Indian J. pure appl. Math., 18 (8) : 763–777, August 1987

2-*D* STATIC RESPONSE OF A TRANSVERSELY ISOTROPIC MULTILAYERED HALF-SPACE TO SURFACE LOADS

NAT RAM GARG

AND

SARVA JIT SINGH

Department of Mathematics, Maharshi Dayanand University, Rohtak 124001

(*Received* 8 *July* 1986; *after revision* 22 *October* 1986)

The two-dimensional problem of the static deformation of a transversely isotropic, multilayered, elastic half-space by surface loads is studied. Both plane strain and antiplane strain cases are considered. The method of layer matrices is used to obtain the field at any point of the medium. Explicit expressions for stresses caused by a surface line load on a transversely isotropic uniform half-space are derived. The present formulation avoids the cumbersome nature of the problem and is quite convenient for numerical computation.

1. INTRODUCTION

Kuo[1] studied the three-dimensional problem of an inclined static load on a circular area of the surface of a multilayered isotropic half-space. Singh[2] solved the corresponding problem for three-dimensional buried sources. The two-dimensional plane strain and antiplane strain problems of the static deformation of a multilayered isotropic half-space by surface loads has been discussed by Garg and Singh[3], who have considered in detail the particular cases of a normal line load and a shear line load.

In the present paper, we have formulated the two-dimensional problem of the static deformation of a transversely isotropic multilayered half-space by surface loads. Both plane strain and antiplane strain cases are considered. The Thomson[4]-Haskell[5] matrix method is used to obtain the required field. The particular cases of a normal line-load and a shear line load are considered in detail. It is shown that in the case of a transversely isotropic uniform half-space the integrals giving the stresses can be evaluated analytically. The corresponding axially-symmetric problem has earlier been discussed by Singh[6].

The importance of the problem considered in the present paper lies in the fact that the crust of the earth is anisotropic and, therefore, it will be useful to study the effect of anisotropy on the static field due to surface loads. It may also find some applications in engineering, since the materials used are not always isotropic.

2. Basic Equations

In the cartesian coordinates (x_1, x_2, x_3), the equations of equilibrium for zero body forces are

$$\frac{\partial p_{11}}{\partial x_1} + \frac{\partial p_{12}}{\partial x_2} + \frac{\partial p_{13}}{\partial x_3} = 0, \qquad \ldots(2.1)$$

$$\frac{\partial p_{12}}{\partial x_1} + \frac{\partial p_{22}}{\partial x_2} + \frac{\partial p_{23}}{\partial x_3} = 0 \qquad \ldots(2.2)$$

$$\frac{\partial p_{13}}{\partial x_1} + \frac{\partial p_{23}}{\partial x_2} + \frac{\partial p_{33}}{\partial x_3} = 0 \qquad \ldots(2.3)$$

where p_{ij} is the stress tensor. If (u_1, u_2, u_3) denote the components of the displacement vector, the strain-displacement relations are

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (i, j = 1, 2, 3). \qquad \ldots(2.4)$$

For a transversely isotropic medium, whose axis of symmetry coincides with the x_3-axis, the stress-strain relations are given by (Payton[7], p. 3)

$$\begin{bmatrix} p_{11} \\ p_{22} \\ p_{33} \\ p_{23} \\ p_{13} \\ p_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{13} \\ 2e_{12} \end{bmatrix} \qquad \ldots(2.5)$$

with

$$c_{66} = (c_{11} - c_{12})/2. \qquad \ldots(2.6)$$

An isotropic solid is a special case of a transversely isotropic solid for which

$$c_{12} = c_{13} = \lambda, \; c_{11} = c_{33} = \lambda + 2\mu, \; c_{44} = c_{66} = \mu \qquad \ldots(2.7)$$

where λ, μ are the Lamé constants.

We shall be considering a two-dimensional approximation in which the displacement components and consequently stresses are independent of x_1 so that $\partial/\partial x_1 \equiv 0$. Under this assumption, the plane strain problem $(u_1 = 0)$ and the antiplane strain problem $(u_2 = u_3 = 0)$ are decoupled and, therefore, can be treated separately. In the following, we shall write (x, y, z) for (x_1, x_2, x_3) and (u, v, w) for (u_1, u_2, u_3).

3. Antiplane Strain Problem

For the antiplane strain problem

$$u = u(y, z), \quad v = w \equiv 0. \qquad \ldots(3.1)$$

The non-zero stresses are

$$p_{12} = c_{66} \frac{\partial u}{\partial y}, \quad p_{13} = c_{44} \frac{\partial u}{\partial z}. \qquad \ldots(3.2)$$

