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STRUCTURAL BEHAVIOUR OF TAPERED RODS SUBJECTED TO AXIAL COMPRESSIVE LOADS IN THE POST-BUCKLING RANGE

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ABSTRACT

The equation governing the behaviour of rods of variable cross-section subjected to compressive loads is deduced using the classical assumption that the plane sections of the rod before bending remain plane even after bending; the stress-strain relationship can be either the conventional Ramberg-Osgood Law or the alternative form developed by Rao and Krishna Murthy; the deformations are considered to be large. The governing equation is nonlinear and an iterative method is used to obtain numerical results. A class of rods whose variation of the cross-section can be represented by $I = I_o(1-\beta s)^n$, (where s is the curvilinear coordinate) are considered to study the post-buckling behaviour. Numerical results have been obtained for various values of the geometric parameters involved in the problem, and the results are in good agreement with those available in the literature.

NOTATION

b_o	width of the column at the fixed end
d_o	diameter (or depth) of column at the fixed end
E	Young's modulus of elasticity
E_t	tangent modulus
E_r	reduced modulus
I_o	moment of inertia of the root section
L	undeformed length of the column
P	compressive load on the column

R	radius of curvature of the rod
s	curvilinear co-ordinate, measured along the axis of the column from the fixed end
v	deflection at any section from the straight position
v_a	deflection from straight position at the free end
x,y	non-dimensional Cartesian co-ordinates, measured from the fixed end
x_a	projected length of the deformed column
α	slope at free end of the column
$'$	denotes differentiation with respect to x
β_1	taper parameter $= 1-b_1/b_o$
β_2	taper parameter $= 1-d_1/d_o$
λ	PL^2/EI_2
θ	slope at any section s of the column
σ_o	axial stress at the fixed end
σ_{os}	axial stress at any station s
σ_1, σ_2	bending stresses
ρ	radius of gyration at the base section

INTRODUCTION

Study of compression members plays an important role in the design of structures. In many cases, these members are of variable cross-section. Although the linear stability analysis of the tapered columns provides an useful indication to the designer about the critical conditions, the post-buckling analysis is of greater value in predicting the load-bearing capacity of the structure after 'buckling' has occurred. The solution for post-buckling of uniform cantilever columns was given by Timoshenko [1]. Bhandari [2] studied the post buckling behaviour of columns with exponential variation of moment of inertia, the tip section having a higher moment of inertia than the root section. More recently, Varadan and Pandalai [3] have given a solution for columns with small taper using the Rayleigh-Ritz procedure.

In this paper, we develop equations for the post-buckling analysis of rods, including the effects of large deformation and elasto-plasticity and present the post-buckling behaviour of slender columns for a wide range of taper ratios. Critical loads of short columns are also included.

FORMULATION

The fundamental assumption in this paper is that the bending moment at any section of the rod is proportional to the local curvature, which is representative of slender rods. Further, a linear stress distribution, corresponding to the use of Young's modulus E on the unloading side and the local tangent modulus E_t on the loading side, is assumed over any cross-section of the rod; this assumption, although not essential, is introduced to make the procedure reasonably simple.

Fig. 1 shows a typical cantilever rod of variable cross-section subjected to an axial compressive load. In the deformed position, the stress system at any cross-section may be considered to be the sum of the uniform axial stress field σ_{os} due to normal component of the load P and the stress distribution due to bending moment M(s) at this section. Referring to Fig. 2, we may write,

$$(1) \qquad \sigma_{or} = \frac{P \cos\theta}{A_s}$$

To find the bending stresses, we note that the stress on the concave side of the column (see Fig. 1) will increase along the stress-strain curve *CD* which is approximated, in this case, by a straight line from C with slope E_t at *C*, whereas on the convex side the stress will reduce along a line *CE* having a slope *E*, the Young's modulus of elasticity; *C* corresponds to σ_{os} in stress-strain law. Noticing that the axial stress resultant of the bending stresses at any cross-section is zero, we have

$$(2a) \qquad \int_0^{h_1} \sigma_1 \, dA - \int_0^{h_2} \sigma_2 \, dA = 0$$

and equilibrium in moments requires

(2b) $$\int_0^{h_1} \sigma_1 (y_1+e)\, dA + \int_0^{h_2} \sigma_2 (y_2-e)\, dA = M(s)$$

