

Optimal Training, Employee Preferences and Moral Hazard

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Abstract

We study an agency model with moral hazard, when the employer offers complementary training/development programs that will increase the productivity of the employee's effort. Since it is costly for an employer to offer training and development opportunities and given that employees are not identical, how will an employer choose the quantity and allocation of such programs? Does the quantity and type of training offered, vary with the employee's aversion to effort? Does more "sincerity" necessarily translate into more employee development? Does more training in fact induce the employee to work harder? In theory the answer could go either way. On the one hand, an employer may wish to leverage the use of such programs to motivate a lazy employee to work harder. Conversely, especially because effort is unobservable, one can argue that she may be better off rewarding a sincere employee with more development opportunities. This work reaches a definite and perhaps unpredictable conclusion. We find that there is an inverse relationship between the optimal quantity of the training program and increased aversion to effort for both a relatively lazy and a relatively sincere employee. This is also true regardless of whether the program is relatively cheap or relatively expensive for the employer to offer. Perhaps surprisingly, there is no qualitative change in the comparative statics results, if the employer can monitor or observe effort.

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1 Introduction

Worker training programs to increase productivity are common in the workplace. Employers often provide a variety of training programs such as seminars, workshops and technical skill development opportunities. There is a rich literature on different types of training programs and their effects on worker productivity. Beginning with Becker (1962), various researchers have examined the effectiveness of investment in human capital to improve worker productivity. More recently, Bartel (1994) finds that there is a robust relationship between training and worker productivity. Barrett and O’Connell (2001) estimate the returns to in-company training and distinguish between general and specific training. They find higher productivity gains from general training and one of their interpretations of this finding is that it is a signal to the employee of his value, and this leads the employee to exert more effort and raise productivity.

This article studies a principal agent model with moral hazard, when the employer (principal) offers a piece rate contract to the employee (agent) and also offers complementary training programs that will increase the productivity of the employee’s effort. Since the employee derives disutility from effort, the training program provides the employer an instrument to decrease the uncertainty in the employee’s contract and hence in his income. This potentially motivates the employee to exert more effort, which, all else equal, translates into higher expected output.

The use of piece rate contracts is ubiquitous in many industries. Lazear (2000) studies the effect of piece rate pay and finds substantially improved performance (relative to fixed wage contracts) among workers who installed windshields. Nagin et al (2002) find that piece rate pay is very effective in call centers but requires careful measurement of output. Shearer (2004) uses data from a field experiment in a tree planting company to estimate gains in productivity that are realized when workers are paid piece rates rather than fixed wages. He estimates productivity gains in the order of approximately 20%. Other examples of industries where piece rate contracts are fairly common are agricultural sharecropping and real estate. Dunlop and Weil (1996) in a study of the apparel industry, find that manufacturers frequently pay by the piece to complete specific parts of garments.

Within this theoretical framework, this work asks the following question: Given that it is costly for an employer to offer training and development opportunities to an employee and given that employees are not identical, how will an employer choose the quantity and allocation of such programs? The difference between the employees is their attitude or aversion to effort. The employer does not observe employee effort but knows the employee’s preferences and most particularly his aversion to effort. Therefore, what is the relationship between the employee “type” (given by the employee’s aversion to effort) and the training and development opportunity offered by the employer? This is a key focus of our work. Does the level and type of training offered, vary with the employee’s aversion to effort? Does more “sincerity” or less aversion to effort, necessarily translate into more (or perhaps more expensive) training opportunities? Does more training in fact induce the employee to work harder?

The answers to these questions are not necessarily obvious. As employers have, over time,

included a variety of human resource management practices such as employee training and development, they are forced to make choices about how to allocate these additional resources for the highest yield. When an employer cannot observe effort, she may be able to leverage the training and development opportunity to motivate a relatively lazy employee to work harder. This argument might be true since more training increases the productivity of the employee's effort and raises expected output. So it is possible that when effort is unobservable to the employer, the lazy employee receives more of the development opportunities. Conversely however, one can argue that especially when the employer cannot observe effort, she must motivate a sincere employee with further development opportunities. Since the moral hazard problem is less severe for a sincere employee, an investment in raising his productivity is more likely to generate higher output. In theory therefore, the answer could go either way. This work reaches a definite and perhaps unpredictable conclusion.