Using (3.1) and (3.2), eqns. (2.1) – (2.3) give one non-trivial equation :

$$\left(s^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u = 0 \qquad \ldots(3.3)$$

where

$$s = (c_{66}/c_{44})^{1/2}. \qquad \ldots(3.4)$$

A solution of (3.3) is of the from

$$u(y, z) = \int_0^\infty (Ae^{-skz} + Be^{skz}) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk \qquad \ldots(3.5)$$

where A, B are functions of k only. From (3.2) and (3.5), we have

$$p_{13}(y, z) = t \int_0^\infty (-Ae^{-skz} + Be^{skz}) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\, dk \qquad \ldots(3.6)$$

where

$$t = (c_{44}\, c_{66})^{1/2}. \qquad \ldots(3.7)$$

Equations (3.5) and (3.6) may be written as

$$u = \int_0^\infty U(z) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk \qquad \ldots(3.8)$$

$$p_{13} = \int_0^\infty T(z) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\, dk \qquad \ldots(3.9)$$

where

$$\begin{bmatrix} U(z) \\ T(z) \end{bmatrix} = \begin{bmatrix} ch\,(skz) & -sh\,(skz) \\ t\, sh\,(skz) & -t\, ch\,(skz) \end{bmatrix} \begin{bmatrix} A + B \\ A - B \end{bmatrix} \qquad (3.10)$$

and ch and sh stand for hyperbolic cosine and sine, respectively.

In the case of an isotropic medium $s = 1$, $t = \mu$ and eqns. (3.5) and (3.6) coincide with the corresponding equations for an isotropic medium obtained by Singh and Garg[8].

4. Plane Strain Problem

For the plane strain problem

$$v = v(y, z), \quad w = w(y, z), \quad u \equiv 0 \qquad \ldots(4.1)$$

and the non-zero strains are

$$e_{22} = \frac{\partial v}{\partial y}, \; 2e_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \; e_{33} = \frac{\partial w}{\partial z}. \qquad \ldots(4.2)$$

Equation (2.5) then yields

$$p_{11} = c_{12}\, e_{22} + c_{13}\, e_{33} \qquad \ldots(4.3)$$

$$p_{22} = c_{11}\, e_{22} + c_{13}\, e_{33} \qquad \ldots(4.4)$$

$$p_{33} = c_{13}\, e_{22} + c_{33}\, e_{33} \qquad \ldots(4.5)$$

$$p_{23} = 2c_{44}\, e_{23} \qquad \ldots(4.6)$$

$$p_{12} = p_{13} = 0. \qquad \ldots(4.7)$$

Using (4.7), We find that eqn. (2.1) is identically satisfied and eqns. (2.2) and (2.3) take the form

$$\frac{\partial p_{22}}{\partial y} + \frac{\partial p_{23}}{\partial z} = 0 \qquad \ldots(4.8)$$

$$\frac{\partial p_{23}}{\partial y} + \frac{\partial p_{33}}{\partial z} = 0. \qquad \ldots(4.9)$$

Therefore, there exists an Airy stress function $F(y, z)$ such that

$$p_{22} = \frac{\partial^2 F}{\partial z^2}, \; p_{23} = -\frac{\partial^2 F}{\partial y\, \partial z}, \; p_{33} = \frac{\partial^2 F}{\partial y^2}. \qquad \ldots(4.10)$$

The equilibrium equations (4.8) and (4.9) are identically satisfied. The compatibility equation is (Sokolnikoff[9], p. 28)

$$\frac{\partial^2 e_{22}}{\partial z^2} + \frac{\partial^2 e_{33}}{\partial y^2} = 2\frac{\partial^2 e_{23}}{\partial y\, \partial z}. \qquad \ldots(4.11)$$

Eliminating e_{22}, e_{33} and e_{23} from (4.4)-(4.6) and (4.11) and then using (4.10), we obtain

$$\left[a\frac{\partial^4}{\partial y^4} + (ac - 2b - b^2)\frac{\partial^4}{\partial y^2\, \partial z^2} + c\frac{\partial^4}{\partial z^4} \right] F = 0 \qquad \ldots(4.12)$$

where

$$a = c_{11}/c_{44}, \; b = c_{13}/c_{44}, \; c = c_{33}/c_{44}. \qquad \ldots(4.13)$$

We may write (4.12) as

$$\left(\alpha^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\beta^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F = 0. \qquad \ldots(4.14)$$

where α and β are given by the relations

$$\alpha^2 + \beta^2 = (ac - 2b - b^2)/c, \; \alpha^2\beta^2 = a/c. \qquad \ldots(4.15)$$

In the case of an isotropic body, (2.7), (4.13) and (4.15) reveal that

$$a = c = (\lambda/\mu) + 2,\ b = \lambda/\mu,\ \alpha = \beta = 1 \qquad \text{...(4.15a)}$$

and eqn. (4.14) becomes biharmonic.

A solution of (4.14) is of the type (assuming $\alpha \neq \beta$)

$$F = \int_0^\infty (A\, e^{-\alpha kz} + B\, e^{\alpha kz} + C\, e^{-\beta kz} + D e^{\beta kz}) \begin{pmatrix} \sin\, ky \\ \cos\, ky \end{pmatrix} \, dk \qquad \text{...(4.16b)}$$

where A, B, C, D may be functions of k.

