(b)$ of the lattice is given [24] by

$$\tilde{d}(b) \simeq 2 - \frac{\log \log b}{\log b} + \text{terms of order } 1/\log b \,. \quad (5.1)$$

It is interesting to note that the first correction terms due to finite b to the $\nu(b=\infty)$ and $\gamma(b=\infty)$ values in equations (4.3) and (4.14) are proportional to $[2-\tilde{d}(b)]$. The next order correction term varies as $1/\log b$, and may be set proportional to $\bar{\varepsilon} = 2 - \bar{d}$, where $\bar{d}$ is the Hausdorff dimension of the lattice. Higher order correction terms can be evaluated systematically, if we knew the exact functional form of the scaling functions $g(x)$ and $h(x)$. These typically involve higher powers of $1/\log b$, i.e. of $(2-\bar{d})$. In addition, there are corrections to scaling contributions to the values of critical exponents. These are typically of the form b^{-y_i}, where y_i are some constants. Since $\bar{\varepsilon} \simeq 1/\log b$, there corrections are of the form $\exp(-c_i/\bar{\varepsilon})$ where c_i are some constants. This emphasises the asymptotic nature of these expansions.

Our approach here should be distinguished from the conventional ε-expansions technique which is often used to develop systematic power-series expansions of critical exponents in powers of $(2-d)$ (say). The latter is a very formal technique which starts by postulating several properties (including translational invariance) satisfied by integrals in non-integral dimensional spaces [25-27]. No constructive examples of non-integral dimensional spaces satisfying all these postulates are known, and it appears that spaces satisfying all these postulated properties necessarily have undesirable features such as non-positive integration measures [28].

In contrast, we start with an explicitly constructed family of fractal lattices for which various thermodynamic limits may be shown to exist, and thermodynamic convexity relations hold. The price paid is a loss of translational invariance, and a strong dependence of the critical exponents on the detailed geometry of the fractals (e.g. wedge angles). This is not necessarily a bad thing. We hope that further studies in this direction will help in understanding the influence of geometry on critical behaviour on fractal lattices.

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