Our model, which we shall discuss in the next section, captures the essentials of this problem. We wish to emphasize that in spite of the apparent simplicity of the model, the comparative statics are by no means trivial to derive. These arguments will be developed in the paper with details given in a technical Appendix.¹

As a benchmark, we first study the complete information case when the employee's effort is observable by the employer. The main results in the complete information case are as follows.

C1. The employer sets the wage directly as a function of the observable effort, and as expected there is a positive relationship between wage and effort.

C2. The greater the employee's aversion to effort, the fewer the training programs provided by the employer. This is true for both a cheap and an expensive program. It is also true for both a relatively lazy and a relatively sincere employee.

The results for the complete information case are used as a benchmark for the results with incomplete information, which are summarized below.

IC1. The employee's optimal effort is positively related to more training by the employer. That is, more training induces the employee to exert more effort. This translates into higher expected output and hence higher expected wage compensation.

IC2. There is an inverse relationship between the optimal quantity of the training program and increased aversion to effort for both a relatively lazy and a relatively sincere employee. That is, an increased aversion to effort will induce the employer to offer less training, regardless of whether the employee is intrinsically more or less averse to effort. This is also true regardless of whether the program is relatively cheap or relatively expensive for the employer to offer.

The article is organized as follows. The model is introduced in Section 2 and the complete information version is analyzed in Section 3. Section 4 presents the analysis of the optimal contract with incomplete information. Section 5 concludes the paper. All proofs are presented in the Appendix in Section 6.

¹One may think about alternative specifications of the wage contract. Bose et al 2009 (forthcoming) have compared the optimality of linear contracts vis a vis the second best contract. That work also yields significant insight into the technical complexities involved in the solution of these models.

2 Model

This model is primarily based on Holmstrom (1979).² Consider a firm owned by a risk neutral principal (employer) and operated by a risk averse agent (employee). The employer observes only the output $x \in [0, \infty)$, produced by the employee but cannot observe the employee's effort, $a \in A$. The employer offers the employee a piece rate contract which consists of a share θ , of the output, and a training/development program p , that increases the employee's marginal productivity of effort. The employer also knows the employee's aversion with respect to effort, δ .

Given effort level a , let $x \in [0, \infty)$ follow the density

$$f(x|a) = f(x; p, a) = \frac{1}{a^p \Gamma(p)} x^{p-1} e^{-x/a}, \text{ for } x \in [0, \infty) \quad (2.1)$$

It is easily seen that the expected value of output $E(x)$, equals ap . That is, a higher p will increase the expected output. The Gamma function, $\Gamma(\cdot)$, is defined as

$$\Gamma(p) = \int_0^{\infty} e^{-x} x^{p-1} dx \text{ for } p > 0. \quad (2.2)$$

The use of the gamma density to capture the stochastic relationship between the employee's unobserved effort and his observed output offers some advantages. The density is quite flexible, and so permits an analysis of a substantial variety of relationships between effort and output. With suitable values of p and a , $f(x|a)$ offers close approximations to many unimodal densities. The use of the gamma density also facilitates the identification of conditions under which the tractable first-order approach can be employed to solve the employer's problem e.g., Rogerson (1985), Jewitt (1988). Holmstrom (1979) uses an exponential density which is a special case of the gamma density for $p = 1$. Generalization with respect to p allows us to study the relationship between p (training development program) and δ (aversion to effort).

The employee's expected utility function is given by

$$U^A(a) = \int_0^{\infty} 2\sqrt{\theta x} f(x|a) dx - a^\delta, \delta > 0.$$

We use the term, employee "type" to refer to the employee's aversion to effort defined by the parameter δ . Hence a higher (lower) δ is indicative of a higher aversion to effort if $a > 1$ ($a < 1$) and so defines a lazy employee.

