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On the connectivity index for lattices of nonintegral dimensionality

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Abstract. We define the connectivity index c for an infinite graph by the requirement that to disconnect a subset of at least V points from the rest of the graph requires the deletion of a minimum of $S(V)$ bonds where $S(V) \sim V^{(c-1)/c}$ for large V. For a d-dimensional hypercubical lattice with d integral, $c = d$. We construct explicit examples of lattices with nonintegral connectivity index c, $1 < c < \infty$. It is argued that the connectivity index is an important parameter determining the critical behaviour of Hamiltonians on these lattices.

Keywords. Graph theory; nonintegral dimension; connectivity index.

1. Introduction

In recent years, much attention has been devoted to studying the variation of critical exponents as a function of d, where d, the dimension of space, is treated as a continuously variable parameter. Wilson and Fisher developed the technique called ϵ-expansion which allows one to write critical exponents, say for Ising-like models as a power-series in ϵ, where $\epsilon = 4d$ (Fisher 1974; Wilson and Kogut 1974). These ϵ-expansion techniques have been pushed to quite high orders (Brezin *et al* 1974; Collet and Eckmann 1978). Similar series expansions in powers of ϵ, where the space dimension is 2-ϵ, 6-ϵ, 5-ϵ, etc. have been developed to describe a wide variety of phase transitions in different physical systems (Belavin and Yurishchev 1973; Harris *et al* 1976; Obukhov 1980). In quantum field theory the space dimension 4-ϵ has been introduced to regularise the ultraviolet divergences in the perturbation theory (Bollini and Giambiagi 1972).

Despite much work done dealing with the computational aspects of the ϵ-expansion technique (only a small part of which was cited above), its conceptual basis has remained quite obscure. Just what physical meaning may be assigned to these ϵ-expansions? We may argue that the appearance of ϵ as a continuous variable is a technical or mathematical artifice, and physically meaningful results correspond only to integral values of ϵ. This argument fails however, as the radius of convergence if these expansions (if they converge at all, presumably they are only asymptotic (Collet and Eckmann 1978) is certainly much less than 1.

In an earlier paper (Dhar 1977, hereafter referred to as I) we attempted to answer this question by explicitly constructing a class of lattices having a nonintegral value of the effective dimensionality. Defining a nearest-neighbour spring model on these lattices, we argued that if the fractional number of eigenmodes having frequency less than ω varies ω^d for small ω, then d should be identified as the (Fourier) dimension of the lattice. While this definition of dimensionality is the only one consistent with the known critical exponents for the spherical model in arbitrary dimensions (Joyce 1972), it has one unsatisfactory feature. The truncated tetrahedron lattice defined by Nelson and Fisher (1975) has the Fourier dimension of $\log_3 5$. The Ising model on this lattice does not have an ordered phase at any temperature. This is in contradiction with the generally accepted result that the lower critical dimension for the Ising model is 1. The same result holds for the truncated n-simplex lattice (see I).

Intuitively, the reason for the nonexistence of a phase transition in the truncated tetrahedron lattice lies in its peculiar connectivity properties. On this lattice, it is possible to disconnect an arbitrarily large set of points from the rest of the graph by just deleting three bonds. On a d-dimensional lattice (with d integral) one needs to break a minimum of $\sim v^{(d-1)/d}$ bonds to disconnect a volume V from the rest of the lattice. The rate of increase of minimum surface with the enclosed volume clearly provides useful information about the connectivity of the lattice. We use this property to define a connectivity index c for lattices, where c may take nonintegral values. For a d-dimensional hypercubical lattice with d integral, the connectivity index c is equal to its Euclidean dimension d. We use the word 'connectivity index' and not 'connectivity dimension' or 'surface dimension' as the concept of dimension has no unique generalisation to nonintegral values. For a nonexhaustive list of mutually inequivalent definitions of dimensionality, and also for earlier references, the book on fractals by Mandelbrot (1977) may be consulted.

The rest of this paper formalizes this concept of connectivity index. In § 2 we give its precise definition and construct examples where it takes arbitrary non-integral values between 1 and ∞. (A similar definition was suggested by Mandelbrot (1977) to characterise the complexity of computer network). In § 3, we argue that for the Ising model, the lower critical value of the *connectivity index* is 1, and hence explain the absence of a phase transition in the Ising model on the truncated n-simplex lattice. The relation of the connectivity index to other definitions of nonintegral dimensions and its role in determining critical behaviour is briefly discussed.

2. Definition of the connectivity index and examples

The connectivity index for an infinite lattice is defined as follows: The lattice is specified by its graph consisting of points, and undirected lines (called bonds) joining the nearest neighbours. We consider only connected graphs. Let V be any finite connected subset of points. The volume of V is the number of points in V. The surface of V, denoted by $S(V)$ is the number of bonds connecting points in V to points outside V. Deletion of these bonds would obviously disconnect the subgraph V from the rest of the points. Let $S_m(v, p)$ be the minimum surface $S(A)$ for all connected subgraphs A, having more than v points and including a particular (arbitrarily chosen) point P. The connectivity index c for the graph is defined by

$$c = \lim_{v\to\infty} \frac{\ln v}{\ln\,[v/S_m(v, P)]}. \tag{1}$$

We shall only be interested in graphs which are sufficiently regular, so that the above limit exists. If the graph is such that the number of nearest neighbours of any point in it is bounded by a finite number n, then the connectivity index c is independent of the choice of the point P. This follows from the fact for any point P' at a finite distance l from the point P, any finite volume A containing P may be increased to include P' as well, with the surface area $S(A)$ increasing by at most nl.

