STRUCTURAL BEHAVIOUR OF TAPERED RODS
SUBJECTED TO AXIAL COMPRESSION LOADS
IN THE POST-BUCKLING RANGE

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ABSTRACT

The equation governing the behaviour of rods of variable cross-section subjected to compressive loads is deduced using the classical assumption that the plane sections of the rod before bending remain plane even after bending; the stress-strain relationship can be either the conventional Ramberg-Osgood Law or the alternative form developed by Rao and Krishna Murthy; the deformations are considered to be large. The governing equation is nonlinear and an iterative method is used to obtain numerical results. A class of rods whose variation of the cross-section can be represented by \( I = I_0 (1 - \beta s)^n \), (where \( s \) is the curvilinear coordinate) are considered to study the post-buckling behaviour. Numerical results have been obtained for various values of the geometric parameters involved in the problem, and the results are in good agreement with those available in the literature.

NOTATION

- \( b_o \): width of the column at the fixed end
- \( d_o \): diameter (or depth) of column at the fixed end
- \( E \): Young's modulus of elasticity
- \( E_t \): tangent modulus
- \( E_r \): reduced modulus
- \( I_o \): moment of inertia of the root section
- \( L \): undeformed length of the column
- \( P \): compressive load on the column
INTRODUCTION

Study of compression members plays an important role in the design of structures. In many cases, these members are of variable cross-section. Although the linear stability analysis of the tapered columns provides an useful indication to the designer about the critical conditions, the post-buckling analysis is of greater value in predicting the load-bearing capacity of the structure after 'buckling' has occurred. The solution for post-buckling of uniform cantilever columns was given by Timoshenko [1]. Bhandari [2] studied the post buckling behaviour of columns with exponential variation of moment of inertia, the tip section having a higher moment of inertia than the root section. More recently, Varadan and Pandalai [3] have given a solution for columns with small taper using the Rayleigh-Ritz procedure.

In this paper, we develop equations for the post-buckling analysis
of rods, including the effects of large deformation and elasto-plasticity, and present the post-buckling behaviour of slender columns for a wide range of taper ratios. Critical loads of short columns are also included.

FORMULATION

The fundamental assumption in this paper is that the bending moment at any section of the rod is proportional to the local curvature, which is representative of slender rods. Further, a linear stress distribution, corresponding to the use of Young's modulus $E$ on the unloading side and the local tangent modulus $E_t$ on the loading side, is assumed over any cross-section of the rod; this assumption, although not essential, is introduced to make the procedure reasonably simple.

Fig. 1 shows a typical cantilever rod of variable cross-section subjected to an axial compressive load. In the deformed position, the stress system at any cross-section may be considered to be the sum of the uniform axial stress field $\sigma_{0r}$ due to normal component of the load $P$ and the stress distribution due to bending moment $M(s)$ at this section. Referring to Fig. 2, we may write,

$\sigma_{or} = \frac{P \cos \theta}{A}$

(1)

To find the bending stresses, we note that the stress on the concave side of the column (see Fig. 1) will increase along the stress-strain curve $CD$ which is approximated, in this case, by a straight line from $C$ with slope $E_t$ at $C$, whereas on the convex side the stress will reduce along a line $CE$ having a slope $E$, the Young's modulus of elasticity; $C$ corresponds to $\sigma_{or}$ in stress-strain law. Noticing that the axial stress resultant of the bending stresses at any cross-section is zero, we have

$\int_0^{h_1} \sigma_1 \, dA - \int_0^{h_2} \sigma_2 \, dA = 0$

(2a)

and equilibrium in moments requires
From simple geometric considerations, we have

\[ \epsilon_1 = \frac{y_1}{R} \text{ and } \epsilon_2 = \frac{y_2}{R} \]

and hence,

\[ \sigma_1 = E\epsilon_1 = \frac{E y_1}{R} \text{ and } \sigma_2 = E\epsilon_2 = \frac{E y_2}{R} \]

Use of (4) in (2a) gives

\[ E S_1 - E S_2 = 0 \]

where \( S_1 \) and \( S_2 \) are the statical moments of the cross-sectional areas to the left and right of the line \( d-d \) in Fig. 3c and are given by

\[
S_1 = \int \limits_0^{h_1} y_1 \, dA, \quad S_2 = \int \limits_0^{h_2} y_2 \, dA
\]