From simple geometric considerations, we have

(3) $$\epsilon_1 = y_1/R \quad and \quad \epsilon_2 = y_2/R$$

and hence,

(4) $$\sigma_1 = E y_1/R \quad and \quad \sigma_2 = E_t\, y_2/R$$

Use of (4) in (2a) gives

(5) $$E\, S_1 - E_t\, S_2 = 0$$

where S_1 and S_2 are the statical moments of the cross-sectional areas to the left and right of the line d-d in Fig. 3c and are given by

(5a) $$S_1 = \int_0^{h_1} y_1\, dA, \quad S_2 = \int_0^{h_2} y_2\, dA$$

Noticing that

(6) $$h_1 + h_2 = d_s$$

one can obtain h_1 and h_2 and hence 'e' from (5) and (6). Using (5) the equation (2b) becomes

(7) $$\frac{EI_{s1} + E_t I_{s2}}{R} = M(s)$$

where I_{s1} and I_{s2} represent the moments of inertia of the cross-section to the right and the left of the line d-d in Fig. 3, about (d-d), and are given by

(7a) $$I_{s_1} = \int_0^{h_1} y^2{}_1\, dA, \quad I_{s_2} = \int_0^{h_2} y_2{}^2\, dA$$

and R is the radius of curvature given by

(7b) $$\frac{1}{R} = \frac{d\theta}{ds}$$

and M(s) the bending at any section s. Eq. (7) is the differential equation governing the behaviour of rods subjected to axial loads in the post buckling range.

TAPERED RODS OF RECTANGULAR CROSS-SECTION

We consider a class of columns of rectangular cross-section, with sides of the cross section given by

$$(8) \qquad d(s) = d_o (1-\beta_1 s)^{n_1}, \qquad b(s) = b_o (1-\beta_2 s)^{n_2},$$

where b_0, d_0 are the width and depth of the rod at the fixed end and β_1 and β_2 are taper parameters. For the analysis of such rods, (7) may be written, in a more convenient form, as

$$(9) \qquad \frac{E_r I_s}{R} = M(s) = P(v_a - v)$$

where E_r is the reduced modulus for the rectangular cross-section given by

$$(10a) \qquad E_r = 4EE_t (E^{1/2} + E_t^{1/2})$$

and I_s is the local plane moment of inertia given by

$$(10b) \qquad I_s = \frac{1}{12} b\, d^3 = \frac{1}{12} b_o\, d_o^3 (1-\beta_1 s)^{n_1} (1-\beta_2 s^3)^{3n_2}$$

which is a function of the curvilinear co-ordinate s.

Using the Ramberg-Osgood stress-strain law, one can write

$$(11) \qquad E_t = E \frac{1}{1 + \frac{3}{7} n \left(\frac{\sigma_s}{\sigma_r}\right)^{n-1}}$$

where n is a shape parameter and σ_r is a reference stress. Substituting the expression for σ_s from (1) and rearranging, we get,

$$(12) \qquad E_t = E \frac{1}{1 + K \left(\frac{\cos\theta}{\phi(s)}\right)^{n-1}}$$

where

$$(13) \qquad K = \frac{3}{7}\, n \left(\frac{\sigma_o}{\sigma_r} \right)^{n-1}, \quad \phi(s) = (1-\beta_1 s)^{n_1} (1-\beta_2 s)^{n_2}$$

$$\sigma_o = P/A^o = P/b_o\, d_o$$

Using (9), (10) and (12), in (8), one finds the curvature as

$$(14) \qquad \frac{1}{R} = \frac{d\theta}{ds} = \frac{P}{EI_o} \frac{(v_a - v)\left[1 + \left\{ 1 + K \left(\frac{\cos\theta}{\phi(s)} \right)^{n-1} \right\} \right]^2}{4\psi(s)}$$

