ASSESSMENT OF ACCURACIES OF FINITE ELEMENT EIGENVALUES

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1. INTRODUCTION

Since the classical finite element displacement method is a piecewise Rayleigh-Ritz method, upper bounds to eigenvalues are assured if consistent and conforming elements are used. But sometimes one may get a lower bound to the eigenvalues by using simplified elements [1], [2]. In either case it is desirable to have an error estimate [1], [3]–[5] in order to assess the accuracy in the solution. Normally it is difficult to estimate the error. Under such circumstances upper and lower bounds to eigenvalues serve the purpose, since the difference of the bounds is an indication of the error.

Recently the authors have suggested a simple scheme of modifying the limits of integrals involved in the Rayleigh–Ritz method to get bounds to eigenvalues [6]. A finite element analogue of this method was introduced in [7]. More recently, a different method intended for a similar purpose has appeared in the literature [8]. The purpose of this paper is to bring out the general applicability of our method and to confirm the procedure by numerical experiments as well as by error estimates in some cases.

2. METHOD

In the case of free vibration problems, following the orthogonality of principal modes, the ith natural frequency can be written as

$$\omega_i^2 = \frac{\mathbf{v}_i^T \mathbf{K} \mathbf{v}_i}{\mathbf{v}_i^T \mathbf{M} \mathbf{v}_i} \tag{1}$$

$$= \frac{\text{Strain energy}}{\text{Kinetic energy per unit circular frequency}},$$

where $\mathbf{v}_i$ is the eigenvector corresponding to the ith natural mode, $\mathbf{K}$ is the assembled stiffness matrix and $\mathbf{M}$ is the assembled mass matrix.

In the first instance, we consider the case when the original finite element method gives an upper bound to ω_i. The nature of the convergence of the sequence can be changed, i.e., it can be made to approach the exact from below, either

(1) by consistently increasing kinetic energy without altering strain energy or

(2) by consistently decreasing strain energy without altering kinetic energy.

Here we choose to modify the kinetic energy as follows: a modified element mass matrix $\bar{\mathbf{m}}$ is constructed by multiplying the element mass matrix $\mathbf{m}$ by a modification function R as

$$\bar{\mathbf{m}} = R\,\mathbf{m} \tag{2}$$

and this is used in the formulation instead of $\mathbf{m}$. R is a function of element displacements and/or number of elements and contains a scalar parameter A. Further it must satisfy the conditions

$$\begin{aligned} R_N &> 1 \quad \text{for any } N, \\ R_{N+1} &< R_N \quad \text{for any } N, \\ R_N &\to 1 \quad \text{as } N \to \infty. \end{aligned} \tag{3}$$

From a study of the modified convergence curves corresponding to various values of A, it is possible to bracket the exact eigenvalue.

If the original finite element method gives a lower bound, then the upper bound can be generated by using a modified mass matrix $\bar{\mathbf{m}}$ defined as

$$\bar{\mathbf{m}} = \frac{1}{R}\,\mathbf{m} \tag{4}$$

instead of that defined in (2). It is obvious that the case of linear structural stability can be treated similarly.

3. Some Numerical Experiments

We have used the above mentioned procedure, with success to generate bounds for eigenvalues of several problems covering vibrations and stability of plates and beams. Here we present briefly a typical illustration to show the application as well as to bring out the salient features of this procedure.

We consider here the problem of transverse oscillations of a tapered cantilever beam. Four degrees of freedom consistent tapered elements [2]

have been used for analysis and this results in upper bounds to the frequency. To apply the present procedure, a modification function R, defined as

$$R = \{1 + A^4(v_r - v_{r-1})^4\}, \tag{5}$$

where v_r is the lateral displacement of the beam at the rth node, will be used. It may be noted here that R satisfies all the required conditions (equation (3)). As R involves the displacements v, the method of numerical solution becomes iterative and the main steps are:

(1) obtain the solution with $R = 1$ ($A = 0$),

(2) assume a suitable value for A,

(3) evaluate R for each element using the eigenvector from the previous step,

(4) construct $\bar{\mathbf{m}}$ for each element,

(5) using $\bar{\mathbf{m}}$ instead of $\mathbf{m}$ obtain a new solution,

(6) repeat the steps 3, 4, 5 until the solution converges,

(7) repeat the procedure for various numbers of terms in order to construct the convergence curve, and

(8) repeat the procedure with different values of A, to construct different modified convergence curves.

Tables 1 and 2 show the trend in convergence for typical values of A in the case of uniform and tapered cantilever beams respectively. This numerical experiment indicates the possibility of generating close bounds by a suitable choice of A, by trial and error.