Noticing that

\[ \epsilon_1 + \epsilon_2 = \epsilon \]

one can obtain \( h_1 \) and \( h_2 \) and hence \( \epsilon \) from (5) and (6). Using (5) the equation (2b) becomes

\[
\frac{EI_{s1}}{R} + \frac{E I_{s2}}{R} = M(s)
\]

where \( I_{s1} \) and \( I_{s2} \) represent the moments of inertia of the cross-section to the right and the left of the line \( d-d \) in Fig. 3, about \( (d-d) \), and are given by

\[
I_{s1} = \int \limits_0^{h_1} y_1^2 \, dA, \quad I_{s2} = \int \limits_0^{h_2} y_2^2 \, dA
\]

and \( R \) is the radius of curvature given by

\[
\frac{1}{R} = \frac{d\theta}{ds}
\]
and $M(s)$ the bending at any section $s$. Eq. (7) is the differential equation governing the behaviour of rods subjected to axial loads in the post-buckling range.

**TAPERED RODS OF RECTANGULAR CROSS-SECTION**

We consider a class of columns of rectangular cross-section, with sides of the cross section given by

$$d(s) = d_o (1 - \beta_1 s)^{n_1},$$

$$b(s) = b_o (1 - \beta_2 s)^{n_2},$$

where $b_o$, $d_o$ are the width and depth of the rod at the fixed end and $\beta_1$ and $\beta_2$ are taper parameters. For the analysis of such rods, (7) may be written, in a more convenient form, as

$$\frac{E_r I_s}{R} = M(s) = P (v_a - v).$$

where $E_r$ is the reduced modulus for the rectangular cross-section given by

$$E_r = 4EE_r (E_1^{1/2} + E_2^{1/2})$$

and $I_s$ is the local plane moment of inertia given by

$$I_s = \frac{1}{12} b d^3 = \frac{1}{12} b_o d_o^3 (1 - \beta_1 s)^{n_1} (1 - \beta_2 s)^{n_2},$$

which is a function of the curvilinear co-ordinate $s$.

Using the Ramberg-Osgood stress-strain law, one can write

$$E_r = E \frac{1}{1 + \frac{3}{7} n \left( \frac{\sigma_s}{\sigma_r} \right)^{n-1}}$$

where $n$ is a shape parameter and $\sigma_r$ is a reference stress. Substituting the expression for $\sigma_s$ from (1) and rearranging, we get,

$$E_r = E \frac{1}{1 + K \left( \frac{\epsilon_{\text{ref}} \phi(s)}{\phi(s)} \right)^n}.$$
where
\[
K = \frac{3}{7} n \left( \frac{\sigma_o}{\sigma_r} \right)^{n-1}, \quad \phi(s) = (1 - \beta_1 s)^{n_1} (1 - \beta_2 s)^{n_2}
\]

\[
\sigma_o = P/A^o = P/b_o d_o
\]

Using (3), (10) and (12), in (8), one finds the curvature as
\[
\frac{1}{K} = \frac{d\theta}{ds} = \frac{P}{4\beta I_o} \left[ 1 + \left\{ 1 + K \left( \frac{\cos \theta}{\phi(s)} \right)^{n-1} \right\} \right]^2
\]

where
\[
\psi(s) = (1 - \beta_1 s)^{n_1} (1 - \beta_2 s)^{n_2}
\]

Introducing the notation,
\[
\lambda_t = \frac{P I^t}{4 \beta I_o}
\]

and recognising
\[
\frac{dx}{ds} = \sin \theta
\]

and integrating (14), we get
\[
\theta(s) = \lambda_t \int^s_0 \left\{ \frac{\sin \theta}{\psi(s)} \right\} \left[ 1 + \left\{ 1 + K \left( \frac{\cos \theta}{\phi(s)} \right)^{n-1} \right\} \right]^2 ds
\]