Using (4.10) and (4.16), the stresses are found to be

$$p_{22} = \int_0^\infty [A\, \alpha^2\, e^{-\alpha kz} + B\, \alpha^2\, e^{\alpha kz} + C\, \beta^2\, e^{-\beta kz} + D\, \beta^2\, e^{\beta kz}]$$

$$\times \begin{pmatrix} \sin\ \ ky \\ \cos\ ky \end{pmatrix} k^2\, d\mathrm{k} \qquad \text{...(4.17)}$$

$$p_{33} = \int_0^\infty [-A\, e^{-\alpha kz} - B\, e^{\alpha kz} - C\, e^{-\beta kz} - D\, e^{\beta kz}] \begin{pmatrix} \sin\ \ ky \\ \cos\ ky \end{pmatrix} k^2 dk \qquad \text{...(4.18)}$$

$$p_{23} = \int_0^\infty [A\alpha\, e^{-\alpha kz} - B\, \alpha\, e^{\alpha kz} + C\, \beta\, e^{-\beta kz} - D\, \beta\, e^{\beta kz}]$$

$$\times \begin{pmatrix} \cos\ ky \\ -\sin\ ky \end{pmatrix} k^2\, dk. \qquad \text{...(4.19)}$$

The expressions for the displacements can be obtained by integrating the stress-displacement relations, which can be written as

$$p_{22} = c_{11}\, \frac{\partial v}{\partial y} + c_{13}\, \frac{\partial w}{\partial z} \qquad \text{...(4.20)}$$

$$p_{33} = c_{13}\, \frac{\partial v}{\partial y} + c_{33}\, \frac{\partial w}{\partial z} \qquad \text{...(4.21)}$$

$$p_{23} = c_{44} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right). \qquad \text{...(4.22)}$$

Solving (4.20) and (4.21), we obtain

$$\frac{\partial v}{\partial y} = \Delta^{-1} (c_{33}\ p_{22} - c_{13}\, p_{33}) \qquad \text{...(4.23a)}$$

$$\frac{\partial w}{\partial z} = \Delta^{-1} (c_{11}\ p_{33} - c_{13} p_{22}) \qquad \text{...(4.23b)}$$

where

$$\Delta = c_{11}\, c_{33} - c_{13}^2\ . \qquad \text{...(4.24)}$$

Integrating (4.23*a*, *b*), we find

$$v = \Delta^{-1} \int (c_{33}\, p_{22} - c_{13}\, p_{33})\, dy + F(z) \qquad \text{...(4.25a)}$$

$$w = \Delta^{-1} \int (c_{11}\, p_{33} - c_{13}\, p_{22})\, dz + G(y) \qquad \text{...(4.25b)}$$

where $F(z)$ and $G(y)$ are arbitrary functions. Equation (4.22) shows that F and G represent a rigid body displacement and can thus be disregarded in the analysis of deformation. Taking $F = G = 0$, (4.17) (4.18) and (4.25a, b) yield

$$v = \int_0^\infty [p_1 (A\, e^{-\alpha kz} + B\, e^{\alpha kz}) + p_2 (C\, e^{-\beta kz} + D\, e^{\beta kz})] \times \begin{pmatrix} -\cos ky \\ \sin ky \end{pmatrix} k\, dk \qquad \text{...(4.26)}$$

$$w = \int_0^\infty [q_1 (A\, e^{-\alpha kz} - B\, e^{-\alpha kz}) + q_2 (C\, e^{-\beta kz} - D\, e^{\beta kz})] \times \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\, dk \qquad \text{...(4.27)}$$

where

$$p_1 = \Delta^{-1} (c_{33}\, \alpha^2 + c_{13}),\; p_2 = \Delta^{-1} (c_{33}\, \beta^2 + c_{13})$$

$$q_1 = \Delta^{-1} [c_{13}\, \alpha + (c_{11}/\alpha)],\; q_2 = \Delta^{-1} [c_{13}\, \beta + (c_{11}/\beta)]. \qquad \text{...(4.28)}$$

We may write (4.18), (4.19), (4.26) and (4.27) in the form

$$v = \int_0^\infty V \begin{pmatrix} -\cos ky \\ \sin ky \end{pmatrix} k\, dk \qquad \text{...(4.29)}$$

$$w = \int_0^\infty W \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\, dk \qquad \text{...(4.30)}$$

$$p_{23} = \int_0^\infty S \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k^2\, dk \qquad \text{...(4.31)}$$

$$p_{33} = \int_0^\infty N \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k^2\, dk. \qquad \text{...(4.32)}$$

The functions V, W, S, N are given by the matrix relation

$$[Y(z)] = [Z(z)]\, [K] \qquad \text{...(4.33)}$$

where

$$[Y(z)] = [V, W, S, N]^T,\; [K] = [A + B, A - B, C + D, C - D]^T \qquad \text{...(4.34)}$$

and $[\ldots]^T$ denotes the transpose of the matrix $[\ldots]$. The matrix $[Z(z)]$ is given below :

$$[Z(z)] = \begin{bmatrix} p_1\, ch(\alpha kz) & -p_1\, sh(\alpha kz) & p_2\, ch(\beta kz) & -p_2\, sh(\beta kz) \\ -q_1\, sh(\alpha kz) & q_1\, ch(\alpha kz) & -q_2\, sh(\beta kz) & q_2\, ch(\beta kz) \\ -\alpha\, sh(\alpha kz) & \alpha\, ch(\alpha kz) & -\beta\, sh(\beta kz) & \beta\, ch(\beta kz) \\ -\, ch(\alpha kz) & sh(\alpha kz) & -ch(\beta kz) & sh(\beta kz) \end{bmatrix}. \qquad \ldots(4.35)$$