It is quite easy to see that the connectivity index for a d-dimensional hypercubical lattice (d integral) is equal to its Euclidean dimension d. We now describe the construction of a class of lattices for which the connectivity index takes non-integral values.

The (m, n) recursive lattice is defined for all integers $m, n > 1$. Its graph is a planar graph which may be obtained from the graph a two-dimensional square lattice by selectively deleting some bonds. The zero order (m, n) recursive lattice graph is shown in figure 1. It has four points and four 'dangling' bonds. These dangling bonds are used to connect the zero order graphs together to form larger graphs. Given an rth order graph, we construct the $(r+1)$th order graph by taking m^2n^2 copies of the rth order graph and arranging them in an $mn \times mn$ array. The dangling bonds of adjacent edges of squares (rth order graphs) are identified in pairs, in a way such that the resulting graph is planar. Of the remaining dangling bonds (those on the perimeter of the $(r+1)$th order square), we retain every mth bond going clockwise along the perimeter, and delete the remaining. The construction is illustrated in figure 2 and 3 for the case $m=n=2$. Clearly, in general, the rth order graph has $4m^{2r}\, n^{2r}$ points and $4n^r$ dangling bonds.

It is easy to see that for this lattice the minimum surface area corresponding to a volume $4m^{2r}\, n^{2r}$ is greater than $2n^r$. Hence using definition (1) it is easily seen that the connectivity index for the (m, n) recursive lattice is $ln\,(m^2n^2)/ln\,(m^2n)$. Choosing m and n judiciously, we obtain a value of the connectivity index arbitrarily close to any preassigned value between 1 and 2.

It is quite easy to construct graphs with higher values of the connectivity index. Let L_1 and L_2 be two graphs with connectivity indices c_1 and c_2 respectively. If L is a direct product graph of L_1 and L_2 (for the definition of the direct product of graphs, see I), the connectivity index of L is c_1+c_2. Forming direct products of the

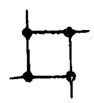

Figure 1. The graph of the zeroth order (m, n) recursive lattice. It has four points and four dangling bonds.

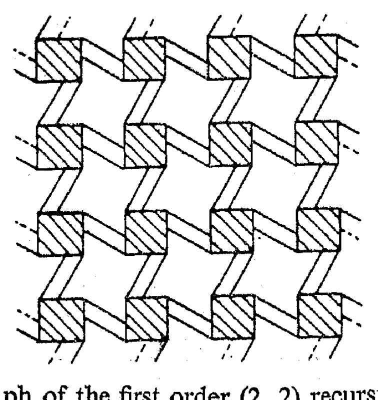

Figure 2. The graph of the first order (2, 2) recursive lattice. The dangling bonds at the surface which have been deleted are represented by dashed lines.

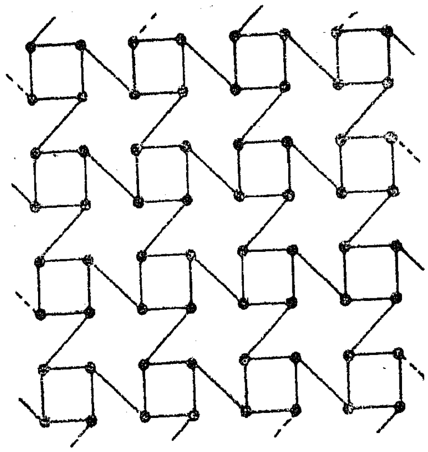

Figure 3. The graph of the second order (2, 2) recursive lattice. The shaded squares represent first order graphs. Only the dangling bonds connecting the different first order graphs are shown explicitly.

(m, n) recursive lattice with itself, or say with a linear chain ($c=1$), we get graphs having the value of connectivity index ranging from 1 to ∞.

3. Discussion

From (1) it follows that the connectivity index for the truncated n-simplex lattice (defined in I) is 1, which is the lower critical dimension for the Ising model. This is in accordance with the result that the Ising model on this lattice does not show any phase transitions but the susceptibility has a strong divergence near zero temperature. Also a Peierls-like argument can be used to show the existence of a phase transition on the (m, n) recursive lattice ($n \geqslant 2$). Clearly, the connectivity index is a useful parameter to characterise the critical behaviour of Hamiltonians on these lattices.

The connectivity index of a lattice need not be equal to its Fourier or Hausdorf dimension. For example, it may be shown that the Fourier as well as the Hausdorf dimension of the (m, n) recursive lattice is 2, not equal to its connectivity index. As pointed out in I, the Fourier and the Hausdorf dimensions of a lattice themselves need not be equal.

The distinction between these definitions is important as the formal ϵ-expansion can be valid for only one of these definitions. In an earlier paper (Dhar 1978) we have shown that critical exponents may be different even for lattices of equal dimensionality. This implies that critical exponents are not functions of dimensionality alone, and series expansions like the ϵ-expansion should involve additional variables. A similar conclusion was reached by Y Gefen *et al* (1980), who have used exact and approximate renormalisation techniques to determine the critical behaviour on some fractal lattices. These authors used a different definition (Hausdorf), and hence concluded that a lower critical dimension does not exist.

Even if we adopt the view that they exhibit the dependence of critical exponents on dimensionality 'with all other variables held fixed', it is important to identify these variables explicitly. Lattices of nonintegral dimensions are necessarily somewhat artificial objects, but if the ϵ-expansion technique is to be physically meaningful, it should be able to predict critical exponents for such lattices. An interesting possibility is that the ϵ-expansion gives correct results for nonintegral dimensional lattices only when all these different characterisations of dimension agree. Construction of such lattices seems to be difficult, if possible. This is an interesting problem deserving further attention.

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