where

$$\psi(s) = (1-\beta_1 s)^{n_1} (1-\beta_2 s)^{3n_2}$$

Introducing the notation,

$$(15) \qquad \lambda_t = \frac{P L^2}{4\, EI_o}$$

and recognising

$$(16) \qquad \frac{dv}{ds} = \sin\theta$$

and integrating (14), we get

$$(17) \qquad \theta(s) = \lambda_t \int_0^s \frac{\int_s^1 \sin\theta\, ds}{\psi(s)} \left[1 + \left\{ 1 + k \left(\frac{\cos\theta}{\phi(s)} \right)^{n-1} \right\} \right]^2 ds$$

The solution of the nonlinear integro-differential equation (17) has to be obtained by a suitable numerical method. In this paper, we use an iterative procedure. We assume a suitable function for θ and use the condition, that assumed slope at the free end must be equal to the one obtained from (17), and this will yield the equilibrium load σ_0 of this configuration. In terms of the slope at free end α, (17) may be rewritten as

$$(18) \qquad \frac{4\alpha E}{\mu^2 \sigma_o} - \int_0^1 \frac{\int_s^1 \sin\theta\, ds}{\psi(s)} \left[1 + \left\{ 1 + \frac{3}{7}\, n \left(\frac{\sigma_o}{\sigma_r} \right)^{n-1} \left(\frac{\cos\theta}{\phi(s)} \right)^{n-1} \right\} \right]^2 ds = 0$$

where

$$\mu = L/\rho \text{ and } \sigma_o = P/b_o d_o$$

The equations corresponding to some specific cases may be deduced from (18) as follows:

(1) *Long uniform column (linear elasticity, large deformations)*

In this case $\beta_1 = \beta_2 = 0$ and hence, $\phi(s) = \psi(s) = 1$. Also, the modulus of elasticity is E and is constant. Hence, (18) reduces

(18a) $$\frac{\alpha E}{\mu^2 \sigma_o} - \int_0^1 \int_s^1 \sin\theta \, ds \, ds = 0$$

(*ii*) *Long tapered column (linear elasticity, large deformation)*

(18b) $$\frac{\alpha E}{\mu^2 \sigma_o} - \int_0^1 \frac{\int_s^1 \sin\theta \, ds}{\psi(s)} ds = 0$$

(*iii*) *Uniform column (elasto-plastic behaviour, large deformation)*

(18c) $$\frac{4 \alpha E}{\mu^2 \sigma_o} - \int_0^1 \left(\int_s^1 \sin\theta \, ds \right) \left[1 + \left\{ 1 + \frac{3}{7} n \left(\frac{\sigma_o}{\sigma_r} \right)^{n-1} (\cos\theta)^{n-1} \right\} \right]^2 ds = 0$$

METHOD OF SOLUTION

We obtain an approximate solution of the governing equations, using an iterative scheme. The approximating function for the slope θ is chosen to be

(19) $$\theta(s) = \sin^{-1} \frac{As}{2}(2 - s)$$

which satisfies the conditions

(19a) $$\theta(0) = 0 \ , \quad \left(\frac{d\theta}{ds} \right)_{s=1} = 0 \ ,$$

and A is a constant; each value of A corresponds to a specific slope at the free end and the corresponding equilibrium load is obtained by solving the corresponding equations (18). The equation (19) has been chosen in such a form that it simplifies the first integral in (18) which is given by

$$\int_0^1 \sin\theta \, ds = A\left[\frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right] \tag{20}$$

Taking

$$A = 2 \sin\alpha \tag{21}$$

one can obtain an interesting limiting case, $\alpha \to 0$, from (18) as

$$\underset{\alpha \to 0}{Lt} \frac{\alpha}{\sin\alpha} = 1 = \frac{P L^2}{2EI_o} \int_0^1 \frac{\left(\frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)}{\psi(s)} \times \left[1 + \frac{3}{7} n \left(\frac{\sigma_o}{\sigma_r}\right)^{n-1} \left(\frac{\cos\theta}{\phi(s)}\right)^{n-1}\right] ds \tag{22}$$

The equation (22) corresponds to critical instability loads of very short columns of variable cross-section.