5. Deformation of a Multilayered Half-space by Surface Loads

We consider a semi-infinite medium made up of $p-1$ parallel, homogeneous, transversely isotropic layers lying over a homogeneous, transversely isotropic half-space. The layers are numbered serially, the layer at the top being layer 1 and the half-space is designated as layer p. We place the origin of the cartesian coordinate system (x, y, z) at the boundary of the semi-infinite medium and the z-axis is drawn into the medium. The nth layer is of thickness d_n and is bounded by interfaces $z = z_{n-1}, z_n$ so that $d_n = z_n - z_{n-1}$. Obviously $z_0 = 0$ and $z_{p-1} = H$, where H is the depth of the last interface.

5.1 *Antiplane Strain Problem*

Introducing the subscript n to the quantities pertaining to the nth layer, (3.10) becomes

$$\begin{bmatrix} U_n(z) \\ T_n(z) \end{bmatrix} = [Z_n(z)] \begin{bmatrix} A_n + B_n \\ A_n - B_n \end{bmatrix} \qquad \ldots(5.1)$$

where the matrix $[Z_n(z)]$ is given by

$$[Z_n(z)] = \begin{bmatrix} ch(s_n kz) & -sh(s_n kz) \\ t_n\, sh(s_n kz) & -t_n\, ch(s_n kz) \end{bmatrix}. \qquad \ldots(5.2)$$

Following Singh[2,16], we obtain (see the Appendix)

$$\begin{bmatrix} U_{n-1}(z_{n-1}) \\ T_{n-1}(z_{n-1}) \end{bmatrix} = [a_n] \begin{bmatrix} U_n(z_n) \\ T_n(z_n) \end{bmatrix} \qquad \ldots(5.3)$$

where the layer matrix $[a_n]$ is given by

$$[a_n] = \begin{bmatrix} ch(s_n k d_n) & -t_n^{-1}\, sh(s_n k d_n) \\ -t_n\, sh(s_n k d_n) & ch(s_n k d_n) \end{bmatrix}. \qquad \ldots(5.4)$$

For the half-space, $B_p = 0$. Equations (5.1) and (5.3) then yield

$$\begin{bmatrix} U_1(0) \\ T_1(0) \end{bmatrix} = [F] \begin{bmatrix} A_p \\ A_p \end{bmatrix} \qquad \ldots(5.5)$$

with

$$[F] = [a_1]\,[a_2] \ldots [a_{p-1}]\,[z_p(H)]. \qquad \ldots(5.6)$$

Equation (5.5) gives the following two equations

$$U_1(0) = (F_{11} + F_{12})\, A_p \qquad \text{...(5.7)}$$

$$T_1(0) = (F_{21} + F_{22})\, A_p. \qquad \text{...(5.8)}$$

From (5.7) or (5.8), A_p is known. Equation (5.7) is applicable when the surface displacement is prescribed and (5.8) is applicable when the surface load is prescribed. We shall confine our discussion to the latter case only.

When the surface load is prescribed the boundary condition is of the form

$$p_{13} = f(y) \text{ at } z = 0. \qquad \text{...(5.9)}$$

We shall write

$$f(y) = \int_0^\infty \bar{f}(k) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk. \qquad \text{...(5.10)}$$

Equations (3.9), (5.8) and (5.10) yield

$$A_p = \bar{f}(k)/[k(F_{21} + F_{22})]. \qquad \text{...(5.11)}$$

The field at any point of the medium can be obtained. For $z_{n-1} < z < z_n$,

$$\begin{bmatrix} U_n(z) \\ T_n(z) \end{bmatrix} = [G(z)] \begin{bmatrix} A_p \\ A_p \end{bmatrix} \qquad \text{...(5.12)}$$

where

$$[G(z)] = [a_n(z_n - z)]\, [a_{n+1}]\, [a_{n+2}] \ldots [a_{p-1}]\, [Z_p(H)] \qquad \text{...(5.13)}$$

and $[a_n(z_n - z)]$ is obtained from $[a_n]$ of (5.4) on replacing d_n by $z_n - z$.