The employer incurs a cost to provide the training program and this is given by $\frac{p^\alpha}{k'}$ where $\alpha > 1$ and $k' > 0$. Here p denotes the level of the program and k' denotes the cost such that a higher (lower) k' will decrease (increase) the cost. While this function is not completely general, it allows for a wide class of strictly convex cost functions. We normalize the cost

²Marino and Zbojnik (2008) use the linear contract principal agent framework in Holmstrom and Milgrom (1987) to study the role of perks that have productive consumption attributes. The focus of their work is perks that can be used for purely personal use.

by defining $k' = \alpha k$ and rewrite this cost as $\frac{p^\alpha}{\alpha k}$ for analytical convenience. Since α is fixed, any change in k is equivalent to a change in k' .

The employer's profit is hence given by

$$L(a, p) = \int_0^\infty (1 - \theta) x f(x|a) dx - \frac{p^\alpha}{\alpha k}.$$

where $(1 - \theta)$ is her share of output x .

3 Complete Information

The model described in the previous section is necessarily for the incomplete information case. When the employer can observe effort, this is modified as follows. She pays the employee a wage w . Hence the employee's utility function is now

$$2\sqrt{w} - a^\delta \quad \text{where } \delta > 0.$$

The employer sets the wage for the employee such that,

$$\int_0^\infty 2\sqrt{w} f(x|a) dx - a^\delta = 0$$

which implies

$$w^* = \frac{a^{2\delta}}{4}.$$

We note that, as expected, employee wage is positively related to observed effort.

For a given output x , the employer's share of output is $x - w^*$ and hence her payoff is given by

$$\begin{aligned} \pi &= \int_0^\infty [x - w^*] f(x|a) dx - \frac{p^\alpha}{\alpha k} \\ &= \int_0^\infty x f(x|a) dx - w^* - \frac{p^\alpha}{\alpha k} \\ &= ap - w^* - \frac{p^\alpha}{\alpha k} \\ &= ap - \frac{a^{2\delta}}{4} - \frac{p^\alpha}{\alpha k}. \end{aligned}$$

The employer will choose the employee's effort level a and program level p to maximize this payoff π .

Lemma 1 below, states the profit maximizing values of p and a . See Subsection 6.1 of Appendix for a proof. The global maximum of π with respect to p and a will be denoted by π^* . The values of p and a at which this global maximum is attained will be denoted by p^* and a^* .

Lemma 1 Suppose $\delta > 1/2$.

(i) If $\alpha > 1 + \frac{1}{2\delta - 1} = \frac{2\delta}{2\delta - 1}$ then

$$p^* = \left(\frac{2}{\delta}\right)^{\frac{1}{2\delta(\alpha-1)-\alpha}} k^{\frac{2\delta-1}{2\delta(\alpha-1)-\alpha}} \quad \text{and} \quad a^* = \left[k^{\frac{1}{\alpha-1}} \left(\frac{2}{\delta}\right)\right]^{\frac{(\alpha-1)}{2\delta(\alpha-1)-\alpha}}. \quad (3.1)$$

and

$$\pi^* = \left(\frac{2}{\delta}\right)^{\frac{1}{2\delta-1}} (p^*)^{\frac{2\delta}{2\delta-1}} \left[1 - \frac{1}{2\delta} - \frac{1}{\alpha}\right].$$

Further, $a^* \geq 1$, if and only if $\delta \leq 2k^{\frac{1}{\alpha-1}}$.

(ii) If $\alpha < 1 + \frac{1}{2\delta-1}$ then there is no interior optimum of π and the maximum value of π is ∞ and is obtained if we let $a^{\delta-1} = \frac{2p}{\delta} \rightarrow \infty$.

(iii) Suppose $\alpha = 1 + \frac{1}{2\delta-1}$ then

(a) if $\left(\frac{2}{\delta}\right)^{\frac{1}{2\delta-1}} > \frac{1}{k}$ the maximum value of π is ∞ and is obtained if we let $a^{2\delta-1} = \frac{2p}{\delta} \rightarrow \infty$.

