PROC. NAT. ACAD. SCI. INDIA, 71 (A), IV, 2001

Reflection and transmission of *P* and *SV* waves at the interface between two monoclinic elastic half-spaces

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Received December 6, 1999; Accepted April 2, 2000

Abstract

Reflection and transmission of quasi –*P* (*qP*) and quasi –*SV* (*qSV*) waves at the interface between two monoclinic elastic half-spaces is discussed. Closed-form expressions for the reflection and transmission coefficients are derived. A method of computing these coefficients is indicated. The present analysis corrects some fundamental errors appearing in a recent paper on the reflection and transmission of *qP* waves at the interface between two monoclinic media.

(Keywords : reflection/transmission/waves/monoclinic media)

Introduction

Musgrave[1] discussed the reflection and transmission of elastic waves at a plane boundary between two anisotropic media of hexagonal type. Dayley and Hron[2] investigated the case when the media involved are transversely isotropic. Keith and Crampin[3] derived a formulation for computing the energy division among *qP*, *qSV* and *qSH* waves generated by plane waves incident on a plane boundary between generally anisotropic media. A comprehensive account was presented for the case of isotropic/orthotropic interface.

A monoclinic medium possesses one plane of elastic symmetry. For wave propagation in the plane of symmetry, *SH* motion is decoupled from the *P-SV* motion. While the particle motion of SH waves is purely transverse, it is neither purely longitudinal nor purely transverse in the case of *P-SV* waves. In a recent paper, Chattopadhyay and Saha[4] discussed the reflection of *qP* waves at the interface between two monoclinic half-spaces. Since the authors assume that *qP* waves are purely longitudinal and *qSV* waves purely transverse, most of the results of this paper, including the expressions for the reflection and transmission coefficients, are erroneous (see also Singh[5]). The aim of the present study is to derive closed-form algebraic expressions for the reflection and transmission coefficients when plane waves of *qP* or *qSV* type are incident at the plane boundary between two monoclinic elastic half-spaces. A method of computing the reflection and transmission coefficients is indicated. Numerical results will be presented in a subsequent publication.

Plane Waves in a Monoclinic Elastic Medium

Consider a homogeneous anisotropic elastic medium of monoclinic type. It has one plane of elastic symmetry and its elastic properties are defined by thirteen elastic moduli. Taking the plane of symmetry as the x_2x_3-plane, the generalized Hooke's law can be expressed in the form

$$\tau_{11} = c_{11} e_{11} + c_{12} e_{22} + c_{13} e_{33} + 2c_{14} e_{23}, \tag{1a}$$

$$\tau_{22} = c_{12} e_{11} + c_{22} e_{22} + c_{23} e_{33} + 2e_{24} e_{23}, \tag{1b}$$

$$\tau_{33} = c_{13} e_{11} + c_{23} e_{22} + c_{33} e_{33} + 2c_{34} e_{23}, \tag{1c}$$

$$\tau_{23} = c_{14} e_{11} + c_{24} e_{22} + c_{34} e_{33} + 2c_{44} e_{23}, \tag{1d}$$

$$\tau_{13} = 2 (c_{55} e_{13} + c_{56} e_{12}), \tag{1e}$$

$$\tau_{12} = 2 (c_{56} e_{13} + c_{66} e_{12}), \tag{1f}$$

where τ_{ij} is the stress tensor and e_{ij} the strain tensor. Further,

$$2e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \tag{2}$$

u_i being the displacement vector.

For plane waves propagating in the x_2x_3-plane

$$u_i = u_i (x_2, x_3, t), \ \partial/\partial x_1 \equiv 0. \tag{3}$$

The equations of motion without body forces are

$$\frac{\partial}{\partial x_j} \tau_{ij} = \rho \frac{\partial^2 u_i}{\partial t^2} \ (i = 1, 2, 3), \tag{4}$$

using the summation convention. From eqn. (1) to (4), we obtain the equations of motion in terms of the displacements in the form

$$c_{66}\,\frac{\partial^2 u_1}{\partial x_2^2} + 2c_{56}\,\frac{\partial^2 u_1}{\partial x_2\,\partial x_3} + c_{55}\,\frac{\partial^2 u_1}{\partial x_3^2} = \rho\,\frac{\partial^2 u_1}{\partial t^2}\,, \tag{5}$$