Inserting the value of A_p given in (5.11) into (5.12) and making use of (3.8) and and (3.9), we find

$$u = \int_0^\infty \bar{f}(k) \left(\frac{G_{11} + G_{12}}{F_{21} + F_{22}}\right) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k^{-1} dk \qquad \text{...(5.14)}$$

$$p_{13} = \int_0^\infty \bar{f}(k) \left(\frac{G_{21} + G_{22}}{F_{21} + F_{22}}\right) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk. \qquad \text{...(5.15)}$$

5.2. *Plane Strain Problem*

For the nth layer, we obtain

$$[Y_{n-1}(z_{n-1})] = [a_n]\, [Y_n(z_n)] \qquad \text{...(5.16)}$$

where the elements of the layer matrix $[a_n]$ are listed in the Appendix. Proceeding as in the case of the antiplane strain problem and putting $B_p = D_p = 0$, we find

$$[V_1(0),\ W_1(0),\ S_1(0),\ N_1(0)]^T = [F][A_p,\ A_p,\ C_p,\ C_p]^T \quad \ldots(5.17)$$

with $[F]$ of (5.6). Equation (5.17) gives the following four equations

$$V_1(0) = (F_{11} + F_{12})\,A_p + (F_{13} + F_{14})\,C_p \quad \ldots(5.18a)$$

$$W_1(0) = (F_{21} + F_{22})\,A_p + (F_{23} + F_{24})\,C_p \quad \ldots(5.18b)$$

$$S_1(0) = (F_{31} + F_{32})\,A_p + (F_{33} + F_{34})\,C_p \quad \ldots(5.19a)$$

$$N_1(0) = (F_{41} + F_{42})\,A_p + (F_{43} + F_{44})\,C_p. \quad \ldots(5.19b)$$

For given displacements at the surface, A_p and C_p are known from (5.18a,b). For given surface loads, A_p and C_p are known from (5.19a,b).

When the surface load is prescribed the boundary conditions are of the form

$$p_{23} = g(y),\ p_{33} = h(y) \text{ at } z = 0. \quad \ldots(5.20)$$

As before, we put

$$g(y) = \int_0^\infty \bar{g}(k) \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} dk \quad \ldots(5.21)$$

$$h(y) = \int_0^\infty \bar{h}(k) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk. \quad \ldots(5.22)$$

Equations (4.31), (4.32), (5.19a,b), (5.21) and (5.22) give the values of A_p and C_p:

$$A_p = (1/\Omega\, k^2)\,[(F_{43} + F_{44})\,\bar{g} - (F_{33} + F_{34})\,\bar{h}] \quad \ldots(5.23a)$$

$$C_p = (1/\Omega\, k^2)\,[(F_{31} + F_{32})\,\bar{h} - (F_{41} + F_{42})\bar{g}). \quad \ldots(5.23b)$$

where

$$\Omega = (F_{31} + F_{32})(F_{43} + F_{44}) - (F_{33} + F_{34})(F_{41} + F_{42}). \quad \ldots(5.24)$$

The field at any point of the medium can be obtained from the relation

$(z_{n-1} < z < z_n)$

$$[V_n(z),\ W_n(z),\ S_n(z),\ N_n(z)]^T = [G(z)]\,[A_p,\ A_p,\ C_p,\ C_p]^T \quad \ldots(5.25)$$

where $[G]$ is defined in (5.13). From (4.29)-(4.32), (5.23a, b) and (5.25), we obtain

$$v = \int_0^\infty [(G_{11} + G_{12})\ \{(F_{43} + F_{44})\,\bar{g} - (F_{33} + F_{34})\,\bar{h}\} + (G_{13} + G_{14})\,\{(F_{31} + F_{32})\,\bar{h} - (F_{41} + F_{42})\bar{g}\}] \begin{pmatrix} -\cos ky \\ \sin ky \end{pmatrix} k^{-1} \times \Omega^{-1}\,dk \quad \ldots(5.26)$$

$$w = \int_0^\infty [(G_{21} + G_{22}) \{(F_{43} + F_{44}) \bar{g} - (F_{33} + F_{34})\bar{h}\}$$

$$+ (G_{23} + G_{24}) \{(F_{31} + F_{32}) \bar{h} - (F_{41} + F_{42}) \bar{g}\}] \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k^{-1}$$

$$\times \Omega^{-1} \, dk \qquad \ldots(5.27)$$

$$p_{23} = \int_0^\infty [(G_{31} + G_{32}) \{(F_{43} + F_{44}) \bar{g} - (F_{33} + F_{34}) \bar{h}\}$$

$$+ (G_{33} + G_{34}) \{(F_{31}+F_{32}) \bar{h} - (F_{41} + F_{42}) \bar{g}\}] \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix}$$

$$\times \Omega^{-1} \, dk \qquad \ldots(5.28)$$

$$p_{33} = \int_0^\infty [(G_{41} + G_{42}) \{(F_{43} + F_{44}) \bar{g} - (F_{33} + F_{34}) \bar{h}\}$$

$$+ (G_{43} + G_{44}) \{(F_{31} + F_{32}) \bar{h} - (F_{41} + F_{42}) \bar{g}\}] \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix}$$

$$\times \Omega^{-1} \, dk. \qquad \ldots(5.29)$$

6. Particular Cases-Surface Loads

The results obtained in Section 5 are of general nature. We now consider a few particular cases in which the surface load is precisely defined.