Based on a simpler trial function

$$v = A x^2 = \frac{\alpha}{2 x_a} x^2 \tag{23}$$

one can obtain, a first approximation to equilibrium axial load of slender rods (material assumed to be elastic, large deformation effect considered) in terms of the slope at free end α, as

$$\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{C_1} + \frac{1}{C_2} \tag{24}$$

where,

$$\lambda_1 = \left(\frac{a x_a^4}{30} + \frac{b x_a^2}{12} + \frac{C}{2}\right)^{-1}$$

$$C_1 = N \beta x_a^3 \left(\frac{33\alpha^2 + 98}{840}\right) \tag{24a}$$

$$C_2 = N \frac{(N+1)}{2} \beta^2 x_a^4 \frac{13\alpha^2 + 28}{560}$$

$$N = n_1 + 3 n_2$$

The details regarding the derivation of (24) are given in the Appendix.

RESULTS AND DISCUSSION

Figs. (5) and (6) show a comparison of the values of λ for the two cases of $n=1$ and $n=4$. The results of [3] are better for small values of β. This is to be expectel since the function assumed in [3] is the exact mode shape of uniform columns. However, the present results are seen to be better for higher values of β. It is interesting to note that the present results and those of [3] are on either side of the more accurate results of Timoshenko, wherever available. Fig. 7. gives the values of λ, for various values of α, in the complete linear taper range of $0 \leqslant \beta \leqslant 1$ for the two cases, $n=1$ and $n=4$. For a column of rectangular cross-section, $n=1$ represents a case of taper in breadth only. The case $n=4$ corresponds to a linearly tapered circular or a rectangular column with linear taper in both directions, By suitably selecting the values of n, cases of non-linear taper can also be represented.

Critical instability loads of short columns (elasto-plastic material, and small deformations) for two typical taper ratios are presented in Fig. 8. For uniform columns, (22) gives a value of $\lambda=2.4$ which compares well with the known exact value of $\pi^2/4$.

APPENDIX-A

The equilbrium equation may be written as (see eq. 8)

(A1) $$EI \frac{v''}{(1+v'^2)^{3/2}} = M$$

In the case of cantilever columns considerd, we have

(A2) $$EI\, v'' = P(v_a - v)\,(1+v'^2)^{3/2}$$

We consider cases for which $\beta_1=\beta_2=\beta$ and the column has a moment of inertia at any section expressible in the form

(A3) $$I(x) = I_o\,(1-\beta x)^n$$

A deflected shape is assumed as $v_o = Ax^2$, so that

(A4) $$v_{o_a} = A\,x_a^2,\quad v'_{o_a} = \alpha = 2\,A\,x_a$$

In the first step, we replace v on the right hand side of (A2) by v_o, so that

(A5) $$v''_1 = \frac{1}{(1-\beta x)^n} A (x^2_a - x^2)(1+4A^2 x^2)^{3/2}$$

As a first approximation, we expand the last term on the right hand side and retain only two terms in the expansion, so that

$$v_1'' = A(x_a^2 - x^2)(1+6A^2 x^2)(1-\beta x)^{-n}$$

or

(A6) $$v_1 = A\int_0^x\int (ax^4+bx^2+c)\left[1+n\beta x+\frac{n(n+1)}{2}\beta^2 x^2+\dots\right]dx\,dx$$

where

$$a=-6A^2;\quad b=6A^2 x_a^2-1;\quad c=x_a^2$$

Depending on the required accuracy, one can retain a suitable number of terms in the expansion of $(1-\beta x)^{-n}$. In the present analysis, only three terms have been considered.