The solution of the nonlinear integro-differential equation (17) has to be obtained by a suitable numerical method. In this paper, we use an iterative procedure. We assume a suitable function for \( \theta \) and use the condition, that assumed slope at the free end must be equal to the one obtained from (17), and this will yield the equilibrium load \( \sigma_a \) of this configuration. In terms of the slope at free end \( a \), (17) may be rewritten as
\[
\frac{4 \pi L^o}{\mu^2 \sigma_o} = \int^1_0 \frac{\sin \theta}{\psi(s)} \left[ 1 + \left\{ 1 + \frac{3}{7} n \left( \frac{\sigma_o}{\sigma_r} \right)^{n-1} \left( \frac{\cos \theta}{\phi(s)} \right)^{n-1} \right\} \right]^2 ds = 0
\]
where

\[ \mu = \frac{L}{\rho} \text{ and } \sigma_o = P\beta_o d_o \]

The equations corresponding to some specific cases may be deduced from (18) as follows:

(i) **Long uniform column (linear elasticity, large deformations)**

In this case \( \beta_1 = \beta_2 = 0 \) and hence, \( \phi(s) = \psi(s) = 1 \). Also, the modulus of elasticity is \( E \) and is constant. Hence, (18) reduces to

\[
\alpha E \left( \int_0^1 \int_0^1 \sin \theta \, ds \, ds = 0 \right)
\]

(ii) **Long tapered column (linear elasticity, large deformation)**

\[
\frac{\alpha E}{\mu^2 \sigma_o} \left[ \int_0^1 \sin \theta \, ds \right] = 0
\]

(iii) **Uniform column (elasto-plastic behaviour, large deformation)**

\[
\frac{4 \alpha E}{\mu^2 \sigma_o} \left[ \left( \int_0^1 \sin \theta \, ds \right) \left[ 1 + \left\{ 1 + \frac{3}{7} n \left( \frac{\sigma_o}{\sigma_r} \right)^n \right\} \right] \right] ds = 0
\]

**METHOD OF SOLUTION**

We obtain an approximate solution of the governing equations, using an iterative scheme. The approximating function for the slope \( \theta \) is chosen to be

\[
\theta(s) = \sin^{-1} \left( \frac{As}{2} (2 - s) \right)
\]

which satisfies the conditions

\[
\theta(0) = 0, \quad \left( \frac{d\theta}{ds} \right)_{s=1} = 0
\]

and \( A \) is a constant: each value of \( A \) corresponds to a specific slope at the free end and the corresponding equilibrium load is obtained.
by solving the corresponding equations (18). The equation (19) has been chosen in such a form that it simplifies the first integral in (18) which is given by

\[ \int_0^1 \sin \theta \, ds = A \left[ \frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6} \right] \]

Taking

\[ A = 2 \sin \alpha \]

one can obtain an interesting limiting case, \( \alpha \to 0 \), from (18) as

\[ \lim_{\alpha \to 0} \frac{\alpha}{\sin \alpha} = 1 = \frac{P L^2}{2EI_0} \int_0^1 \left( \frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6} \right) \frac{\psi(s)}{\phi(s)} \left[ 1 + \frac{3}{7} n \left( \frac{\sigma_o}{\sigma_r} \right)^{n-1} \left( \frac{\cos \theta}{\phi(s)} \right)^{n-1} \right] ds \]

The equation (22) corresponds to critical instability loads of very short columns of variable cross-section.

Based on a simpler trial function

\[ v = Ax^2 = \frac{\alpha}{2} x^2 \]

one can obtain, a first approximation to equilibrium axial load of slender rods (material assumed to be elastic, large deformation effect considered) in terms of the slope at free end \( \alpha \), as

\[ \lambda = \frac{1}{\lambda_1} + \frac{1}{C_1} + \frac{1}{C_2} \]

where,

\[ \lambda_1 = \left( \frac{a x_a^4 + b x_a^2 + C}{30} + \frac{C}{12} + \frac{C}{2} \right)^{-1} \]

\[ C_1 = N \beta r_a^3 \left( \frac{33 \alpha^2 + 98}{840} \right) \]

\[ C_2 = N \left( \frac{N+1}{2} - \beta^2 \alpha^4 \right) a^2 \frac{13 \alpha^2 + 28}{560} \]

\[ N = n_i + 3 n_i \]

The details regarding the derivation of (24) are given in the Appendix.
RESULTS AND DISCUSSION

Figs. (5) and (6) show a comparison of the values of $\lambda$ for the two cases of $n = 1$ and $n = 4$. The results of [3] are better for small values of $\beta$. This is to be expected since the function assumed in [3] is the exact mode shape of uniform columns. However, the present results are seen to be better for higher values of $\beta$. It is interesting to note that the present results and those of [3] are on either side of the more accurate results of Timoshenko, wherever available. Fig. 7, gives the values of $\lambda$, for various values of $\alpha$, in the complete linear taper range of $0 \leq \beta \leq 1$ for the two cases, $n = 1$ and $n = 4$. For a column of rectangular cross-section, $n = 1$ represents a case of taper in breadth only. The case $n = 4$ corresponds to a linearly tapered circular or a rectangular column with linear taper in both directions. By suitably selecting the values of $n$, cases of non-linear taper can also be represented.