6.1. *Antiplane Strain Problem*

Let

$$f(y) = -R\,\delta(y) \qquad \ldots(6.1)$$

where $\delta(y)$ denotes the Dirac delta function. We use the representation

$$\delta(y) = (1/\pi) \int_0^\infty \cos ky \, dk. \qquad \ldots(6.2)$$

From (5.10), (6.1) and (6.2), we find that

$$\bar{f}(k) = -R/\pi \qquad \ldots(6.3)$$

and that the lower solution must be taken. Putting this value of $\bar{f}(k)$ in (5.14) and (5.15), we obtain

$$u = \frac{-R}{\pi} \int_0^\infty \left(\frac{G_{11} + G_{12}}{F_{21} + F_{22}} \right) \cos ky \, k^{-1} \, dk \qquad \ldots(6.4)$$

$$p_{13} = \frac{-R}{\pi} \int_0^\infty \left(\frac{G_{21} + G_{22}}{F_{21} + F_{22}} \right) \cos ky \, dk. \qquad \ldots(6.5)$$

6.2. *Plane Strain Problem*

For plane strain problem, we shall be considering the particular cases of a normal line load and a shear line load.

Normal Line Load—Let a normal line load p per unit length be applied at the origin to the surface $z = 0$ in the positive direction of the z- axis. Then

$$g(y) = 0, \; h(y) = -p\,\delta(y) \qquad \ldots(6.6)$$

where $\delta(y)$ is defined in (6.2). From (5.21), (5.22) and (6.6), we obtain

$$\bar{g}(k) = 0, \; \bar{h}(k) = -P/\pi \qquad \ldots(6.7)$$

and that the lower solution must be taken. The displacements and stresses at any point of the medium are given by (5.26) — (5.29) and (6.7) :

$$v = (-P/\pi) \int_0^\infty [(G_{13} + G_{14})(F_{31} + F_{32}) - (G_{11} + G_{12})(F_{33} + F_{34})] \times k^{-1} \Omega^{-1} \sin ky \, dk \qquad \ldots(6.8)$$

$$w = (-P/\pi) \int_0^\infty [(G_{23} + G_{24})(F_{31} + F_{32}) - (G_{21} + G_{22})(F_{33} + F_{34})] \times k^{-1} \Omega^{-1} \cos ky \, dk \qquad \ldots(6.9)$$

$$p_{23} = (P/\pi) \int_0^\infty [(G_{33} + G_{34})(F_{31} + F_{32}) - (G_{31} + G_{32})(F_{33} + F_{34})] \times \Omega^{-1} \sin ky \, dk \qquad \ldots(6.10)$$

$$p_{33} = (-P/\pi) \int_0^\infty [(G_{43} + G_{44})(F_{31} + F_{32}) - (G_{41} + G_{42})(F_{33} + F_{34})] \times \Omega^{-1} \cos ky \, dk. \qquad \ldots(6.11)$$

Shear Line Load—Suppose that a shear line load Q per unit length is applied at the origin to the surface $z = 0$ in the positive direction of the y-axis. Then

$$g(y) = -Q\,\delta(y), \; h(y) = 0. \qquad \ldots(6.12)$$

From (5.21), (5.22) and (6.12), we find

$$\bar{g}(k) = -Q/\pi, \; \bar{h}(k) = 0 \qquad \ldots(6.13)$$

and that the upper solution must be taken. The displacements and stresses are given by (5.26) — (5.29) and (6.13):

$$v = (Q/\pi) \int_0^\infty [(G_{11} + G_{12})(F_{43} + F_{44}) - (G_{13} + G_{14})(F_{41} + F_{42})] \times k^{-1} \Omega^{-1} \cos ky \, dk \quad \ldots(6.14)$$

$$w = (-Q/\pi) \int_0^\infty [(G_{21} + G_{22})(F_{43} + F_{44}) - (G_{23} + G_{24})(F_{41} + F_{42})] \times k^{-1} \Omega^{-1} \sin ky \, dk \quad \ldots(6.15)$$

$$p_{23} = (-Q/\pi) \int_0^\infty [(G_{31} + G_{32})(F_{43} + F_{44}) - (G_{33} + G_{34})(F_{41} + F_{42})] \times \Omega^{-1} \cos ky \, dk \quad \ldots(6.16)$$

$$p_{33} = (-Q/\pi) \int_0^\infty [(G_{41} + G_{42})(F_{43} + F_{44}) - (G_{43} + G_{44})(F_{41} + F_{42})] \times \Omega^{-1} \sin ky \, dk. \quad \ldots(6.17)$$

7. Uniform Half-Space

In Section 6, we have derived the displacements and stresses at any point of the medium caused by surface loads acting on the surface of a transversely isotropic multilayered half-space. These results are in the form of integrals over the variable k. These integrals can be evaluated numerically by using the method suggested by Jovanovich *et al.*[11,12]. In the case of a transversely isotropic uniform half-space the integrals giving the stresses can be integrated analytically. The basic transform integrals used are listed below :

$$\int_0^\infty e^{-\xi k} \sin ky \, dk = \frac{y}{y^2 + \xi^2} \quad \ldots(7.1)$$

$$\int_0^\infty e^{-\xi k} \cos ky \, dk = \frac{\xi}{y^2 + \xi^2} \quad \ldots(7.2)$$