$$c_{22}\,\frac{\partial^2 u_2}{\partial x_2^2} + c_{44}\,\frac{\partial^2 u_2}{\partial x_3^2} + c_{24}\,\frac{\partial^2 u_3}{\partial x_2^2} + c_{34}\,\frac{\partial^2 u_3}{\partial x_3^2} + 2c_{24}\,\frac{\partial^2 u_2}{\partial x_2\,\partial x_3}$$

$$+\,(c_{23} + c_{44})\,\frac{\partial^2 u_3}{\partial x_2\,\partial x_3} = \rho\,\frac{\partial^2 u_2}{\partial t^2}\,, \tag{6}$$

$$c_{24}\,\frac{\partial^2 u_2}{\partial x_2^2} + c_{34}\,\frac{\partial^2 u_2}{\partial x_3^2} + c_{44}\,\frac{\partial^2 u_3}{\partial x_2^2} + c_{33}\,\frac{\partial^2 u_3}{\partial x_3^2} + 2c_{34}\,\frac{\partial^2 u_3}{\partial x_2\,\partial x_3}$$

$$+\,(c_{23} + c_{44})\,\frac{\partial^2 u_2}{\partial x_2\,\partial x_3} = \rho\,\frac{\partial^2 u_3}{\partial t^2}\,. \tag{7}$$

From eqn. (5) to (7), it is obvious that the u_1 motion representing SH waves is decoupled from the $(u_2\,,\,u_3)$ motion representing qP and qSV waves.

Let $\boldsymbol{p}(0, p_2\,, p_3)$ denote the unit propagation vector, c the phase velocity and k the wave number of plane waves propagating in the x_2x_3 −plane. We seek plane wave solutions of the equations of motion (6) and (7) of the form

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = A \begin{pmatrix} d_2 \\ d_3 \end{pmatrix} \exp\,[ik\,(ct - x_2p_2 - x_3p_3)]\,, \tag{8}$$

where $\boldsymbol{d}$ $(0, d_2, d_3)$ is the unit displacement vector, also known as the polarization vector. Inserting the expressions for u_2 and u_3 in the equations of motion (6) and (7), we obtain

$$(U - \rho c^2)\, d_2 + V d_3 = 0\,, \tag{9}$$

$$V d_2 + (Z - \rho c^2)\, d_3 = 0\,, \tag{10}$$

where

$$\begin{aligned} U(p_2, p_3) &= c_{22} p_2^2 + c_{44} p_3^2 + 2c_{24} p_2 p_3\,, \\ V(p_2, p_3) &= c_{24} p_2^2 + c_{34} p_3^2 + (c_{23} + c_{44})\, p_2 p_3\,, \\ Z(p_2, p_3) &= c_{44} p_2^2 + c_{33} p_3^2 + 2c_{34} p_2 p_3\,. \end{aligned} \tag{11}$$

Eqn. (9) and (10) yield

$$d_2 / d_3 = V / (\rho c^2 - U) = (\rho c^2 - Z) / V\,. \tag{12}$$

Therefore, ρc^2 satisfies the quadratic equation

$$\rho^2 c^4 - (U + Z)\, \rho c^2 + (UZ - V^2) = 0, \tag{13}$$

with solutions

$$2\rho c^2 (p_2, p_3) = (U + Z) \pm [(U - Z)^2 + 4V^2]^{1/2}. \tag{14}$$

The upper sign in eqn. (14) is for qP waves and the lower sign is for qSV waves.

It has been shown by Singh[5] that eqn. (8) will represent a pure longitudinal or transverse wave if

$$\begin{aligned} c_{24} p_2^4 &+ (c_{23} - c_{22} + 2c_{44})\, p_2^3 p_3 - 3(c_{24} - c_{34})\, p_2^2 p_3^2 \\ &- (c_{23} - c_{33} + 2c_{44})\, p_2 p_3^3 - c_{34} p_3^4 = 0\,. \end{aligned} \tag{15}$$