The value of λ is obtained from the condition that the assumed tip-deflection v_{o_u} must be equal to v_1 at $x=x_a$. The expression for λ can be written as

(A7) $$\frac{1}{\lambda}=\frac{1}{\lambda(1)}+\frac{1}{c_1}+\frac{1}{c_2}$$

where

$$\lambda(1)=\left(\frac{a\,x_a^4}{30}+\frac{b\,x_a^2}{12}+\frac{c}{2}\right)^{-1},\quad c_1=n\beta x_a^3\left(\frac{33\alpha^2+98}{840}\right),$$

$$c_2=\frac{n(n+1)}{2}x_1^4\,\beta^2\left(\frac{13\alpha^2+28}{560}\right)$$

In the above equation, the value of x_a corresponding to the assumed function is used and it is obtained as follows:

(A8) $$\int_0^{x_a}\sqrt{1+\left(\frac{dv}{dx}\right)^2}\,dx=\int_0 ds=1$$

Using (A5), in the above expression, we get

(A9) $$A=\tfrac{1}{4}\left[\alpha\sqrt{1+\alpha^2}+ln\,(\alpha+\sqrt{1+\alpha^2})\right]$$

(A10) $$x_x=\frac{\alpha}{2A}=\frac{2\alpha}{\alpha\sqrt{1+\alpha^2}+ln\,(\alpha+\sqrt{1+\alpha^2})}$$