For a half-space, $p = 1$ and

$$[F] = [Z(0)], \; [G] = [Z(z)] \quad \ldots(7.3)$$

7.1. *Antiplane Strain Problem*

In this case,

$$[F] = \begin{bmatrix} 1 & 0 \\ 0 & -t \end{bmatrix} \quad \ldots(7.4)$$

and

$$[G] = \begin{bmatrix} ch\,(skz) & -sh\,(skz) \\ t\,sh\,(skz) & -t\,ch\,(skz) \end{bmatrix}. \quad \ldots(7.5)$$

Using (6.5), (7.2), (7.4) and (7.5) we find

$$p_{13} = \frac{-R}{\pi}\left[\frac{sz}{y^2 + s^2z^2}\right]. \quad \ldots(7.6)$$

In the case of an isotropic half-space $s = 1$ and (7.6) reduces to

$$p_{13} = \frac{-R}{\pi}\left[\frac{z}{y^2 + z^2}\right]. \quad \ldots(7.7)$$

7.2. *Plane Strain Problem*

In this case, using (4.35) and (7.3), we obtain

$$[F] = \begin{bmatrix} p_1 & 0 & p_2 & 0 \\ 0 & q_1 & 0 & q_2 \\ 0 & \alpha & 0 & \beta \\ -1 & 0 & -1 & 0 \end{bmatrix}. \quad \ldots(7.8)$$

Normal Line Load—Making use of (4.35), (5.24), (6.10), (6.11), (7.1), (7.2) and (7.8), we find

$$p_{23}\,(z) = \frac{\alpha\beta p}{\pi(\beta - \alpha)}\left[\frac{y}{y^2 + \beta^2z^2} - \frac{y}{y^2 + \alpha^2z^2}\right] \quad \ldots(7.9)$$

$$p_{33}\,(z) = \frac{\alpha\beta p}{\pi(\beta - \alpha)}\left[\frac{z}{y^2 + \beta^2z^2} - \frac{z}{y^2 + \alpha^2z^2}\right]. \quad \ldots(7.10)$$

The stresses for an isotropic uniform half-space can be deduced from (7.9) and (7.10). First cancelling the factor $(\beta - \alpha)$ from the numerator and denominator and then putting $\alpha = \beta = 1$, we shall find

$$p_{23} = \frac{-2p}{\pi}\left[\frac{yz^2}{(y^2 + z^2)^2}\right],\ p_{33} = \frac{-2p}{\pi}\left[\frac{z^3}{(y^2 + z^2)^2}\right]. \quad \ldots(7.11)$$

The results obtained here coincide with the corresponding results of Snendon[13] (p. 409).

Shear Line Load Using (4.35), (5.24), (6.16), (6.17), (7.1) (7.2) and (7.8), we obtain

$$p_{23}\,(z) = \frac{Q}{\pi(\beta - \alpha)}\left[\frac{\alpha^2 z}{y^2 + \alpha^2z^2} - \frac{\beta^2 z}{y^2 + \beta^2z^2}\right] \quad \ldots(7.12)$$

$$p_{33}\,(z) = \frac{-Q}{\pi(\beta - \alpha)}\left[\frac{y}{y^2 + \alpha^2z^2} - \frac{y}{y^2 + \beta^2z^2}\right]. \quad \ldots(7.13)$$

For an isotropic uniform half-space, (7.12) and (7.13) take the form

$$p_{23} = \frac{-2Q}{\pi}\left[\frac{y^2z}{(y^2 + z^2)^2}\right],\ p_{33} = \frac{-2Q}{\pi}\left[\frac{yz^2}{(y^2 + z^2)^2}\right]. \quad \ldots(7.14)$$

ACKNOWLEDGEMENT

One of the authors (S. J. S.) is thankful to the University Grants Commission, New Delhi for financial support in the form of a National Fellowship.

REFERENCES

1. J. T. Kuo, *J. Geophys. Res.* **74** (1969), 3195-3207.
2. S. J. Singh, *J. Geophys. Res.* **75** (1970), 3257-63.
3. N. R. Garg, and S. J. Singh, *Bull. Ind. Soc. Earthq. Tech.* **22** (1985), 39-52.
4. W. T. Thomson, *J. Apply. Phys.* **21** (1950), 89-93.
5. N. A. Haskell, *Bull. Seism. Soc. Am* **43** (1953), 17-34.
6. S. J. Singh, *Phys. Earth. Planet. Inter.* **42** (1986), 263-73.
7. R. G. Payton, *Elastic Wave Propagation in Transversely Isotropic Media.* Martinus Nijhoff Publishers, Netherlands, 1983.
8. S. J. Singh, and N. R. Garg, *Phys. Earth. Planet. Inter.* **40** (1985), 135-45.
9. I. S. Sokolnikoff, *Mathematical Theory of Elasticity.* Tata-McGraw-Hill Publishing Co., New Delhi, 1978.
10. S. J. Singh, *Acta Geophys. Pol.* **33** (1985), 123-34.
11. D. M. Jovanovich, M. I.Husseini, and M. A. Chinnery, *Geophys. J. R. Astron. Soc.* **39** (1974) 205-17.
12. D. M. Jovanovich, M. I Husseini, and M. A. Chinnery, *Geophys. J. R. Astron. Soc.* **39** (1974), 219-39.
13. I. N. Sneddon, *Fourier Transforms.* McGraw-Hill Book Co. Inc. New York, 1951.