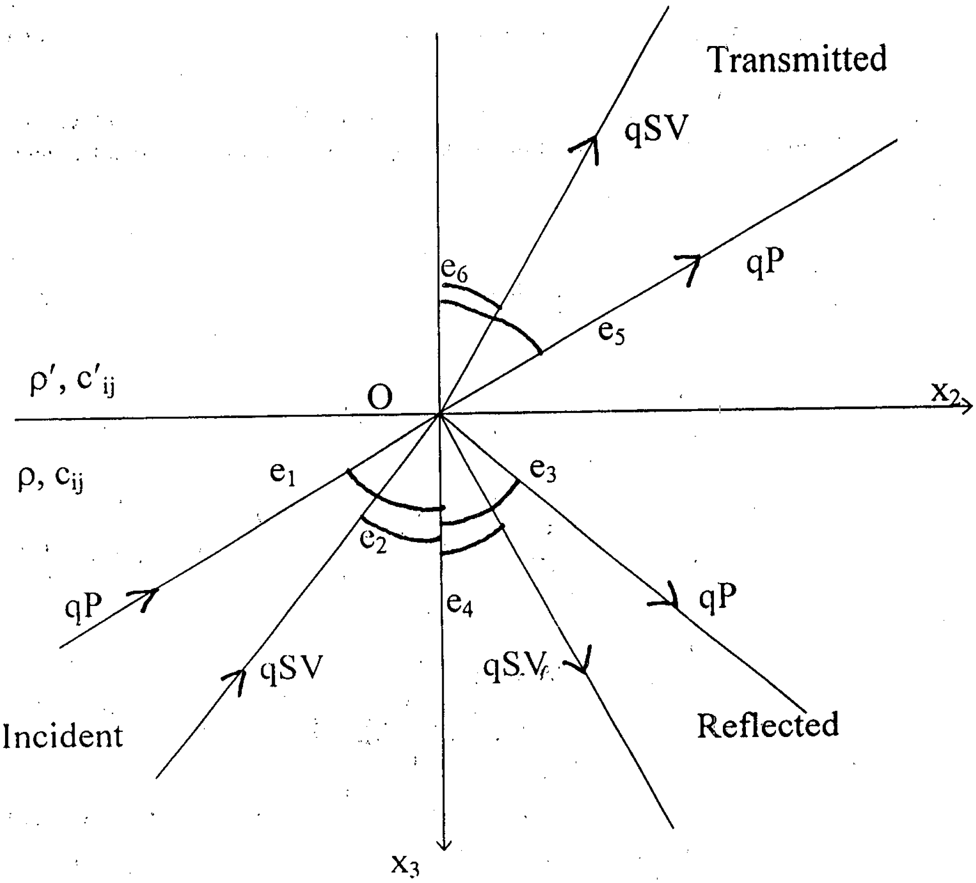

Fig. 1 – Reflection and transmission of qP and qSV waves at the plane interface ($x_3 = 0$) between two monoclinic half-spaces.

Eqn. (15) gives the directions of propagation for which P waves are purely longitudinal and SV waves purely transverse.

Reflection and Transmission of qP and qSV Waves

Consider a homogeneous, monoclinic, elastic half-space occupying the region $x_3 > 0$ in welded contact with another homogeneous, monoclinic, elastic half-space $x_3 < 0$ (Fig. 1). The identical plane of elastic symmetry of the two media is taken as the x_2x_3 – plane. Plane qP or qSV waves are incident at the interface $x_3 = 0$ from the half-space $x_3 > 0$. We consider plane strain problem for which

$$u_1 = 0,\ u_2 = u_2(x_2, x_3, t),\ u_3 = u_3(x_2, x_3, t). \tag{16}$$

Incident qP or qSV waves will generate reflected qP and qSV waves in the half-space $x_3 > 0$ and transmitted qP and qSV waves in the half-space $x_3 < 0$. The total displacement field is given by

$$u_2 = \sum_{j=1}^{4} A_j e^{iP_j}, \qquad u_3 = \sum_{j=1}^{4} B_j e^{iP_j} \tag{17}$$

for $x_3 > 0$, and

$$u_2' = \sum_{j=5}^{6} A_j e^{iP_j}, \qquad u_3' = \sum_{j=5}^{6} B_j e^{iP_j} \tag{18}$$

for $x_3 < 0$, where

$$P_j = \omega\,[t - (x_2 \sin e_j - x_3 \cos e_j) / c_j], \qquad (j = 1,2,5,6;\ \text{no summation}) \tag{19}$$

$$P_j = \omega\,[t - (x_2 \sin e_j + x_3 \cos e_j) / c_j], \qquad (j = 3, 4) \tag{20}$$