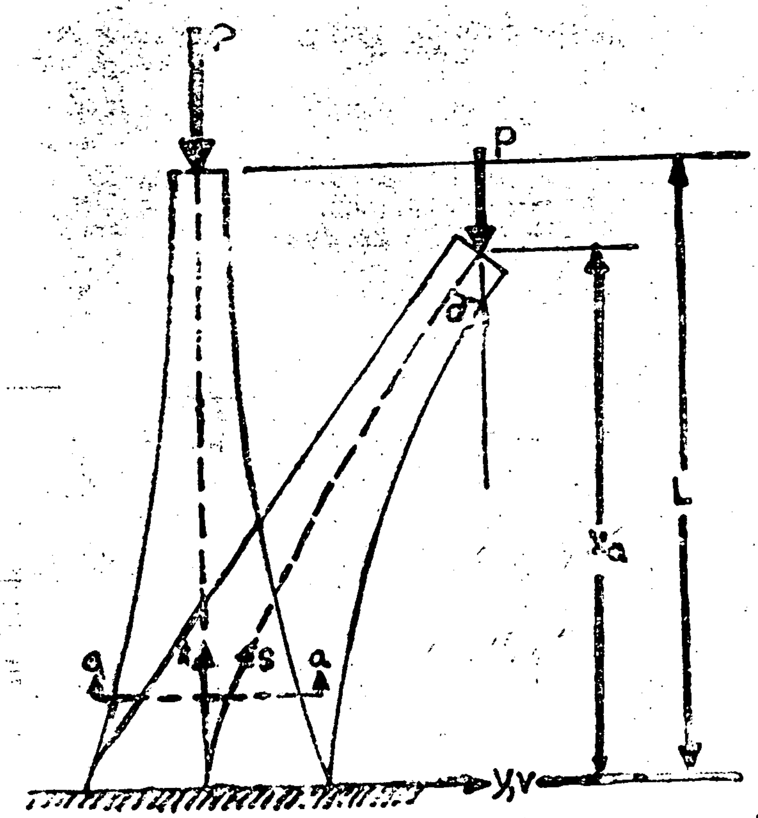

FIG. 1 A TAPERED COLUMN

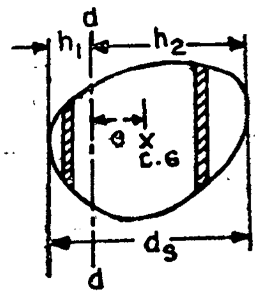

FIG. 2. SECTION a-a

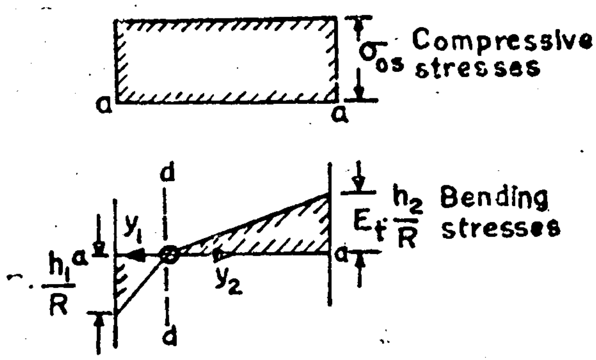

FIG. 3. STRESSES AT a-a

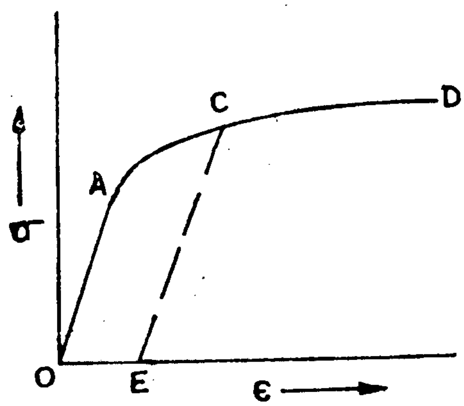

FIG. 4. STRESS-STRAIN CURVE

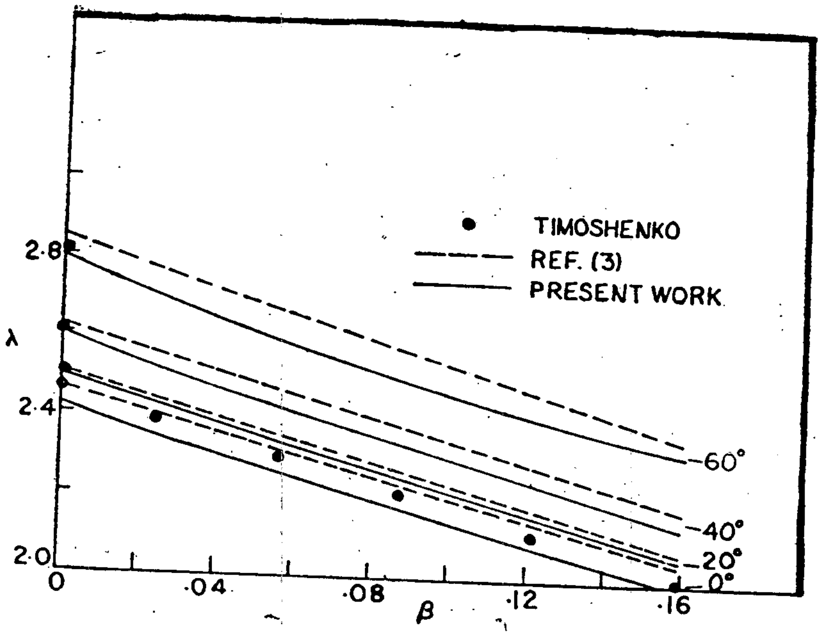

FIG. 5 – COMPARISON OF λ (n = 4)