Critical instability loads of short columns (elasto-plastic material, and small deformations) for two typical taper ratios are presented in Fig. 8. For uniform columns, (22) gives a value of $\lambda = 2.4$ which compares well with the known exact value of $\pi^2/4$.

APPENDIX-A

The equilibrium equation may be written as (see eq. 8)

$$EI \frac{v''}{(1 + v^2)^{3/2}} = M$$

(A1)

In the case of cantilever columns considered, we have

$$EI \ v'' = P(v_a - v) (1 + v^2)^{3/2}$$

(A2)

We consider cases for which $\beta_1 = \beta_2 = \beta$ and the column has a moment of inertia at any section expressible in the form

$$I(x) = I_o (1 - \beta x)^n$$

(A3)

A deflected shape is assumed as $v_o = Ax^2$, so that

$$v'_o = A \ x_o, \ v''_o = A = 2 \ A \ x_o$$

(A4)

In the first step, we replace $v$ on the right hand side of (A2) by $v_o$, so that
As a first approximation, we expand the last term on the right-hand side and retain only two terms in the expansion; so that

\[ v''_1 = A \left( x_a^2 - x^2 \right) \left( 1 + 4 A^2 x^2 \right)^{1/2} \]

or

\[ v_1 = A \int_0^x \int_0^x \left( a x^4 + b x^2 + c \right) \left[ 1 + n \beta x + \frac{n(n+1)}{2} \beta^2 x^2 + \ldots \right] dx \, dx \]

where

\[ a = -6A^2; \quad b = 6A^2 x_a^2 - 1; \quad c = x_a^2 \]

Depending on the required accuracy, one can retain a suitable number of terms in the expansion of \((1-\beta x)^{-n}\). In the present analysis, only three terms have been considered.

The value of \( \lambda \) is obtained from the condition that the assumed tip-deflection \( v_a \) must be equal to \( v_1 \) at \( x = x_a \). The expression for \( \lambda \) can be written as

\[ \frac{1}{\lambda} = \frac{1}{\lambda(1)} + \frac{1}{c_1} + \frac{1}{c_2} \]

where

\[ \lambda(1) = \left( \frac{a x_a^4}{30} + \frac{b x_a^2}{12} + \frac{c}{2} \right)^{-1}, \quad c_1 = n \beta x_a^3 \left( \frac{33a^2 + 98}{840} \right) \]

\[ c_2 = \frac{n(n+1)}{2} x_a^4 \beta^2 \left( \frac{13a^2 + 28}{560} \right) \]

In the above equation, the value of \( x_a \) corresponding to the assumed function is used and it is obtained as follows:

\[ \int_0^{x_a} \sqrt{1 + \left( \frac{dn}{dx} \right)^2} \, dx = \int_0^1 dx = 1 \]

Using (A5), in the above expression, we get

\[ A = \frac{1}{4} \left[ a \sqrt{1 + a^2} + ln \left( a + \sqrt{1 + a^2} \right) \right] \]

\[ x_a = \frac{a}{2A} = \frac{2a}{\alpha \sqrt{1 + \alpha^2} + ln \left( \alpha + \sqrt{1 + \alpha^2} \right)} \]
FIG. 1 A TAPERED COLUMN

FIG. 2. SECTION a-a

Compressive stresses

Bending stresses

FIG. 3. STRESSES AT a-a

FIG. 4. STRESS-STRAIN CURVE

Tapered rods in post-buckling range
FIG. 5 - COMPARISON OF $\lambda$ ($n = 4$)

FIG. 6 - COMPARISON OF $\lambda$ ($n = 1$)
Tapered rods in post-buckling range

FIG. 7 — VARIATION $\lambda$ WITH $\beta$

FIG. 8 CRITICAL LOADS OF SHORT COLUMNS
REFERENCES