APPENDIX

(i) *Derivation of equation* (5.3)

Putting $z = z_{n-1}, z_n$ in (5.1), we obtain

$$\begin{bmatrix} U_n(z_n) \\ T_n(z_n) \end{bmatrix} = [Z_n(z_n)] \begin{bmatrix} A_n + B_n \\ A_n - B_n \end{bmatrix}, \quad \text{...(A.1)}$$

$$\begin{bmatrix} U_n(z_{n-1}) \\ T_n(z_{n-1}) \end{bmatrix} = [Z_n(z_{n-1})] \begin{bmatrix} A_n + B_n \\ A_n - B_n \end{bmatrix}. \quad \text{...(A.2)}$$

Eliminating $A_n + B_n$, $A_n - B_n$ from (A.1) and (A.2), we find

$$\begin{bmatrix} U_n(z_{n-1}) \\ T_n(z_{n-1}) \end{bmatrix} = [a_n] \begin{bmatrix} U_n(z_n) \\ T_n(z_n) \end{bmatrix}, \quad \text{...(A.3)}$$

where the layer matrix $[a_n]$ is given by (Singh[2])

$$[a_n] = [Z_n(z_{n-1})]\,[Z_n(z_n)]^{-1}$$
$$= [Z_n(-d_n)]\,[Z_n(0)]^{-1}. \quad \text{...(A.4)}$$

The matrix $[Z_n(-d_n)]$ is obtained from $[Z_n(z)]$, given in equation (5.2), on replacing z by $-d_n$.

The boundary conditions at the interface $z = z_{n-1}$ yield

$$\begin{bmatrix} U_n (z_{n-1}) \\ T_n (z_{n-1}) \end{bmatrix} = \begin{bmatrix} U_{n-1} (z_{n-1}) \\ T_{n-1} (z_{n-1}) \end{bmatrix}. \quad \ldots(A.5)$$

Equations (A-3) and (A-5) yield

$$\begin{bmatrix} U_{n-1} (z_{n-1}) \\ T_{n-1} (z_{n-1}) \end{bmatrix} = [a_n] \begin{bmatrix} U_n (z_n) \\ T_n (z_n) \end{bmatrix} \quad \ldots(A.6)$$

which coincides with (5.3). Equations (5.2) and (A.4) yield equation (5.4).

(ii) *Elements of the layer matrix* $[a_n]$ *for plane strain*

From (4.35), we obtain

$$[Z(0)]^{-1} = \begin{bmatrix} -1/\Omega_1 & 0 & 0 & -p_2/\Omega_1 \\ 0 & \beta/\Omega_2 & -q_2/\Omega_2 & 0 \\ 1/\Omega_2 & 0 & 0 & p_1/\Omega_1 \\ 0 & -\alpha/\Omega_2 & q_1/\Omega_2 & 0 \end{bmatrix}$$

where

$$\Omega_1 = p_2 - p_1, \; \Omega_2 = q_1 \beta - q_2 \alpha.$$

Following Singh[2], it is found that the elements of the layer matrix $[a_n]$ for the plane strain problem are (omitting the subscript n) :

$$(11) = (-p_1 \, ch \, \theta + p_2 \, ch \, \phi)/\Omega_1, \quad (12) = (p_1 \beta \, sh \, \theta - q_2 \alpha \, sh \, \phi)/\Omega_2$$

$$(13) = (-p_1 q_2 \, sh \, \theta + p_2 q_1 \, sh \, \phi)/\Omega_2, \quad (14) = p_1 p_2 (-ch \, \theta + ch \, \phi)/\Omega_1$$

$$(21) = (-q_1 \, sh \, \theta + q_2 \, sh \, \phi)/\Omega_1, \quad (22) = (q_1 \beta \, ch \, \theta - q_2 \alpha \, ch \, \phi)/\Omega_2$$

$$(23) = q_1 q_2 (-ch \, \theta + ch \, \phi)/\Omega_2, \quad (24) = (-q_1 p_2 \, sh \, \theta + q_2 p_1 \, sh \, \phi)/\Omega_1$$

$$(31) = (-\alpha \, sh \, \theta + \beta \, sh \, \phi)/\Omega_1, \quad (32) = \alpha\beta (ch \, \theta - ch \, \phi)/\Omega_2$$

$$(33) = (-\alpha q_2 \, ch \, \theta + \beta q_1 \, ch \, \phi)/\Omega_2, \quad (34) = (-\alpha p_2 \, sh \, \theta + \beta p_1 \, sh \, \phi)/\Omega_1$$

$$(41) = (ch \, \theta - ch \, \phi)/\Omega_1, \quad (42) = (-\beta \, sh \, \theta + \alpha \, sh \, \phi)/\Omega_2$$

$$(43) = (q_2 \, sh \, \theta - q_1 \, sh \, \phi)/\Omega_2, \quad (44) = (p_2 \, ch \, \theta - p_1 \, ch \, \phi)/\Omega_1$$

where

$$\theta = \alpha kd, \; \phi = \beta kd.$$