ω being the angular frequency. We distinguish quantities corresponding to various waves by using the subscript (1) for incident qP waves, (2) for incident qSV waves, (3) for reflected qP waves, (4) for reflected qSV waves, (5) for transmitted qP waves and (6) for transmitted qSV waves. Thus, for example, for the incident qP waves, c_1 denotes the phase velocity, e_1 the angle of incidence, $P_1(x_2, x_3, t)$ the phase factor, A_1 the amplitude factor of the u_2 component of the displacement and B_1 that of the u_3 component. The displacement components in the half-space $x_3 < 0$ are denoted by u_2' and u_3'.

Since each of the incident qP, incident qSV, reflected qP, reflected qSV, transmitted qP and transmitted qSV waves must satisfy the equations of motion, we have, as in eqn. (12) and (14),

$$A_i = F_i B_i \ (i = 1, 2, \ldots, 6), \tag{21}$$

where

$$F_i = V_i / (\rho c_i^2 - U_i) = (\rho c_i^2 - Z_i) / V_i , \quad (i = 1, 2, 3, 4), \tag{22a}$$

$$F_i = V_i / (\rho' c_i^2 - U_i) = (\rho' c_i^2 - Z_i) / V_i , \quad (i = 5, 6), \tag{22b}$$

$$2\rho c_i^2 = (U_i + Z_i) + [(U_i - Z_i)^2 + 4V_i^2]^{1/2} , \quad (i = 1, 3) , \tag{23a}$$

$$2\rho c_i^2 = (U_i + Z_i) - [(U_i - Z_i)^2 + 4V_i^2]^{1/2} , \quad (i = 2, 4) \tag{23b}$$

$$2\rho' c_5^2 = U_5 + Z_5 + [(U_5 - Z_5)^2 + 4V_5^2]^{1/2} , \tag{24a}$$

$$2\rho' c_6^2 = U_6 + Z_6 - [(U_6 - Z_6)^2 + 4V_6^2]^{1/2}. \tag{24b}$$

The expressions for U_i, V_i and Z_i are obtained from the expressions for U, V and Z given in eqn. (11) on substituting suitable values for (p_2 , p_3). For incident qP waves, $p_2 = \sin e_1 , p_3 = - \cos e_1$; for incident qSV waves, $p_2 = \sin e_2, p_3 = - \cos e_2$; for reflected qP waves, $p_2 = \sin e_3$, $p_3 = \cos e_3$; for reflected qSV waves, $p_2 = \sin e_4, p_3 = \cos e_4$; for transmitted qP waves, $p_2 = \sin e_5, p_3 = - \cos e_5$; and, for transmitted qSV waves, $p_2 = \sin e_6, p_3 = - \cos e_6$ (see Fig. 1). We thus obtain

$$U_1 = c_{22} \sin^2 e_1 + c_{44} \cos^2 e_1 - 2c_{24} \sin e_1 \cos e_1 ,$$

$$V_1 = c_{24} \sin^2 e_1 + c_{34} \cos^2 e_1 - (c_{23} + c_{44}) \sin e_1 \cos e_1 ,$$

$$Z_1 = c_{44} \sin^2 e_1 + c_{33} \cos^2 e_1 - 2c_{34} \sin e_1 \cos e_1 ; \tag{25}$$

$$U_3 = c_{22} \sin^2 e_3 + c_{44} \cos^2 e_3 + 2c_{24} \sin e_3 \cos e_3 ,$$

$$V_3 = c_{24} \sin^2 e_3 + c_{34} \cos^2 e_3 + (c_{23} + c_{44}) \sin e_3 \cos e_3 ,$$

$$Z_3 = c_{44} \sin^2 e_3 + c_{33} \cos^2 e_3 + 2c_{34} \sin e_3 \cos e_3 ; \tag{26}$$

$$U_5 = c'_{22} \sin^2 e_5 + c'_{44} \cos^2 e_5 - 2c'_{24} \sin e_5 \cos e_5 ,$$

$$V_5 = c'_{24} \sin^2 e_5 + c'_{34} \cos^2 e_5 - (c'_{23} + c'_{44}) \sin e_5 \cos e_5 ,$$

$$Z_5 = c'_{44} \sin^2 e_5 + c'_{33} \cos^2 e_5 - 2c'_{34} \sin e_5 \cos e_5 . \tag{27}$$