TABLE 1. Frequency parameter λ of a uniform cantilever beam. — I mode. Exact value = 12·36236.

N	$A = 0{\cdot}0$	$A = 0{\cdot}2$	$A = 0{\cdot}4$	$A = 0{\cdot}6$
2	12·3743	12·3707	12·3168	12·0830
4	12·3632	12·3629	12·3592	12·3434
6	12·3625	12·3625	12·3617	12·3585
8	12·3624	12·3624	12·3621	12·3611
Nature of bound*	UB	UB	LB	LB

* UB = upper bound; LB = lower bound.

TABLE 2. Frequency parameter λ of a tapered cantilever beam. ($TR=0{\cdot}8$) with linear taper in depth $-I$ mode.

N	$A=0{\cdot}0$	$A=0{\cdot}2$	$A=0{\cdot}4$	$A=0{\cdot}6$
2	13·4542	13·4496	13·3810	13·0834
4	13·4459	13·4456	13·4408	13·4199
6	13·4452	13·4452	13·4442	13·4400
8	13·4451	13·4451	13·4447	13·4434
Nature of bound*	UB	UB	LB	LB

* UB = Upper Bound; LB = Lower Bound.

4. Choice of the Scalar Parameter A

Our experience indicates that it is economical to choose A initially, by trial and error, using the solution with only a few elements. Two successive approximations to the eigenvalues are worked out with various values of A, then A is chosen such that the higher approximation is greater (lower) than the lower one to get a lower (upper) bound. In the case of beams 2 element and 4 element solutions are used to give a satisfactory value of A.

5. Discretization Errors

We calculate the discretization errors involved in the finite element idealization for a number of eigenvalue problems. From the knowledge of the error estimates it is possible to prove convergence of method in each case. Here, the modification function R is assumed as

$$R = \left\{1 \pm \frac{A}{N^c}\right\}, \qquad c \geqslant 1, \tag{6}$$

where N^c is respectively the number of elements or number of elements per side for one- and two-dimensional problems. This satisfies all the required conditions (equation (3)). To calculate the discretization errors, we use the procedure given in [4]. For the sake of completeness and clarity, we explain the method here considering a simple problem, namely, vibrations of bars.

6. Axial Vibrations of Bars

The stiffness matrix $\mathbf{k}$ and modified lumped mass matrix $\bar{\mathbf{m}}$ of the bar element are given by

$$\mathbf{k} = \frac{E\bar{A}}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{7}$$

and

$$\bar{\mathbf{m}} = \frac{R\rho l}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tag{8}$$

where E is Young's modulus, $\bar{A}$ the area of cross-section of the bar element and ρ the mass per unit length. The length l of the element is equal to

$$l = \frac{L}{N}, \tag{9}$$

where L is the length of the bar and N is the number of elements into which the bar is idealized.

Following the procedure given in [4], one can write the equilibrium equation of the bar at the rth node, from (7) and (8), as

$$\frac{E\bar{A}}{l}(-u_{r-1} + u_r) + \frac{E\bar{A}}{l}(u_r - u_{r+1}) - lR\rho\omega^2 u_r = 0. \tag{10}$$

Expanding the displacements u_{r+1} and u_{r-1} about u_r by means of Taylor's series, we get

$$u'' + \frac{l^2}{12}u'''' + O(l^4) + \ldots + \frac{R\rho\omega^2 u}{E\bar{A}} = 0, \tag{11}$$

where primes denote the order of differentiation with respect to the axial co-ordinate. As $l \to 0$, i.e., as $N \to \infty$, (11) becomes

$$u'' + \frac{\rho\omega^2}{E\bar{A}}u = 0 \tag{12}$$

which is the exact differential equation for the vibration of bars. By comparing (11) and (12), one can see that the error in the differential equation, due to discretization, is $(l^2/12)u'''' + O(l^4) + \ldots$. The error term containing the smallest power of l will be called the principal error. The remaining error terms are negligible for small values of l compared to the principle error.

Considering a bar with fixed ends $[u(0) = u(L) = 0]$, one can take the known exact solution

$$u = u_0 \sin\frac{M\pi X}{L}, \tag{13}$$

where M is the number of half waves in the x-direction.

TABLE 3. Discretization errors.