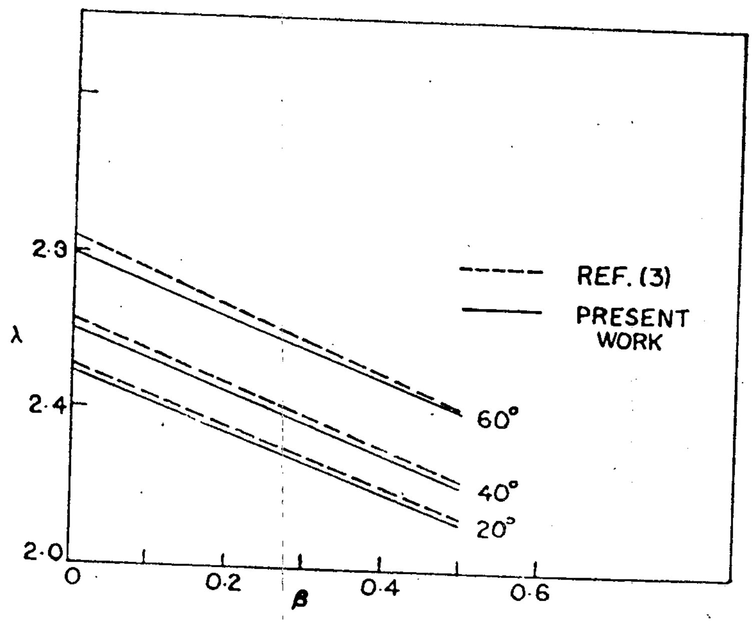

FIG. 6 – COMPARISON OF λ (n = 1)

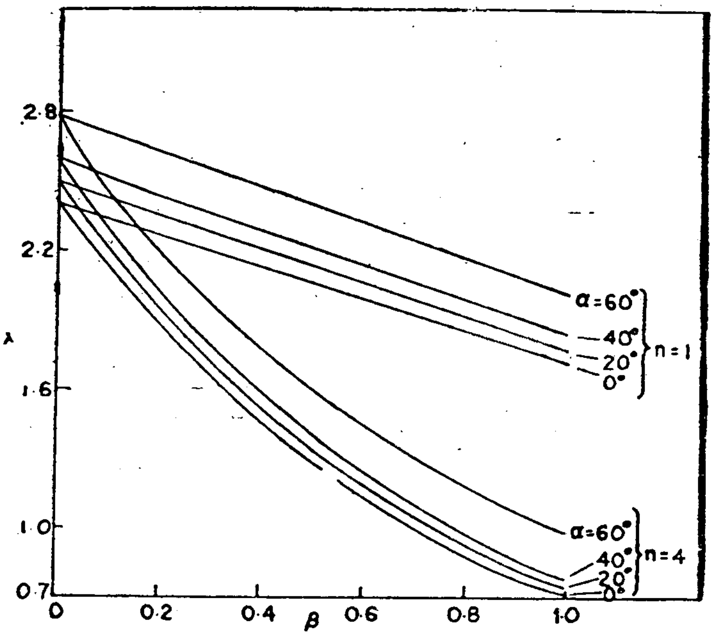

FIG. 7 — VARIATION λ WITH β

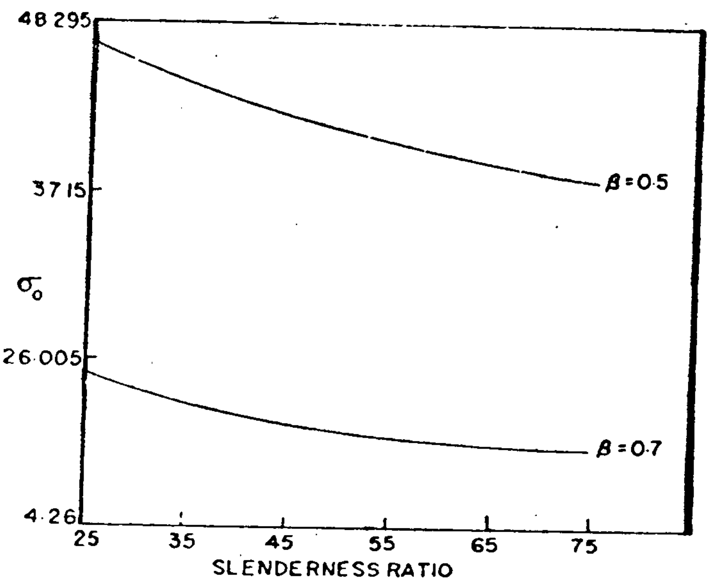

FIG. 8. CRITICAL LOADS OF SHORT COLUMNS

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