(U_2 , V_2 , Z_2) are obtained from (U_1 , V_1 , Z_1) on replacing e_1 by e_2, (U_4 , V_4 , Z_4) are obtained from (U_3 , V_3 , Z_3) on replacing e_3 by e_4 and (U_6 , V_6 , Z_6) are obtained from (U_5 , V_5 , Z_5) on replacing e_5 by e_6.

The total displacement field given by eqn. (17) and (18) must satisfy the boundary conditions,

$$u_2 = u'_2,\ u_3 = u'_3 ,\ \tau_{23} = \tau'_{23} ,\ \tau_{33} = \tau'_{33} \text{ at } x_3 = 0. \tag{28}$$

Since the boundary conditions (28) are to be satisfied for all values of x_2 , we must have

$$P_1 (x_2 , 0) = P_2(x_2 , 0) = P_3 (x_2 , 0) = P_4(x_2 , 0) = P_5 (x_2 , 0) = P_6 (x_2 , 0). \tag{29}$$

Eqn. (19), (20) and (29) imply

$$\frac{\sin e_1}{c_1 (e_1)} = \frac{\sin e_2}{c_2 (e_2)} = \frac{\sin e_3}{c_3 (e_3)} = \frac{\sin e_4}{c_4 (e_4)} = \frac{\sin e_5}{c_5 (e_5)} = \frac{\sin e_6}{c_6 (e_5)} = 1 / c_a , \tag{30}$$

where c_a is the apparent phase velocity. This is the form of Snell's law for monoclinic media.

From eqn. (3a), (25) and (26), we note that even if $e_1 = e_3$, $c_1 \neq c_3$. Therefore, from eqn. (30), the angle of reflection of qP waves is not equal to the angle of incidence of qP waves. Similarly, the angle of reflection of qSV waves is not equal to the angle of incidence of qSV waves. Chattopadhyay and Saha[4] assume that the angle of reflection of qP (qSV) waves is equal to the angle of incidence of qP (qSV) waves. Therefore, the reflection and the transmission coefficients obtained by Chattopadhyay and Saha[4] are not correct.

Using the relations (21), (29) and (30) the bounday conditions (28) yield

$$B_1 + B_2 + B_3 + B_4 - B_5 - B_6 = 0\,, \tag{31a}$$

$$F_1B_1 + F_2B_2 + F_3B_3 + F_4B_4 - F_5B_5 - F_6B_6 = 0\,, \tag{31b}$$

$$a_1B_1 + a_2B_2 + a_3B_3 + a_4B_4 - a_5B_5 - a_6B_6 = 0\,, \tag{31c}$$

$$b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4 - b_5B_5 - b_6B_6 = 0\,, \tag{31d}$$

where

$$a_1 = c_{24}F_1 + c_{44} - (c_{44}F_1 + c_{34}) \cot e_1\,,$$

$$a_2 = c_{24}F_2 + c_{44} - (c_{44}F_2 + c_{34}) \cot e_2\,,$$

$$a_3 = c_{24}F_3 + c_{44} + (c_{44}F_3 + c_{34}) \cot e_3\,,$$

$$a_4 = c_{24}F_4 + c_{44} + (c_{44}F_4 + c_{34}) \cot e_4\,,$$

$$a_5 = c'_{24}F_5 + c'_{44} - (c'_{44}F_5 + c'_{34}) \cot e_5\,,$$

$$a_6 = c'_{24}F_6 + c'_{44} - (c'_{44}F_6 + c'_{34}) \cot e_6\,,$$

$$b_1 = c_{23}F_1 + c_{34} - (c_{34}F_1 + c_{33}) \cot e_1\,,$$

$$b_2 = c_{23}F_2 + c_{34} - (c_{34}F_2 + c_{33}) \cot e_2\,,$$

$$b_3 = c_{23}F_3 + c_{34} + (c_{34}F_3 + c_{33}) \cot e_3\,,$$

$$b_4 = c_{23}F_4 + c_{34} + (c_{34}F_4 + c_{33}) \cot e_4 ,$$

$$b_5 = c'_{23}F_5 + c'_{34} - (c'_{34}F_5 + c'_{33}) \cot e_5 ,$$

$$b_6 = c'_{23}F_6 + c'_{34} - (c'_{34}F_6 + c'_{33}) \cot e_6 .$$

Incident qP waves :