Problem	Modification function R	Principal error	Upper bound	Lower bound
1. Axial vibrations of bars				
(a) Lumped mass matrix	$1 - A(M\pi/N)^2$	$\omega_{ex}^2(M\pi/N)^2(A - \frac{1}{12})$	$A > \frac{1}{12}$	$A < \frac{1}{12}$
(b) Consistent mass matrix [9]	$1 + A(M\pi/N)^2$	$\omega_{ex}^2(M\pi/N)^2(\frac{1}{12} - A)$	$A < \frac{1}{12}$	$A > \frac{1}{12}$
2. Transverse vibrations of simply supported beams				
(a) Lumped mass matrix	$1 - A(M\pi/N)^4$	$\omega_{ex}^2(M\pi/N)^4(A - \frac{1}{720})$	$A > \frac{1}{720}$	$A < \frac{1}{720}$
(b) Simplified mass matrix [1]	$1 + A(M\pi/N)^2$	$\omega_{ex}^2(M\pi/N)^2(\frac{1}{6} - A)$	$A < \frac{1}{6}$	$A > \frac{1}{6}$
(c) Consistent mass matrix [9]	$1 + A(M\pi/N)^4$	$\omega_{ex}^2(M\pi/N)^4(\frac{1}{720} - A)$	$A < \frac{1}{720}$	$A > \frac{1}{720}$
3. Stability of simply supported columns				
(a) Simplified geometric stiffness matrix [2]	$1 + A(M\pi/N)^2$	$P_{ex}(M\pi/N)^2(\frac{1}{12} - A)$	$A < \frac{1}{12}$	$A > \frac{1}{12}$
(b) Consistent geometric stiffness matrix [9]	$1 + A(M\pi/N)^4$	$P_{ex}(M\pi/N)^2(\frac{1}{720} - A)$	$A < \frac{1}{720}$	$A > \frac{1}{720}$
4. Vibrations of simply supported square plate with lumped mass approximation*	$1-A[\pi^2/(M^2+\overline{M}^2)](M\overline{M}/N)^2$	$\omega_{ex}^2(A-\alpha)[\pi^2/(M^2+\overline{M}^2)](M\overline{M}/N)^2$	$A > \alpha$	$A < \alpha$

* $\alpha = \frac{1}{6} - \frac{\nu}{12} + \frac{\nu^2}{48}$ for Melosh's model [10]; $\alpha = \frac{1}{6}(1 + \nu)$ for ACM model [11], [12].

Substituting (13) in (11) and ignoring terms of the order l^4 and higher, we get

$$\omega^2 = \frac{\omega_{ex}^2 [1 - \frac{1}{12}(M^2\pi^2)/N^2]}{R} \tag{14}$$

where

$$\omega_{ex}^2 = \left(\frac{M\pi}{L}\right)^2 \frac{E\bar{A}}{\rho}. \tag{15}$$

For convenience, we write the modification function R in the form

$$R = \left\{1 - A\left(\frac{M\pi}{N}\right)^2\right\}. \tag{16}$$

From (14) and (16), we get

$$\omega^2 = \omega_{ex}^2 \left\{1 + \left(A - \frac{1}{12}\right)\left(\frac{M\pi}{N}\right)^2\right\}. \tag{17}$$

and the principal error ε is given by

$$\varepsilon = \omega^2 - \omega_{ex}^2 = \omega_{ex}^2 \left(\frac{M\pi}{N}\right)^2 \left(A - \frac{1}{12}\right). \tag{18}$$

From (18) it is clear that for $A < 1/12$, ε is negative and for $A > 1/12$, ε is positive. Thus, the use of the modified mass matrix with $A > 1/12$ gives a positive principal error and therefore an upper bound to the frequency, whereas the use of the unmodified mass matrix gives a lower bound to the frequency. The use of modified mass matrices with $A < 1/12$ will give a refined frequency, but it is still a lower bound.

The above procedure is used to work out the discretization errors with modified mass or geometric stiffness matrices in a number of eigenvalue problems and the results are given in Table 3.

7. Conclusions

We have presented here a simple method for generating bounds for finite element eigenvalues. The method is based on the use of a modification function to modify the mass (or geometric stiffness) matrix. By an appropriate choice of two values for the scalar parameter in the modification function it is possible to generate two converging sequences approaching the exact from either side. Discretization errors worked out in several cases confirm this proposition. For practical applications it is economical to choose the scalar parameter A initially with a small number of elements.

It has been our experience that an adhoc choice of the scalar parameter may sometimes gives a convergence curve crossing the exact. But it crosses the exact once and only once and this crossing occurs when the number of elements are very small (2 or 3 in the case of beams [13]). This feature does not effect the main conclusions; namely, that one can always choose two values for the scalar parameter, so that the corresponding convergence curves bracket the exact solution. However, a more elaborate definition to R, in terms of more scalar parameters, can lead a greater control of the convergence curve.

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