In the case of incident qP waves, $A_2 = B_2 = 0$ and A_1 , B_1 are supposed to be known. Eqn. (31a, b, c, d) then constitute a set of four simultaneous equations in four unknowns, namely, B_3 , B_4 , B_5 and B_6 . These equations can be solved by Cramer's rule. We find

$$B_i / B_1 = \Delta_i^P / \Delta \quad (i = 3, 4, 5, 6), \tag{32}$$

where Δ and Δ_i^P are defined in Appendix A. Using eqn. (21), we find

$$\frac{A_i}{A_1} = \frac{F_i}{F_1}\left(\frac{B_i}{B_1}\right) = \frac{F_i}{F_1}\left(\frac{\Delta_i^P}{\Delta}\right) (i = 3, 4, 5, 6; \text{ no summation over } i). \tag{33}$$

Incident qSV waves :

For incident qSV waves, $A_1 = B_1 = 0$ and A_2 , B_2 are supposed to be known. The amplitude ratios are found to be

$$B_i / B_2 = \Delta_i^S / \Delta , \tag{34}$$

$$\frac{A_i}{A_2} = \frac{F_i}{F_2}\left(\frac{B_i}{B_2}\right) = \frac{F_i}{F_2}\left(\frac{\Delta_i^S}{\Delta}\right) \quad (i = 3, 4, 5, 6), \tag{35}$$

where Δ_i^S are defined in Appendix A.

Isotropic half-spaces :

For an isotropic medium,

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu ,$$

$$c_{12} = c_{13} = c_{23} = \lambda\,,\ \ c_{44} = c_{55} = c_{66} = \mu\,,$$

$$c_{14} = c_{24} = c_{34} = c_{56} = 0\,, \tag{36}$$

where λ, μ are the Lamé parameters. Using these values for c_{ij} and similar values for c'_{ij} we obtain

$$c_1 = c_3 = [(\lambda + 2\mu)/\rho]^{1/2} = \alpha\,,\ \ c_2 = c_4 = (\mu/\rho)^{1/2} = \beta\,,$$

$$c_5 = [(\lambda' + 2\mu')/\rho']^{1/2} = \alpha'\,,\ c_6 = (\mu'/\rho')^{1/2} = \beta'\,,$$

$$e_1 = e_3 = e,\ \ e_2 = e_4 = f,\ \ e_5 = e',\ \ e_6 = f',$$

$$\frac{\sin e}{\alpha} = \frac{\sin f}{\beta} = \frac{\sin e'}{\alpha'} = \frac{\sin f'}{\beta'}\,,$$

$$F_1 = -F_3 = -\tan e,\ F_2 = -F_4 = \cot f,\ \ F_5 = -\tan e'\,,\ \ F_6 = \cot f'\,,$$

$$a_1 = a_3 = 2\mu\,,\ a_2 = a_4 = -\mu\cos 2f/\sin^2 f,\ a_5 = 2\mu'\,,\ a_6 = -\mu'\cos 2f'/\sin^2 f'\,,$$

$$b_1 = -b_3 = -2\mu\,(\alpha/\beta)^2\ \cos 2f/\sin 2e\,,$$

$$b_2 = -b_4 = -2\mu\cot f,\ b_5 = -2\mu'\,(\alpha'/\beta')^2\cos 2f'/\sin 2e',\ b_6 = -2\mu'\cot f' \tag{37}$$

Putting these values in eqn. (31a, b, c, d), we get results equivalent to the corresponding results given by Ben-Menahem and Singh[6] (eqn. (3.54) and (3.56)) for isotropic media.

Discussion and Conclusions

The reflection and transmission coefficients given by Chattopadhyay and Saha[4] for *qP* waves incident at the plane boundary between two monoclinic elastic half-spaces are incorrect because of two erroneous assumptions made by these authors, namely, *qP* waves are longitudinal (*qSV* waves are transverse) and the angle of reflection of *qP* (*qSV*) waves is equal to the angle of incidence of qP (qSV) waves. In the present study, we have obtained the correct reflection and transmission coefficients by solving the problem *ab initio*.

Eqn. (32) and (33) give the amplitude ratios when plane qP waves are incident at the plane boundary between two monoclinic elastic half-spaces. In these equations, A_i / A_1 are the amplitude ratios for the horizontal component of the displacement and B_i / B_1 are the amplitude ratios for the vertical component of the displacement. Similarly, eqn. (34) and (35) give the amplitude ratios for incident qSV waves. From eqn. (17) and (21), we note that, for example, the total displacement of the incident qP waves is

$$\left(A_1^2 + B_1^2\right)^{1/2} e^{iP_1} = \left(1 + F_1^2\right)^{1/2} B_1 \, e^{iP_1} .$$

Therefore, the reflection coefficients can be expressed in the form

$$R_{PP} = \left(\frac{1 + F_3^2}{1 + F_1^2}\right)^{1/2} \cdot \frac{B_3}{B_1}, \quad R_{PS} = \left(\frac{1 + F_4^2}{1 + F_1^2}\right)^{1/2} \cdot \frac{B_4}{B_1} \tag{38}$$

for incident qP waves, and

$$R_{SP} = \left(\frac{1 + F_3^2}{1 + F_2^2}\right)^{1/2} \cdot \frac{B_3}{B_2}, \quad R_{SS} = \left(\frac{1 + F_4^2}{1 + F_2^2}\right)^{1/2} \cdot \frac{B_4}{B_2} \tag{39}$$

for incident qSV waves. Similar expressions can be written for the transmission coefficients. The reflection and transmission coefficients are in terms of the six angles e_i and the six velocities $c_i\,(e_i)$, $i = 1, 2, \ldots, 6$. For an incident qP wave, e_1 and, therefore, $c_1\,(e_1)$ is supposed to be known. One has to compute $e_i\,(i = 3, 4, 5, 6)$ for given e_1. The velocities $c_i\,(e_i)$ can then be computed from explicit algebraic formulae. We give below the procedure for computing e_i for given e_1 in the case of incident qP waves and for given e_2 in the case of incident qSV waves.

The Snell's law for a monoclinic medium is given by eqn. (30) in which the apparent velocity c_a can be written as $c_a = c/p_2$, where $\mathbf{p}\,(0, p_2, p_3)$ is the propagation vector. We define dimensionless apparent velocity c through the relation

$$\bar{c} = c_d/\beta = c/(p_2\beta), \tag{40}$$

where $\beta = (c_{44} / \rho)^{1/2}$. Eqn. (13) then becomes

$$\bar{c}^4 - (\bar{U} + \bar{Z})\,\bar{c}^2 + (\bar{U}\bar{Z} - \bar{V}^2) = 0, \tag{41}$$

where

$$\bar{U} = U/(c_{44}p_2^2) = p^2 + 2\,\bar{c}_{24}p + \bar{c}_{22},$$

$$\bar{V} = V/(c_{44}p_2^2) = \bar{c}_{34}p^2 + (1 + \bar{c}_{23})\,p + \bar{c}_{24},$$

$$\bar{Z} = Z/(c_{44}p_2^2) = \bar{c}_{33}\,p^2 + 2\,\bar{c}_{33}\,p + 1,$$

$$p = p_3/p_2, \quad \bar{c}_{ij} = c_{ij}/c_{44}. \tag{42}$$

For incident *qP* waves, $p = -\cot e_1$; for incident *qSV* waves, $p = -\cot e_2$; for reflected *qP* waves, $p = \cot e_3$; for reflected *qSV* waves, $p = \cot e_4$; for transmitted *qP* waves, $p = -\cot e_5$; for transmitted *qSV* waves, $p = -\cot e_6$. For a given p, eqn. (41) can be solved for $\bar{c}^2$, the two roots corresponding to *qP* and *qSV* waves. However, for a given $\bar{c}$, eqn. (41) is a bi-quadratic in p, corresponding to incident *qP*, incident *qSV*, reflected *qP* and reflected *qSV*. The positive roots corresponding to the reflected waves and the negative roots corresponding to the incident waves. On inserting the expressions for $\bar{U}$, $\bar{Z}$ and $\bar{V}$ from eqn. (42) into eqn. (41), the bi-quadratic in p becomes

$$g_0 p^4 + g_1 p^3 + g_2 p^2 + g_3 p + g_4 = 0, \tag{43}$$

where $g_0 = \bar{c}_{33} - \bar{c}_{34}^2$,

$$g_1 = 2\,(\bar{c}_{24}\,\bar{c}_{33} - \bar{c}_{23}\,\bar{c}_{34}),$$

$$g_2 = 1 + \bar{c}_{22}\,\bar{c}_{33} + 2\,\bar{c}_{24}\,\bar{c}_{34} - (1 + \bar{c}_{23})^2 - (1 + \bar{c}_{33})\,\bar{c}^2,$$

$$g_3 = 2[\bar{c}_{22}\,\bar{c}_{34} - \bar{c}_{23}\,\bar{c}_{24} - (\bar{c}_{24} + \bar{c}_{34})\,\bar{c}^2],$$

$$g_4 = \bar{c}^4 - (1 + \bar{c}_{22})\,\bar{c}^2 + \bar{c}_{22} - \bar{c}_{24}^2. \tag{44}$$

If we define $q = 1/p = p_2/p_3$, the bi-quadratic transforms to

$$g_4 q^4 + g_3 q^3 + g_2 q^2 + g_1 q + g_0 = 0. \tag{45}$$

For angles of incidence, for which both reflected qP and reflected qSV waves exist, eqn. (45) will possess two positive roots, the smaller positive root (say q_4) corresponding to reflected SV and the larger positive root (q_3) corresponding to reflected qP. Further,

$$e_3 = \tan^{-1}(q_3), \quad e_4 = \tan^{-1}(q_4). \tag{46}$$

A similar procedure can be set up for finding e_5 and e_6.

Acknowledgements

The authors are grateful to the Council of Scientific and Industrial Research, New Delhi for financial support through the Emeritus Scientist Scheme awarded to SJS and to the Univesity Grants Commission, New Delhi for Junior Research Fellowship awarded to SK.

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Appendix A

$$\Delta = \begin{vmatrix} 1 & 1 & -1 & -1 \\ F_3 & F_4 & -F_5 & -F_6 \\ a_3 & a_4 & -a_5 & -a_6 \\ b_3 & b_4 & -b_5 & -b_6 \end{vmatrix}$$

$$\Delta_3^P = \begin{vmatrix} -1 & 1 & -1 & -1 \\ -F_1 & F_4 & -F_5 & -F_6 \\ -a_1 & a_4 & -a_5 & -a_6 \\ -b_1 & b_4 & -b_5 & -b_6 \end{vmatrix}$$

$$\Delta_4^P = \begin{vmatrix} 1 & -1 & -1 & -1 \\ F_3 & -F_1 & -F_5 & -F_6 \\ a_3 & -a_1 & -a_5 & -a_6 \\ b_3 & -b_1 & -b_5 & -b_6 \end{vmatrix}$$

$$\Delta_5^P = \begin{vmatrix} 1 & 1 & -1 & -1 \\ F_3 & F_4 & -F_1 & -F_6 \\ a_3 & a_4 & -a_1 & -a_6 \\ b_3 & b_4 & -b_1 & -b_6 \end{vmatrix}$$

$$\Delta_6^P = \begin{vmatrix} 1 & 1 & -1 & -1 \\ F_3 & F_4 & -F_5 & -F_1 \\ a_3 & a_4 & -a_5 & -a_1 \\ b_3 & b_4 & -b_5 & -b_1 \end{vmatrix}$$

Δ_3^S is obtained from Δ_3^P on replacing the elements $\{-1, -F_1, -a_1, -b_1\}$ in the first column by the elements $\{-1, -F_2, -a_2, -b_2\}$. Δ_i^S $(i = 4, 5, 6)$ are similarly defined.