(b) if $\left(\frac{2}{\delta}\right)^{\frac{1}{2\delta-1}} \leq \frac{1}{k}$ the maximum value of π is 0 and is obtained at $a^* = p^* = 0$.

In view of Lemma 1, we will henceforth always impose the restriction $\alpha > 1 + \frac{1}{2\delta-1}$. This ensures that the employer's cost of providing the training program is sufficiently convex. In examining the optimal contract, the first question of interest is the relationship between the employee effort a , and his aversion to effort. Lemma 2 below derives the the behavior of $\frac{\partial a^*}{\partial \delta}$. Its proof is given in Section 6.2. The relationship between a^* and δ is summarized in Proposition 1.

Lemma 2 Assume that $\delta > 1/2$ and $\alpha > 1 + \frac{1}{2\delta-1}$.

(i) If $k < \left(\frac{\delta}{2e}\right)^{\alpha-1}$ then $\frac{\partial a^*}{\partial \delta} > 0$.

(ii) If $k > \max\left\{\frac{2e}{\delta}, \left(\frac{\delta}{2}\right)^{\alpha-1}, e^{\frac{\alpha-1}{\alpha}}\right\}$ then $\frac{\partial a^*}{\partial \delta} < 0$.

Note that from Lemma 1 (i), if $2ek^{\frac{1}{\alpha-1}} < \delta$, then $a^* \leq 1$. That is, a higher δ indicates less aversion to effort. Likewise, if $2k^{\frac{1}{\alpha-1}} > \delta$, then $a^* \geq 1$. In other words, in this case, a higher δ indicates more aversion to effort. As an illustration of the Lemma, suppose $\alpha = 2$. Then we need $\delta > 1$ and $\frac{\partial a^*}{\partial \delta} > 0$ if $2ek < \delta$ and $\frac{\partial a^*}{\partial \delta} < 0$ if $2e/k < \delta < 2k$. Figure 1 explains the result of this Lemma when $\alpha = 4$. In general, the conditions in the above Lemma on the relative values of k and δ , that determine the signs of the derivatives, are sufficient but are not exhaustive. One may garner more precise (but cumbersome) conditions by following the proof of the Lemma. The above Lemma 1 and 2 lead to the following Proposition.

Proposition 1 (i) *With complete information, the employer sets wage as a function of the observable effort and there is a positive relationship between w^* and a .*

(ii) *If δ is sufficiently small (large), relative to k , the relationship between effort, a^* , and increasing aversion to effort, δ , is negative.*

When the employer observes effort, Proposition 1 finds that there is an inverse relationship between effort and increasing aversion to effort regardless of whether the training program is sufficiently cheap or expensive. As we will see from Proposition 2, this choice of effort by the employee is consistent with the employer's response in terms of programs offered, under similar relative parameter conditions.

Lemma 3 establishes the relationship between p^* and δ and it is qualitatively summarized in Proposition 2. The proof of Lemma 3 is provided in Section 6.3.

Lemma 3 *Suppose $\alpha > 1 + \frac{1}{2\delta-1}$.*

(i) *If $k > \left(\frac{\delta}{2}\right)^{\alpha-1}$, then $\frac{\partial p^*}{\partial \delta} < 0$.*

(ii) *If $k < e^{\frac{\alpha}{2\delta}} \left(\frac{\delta}{2e}\right)^{\alpha-1}$, then $\frac{\partial p^*}{\partial \delta} > 0$.*

A sufficient condition for (ii) is $k < \left(\frac{\delta}{2e}\right)^{\alpha-1}$. Again, suppose $\alpha = 2$. Then we need $\delta > 1$. It then follows that $\frac{\partial p^*}{\partial \delta} < 0$ if $\delta < 2k$ and $\frac{\partial p^*}{\partial \delta} > 0$ if $\delta > 2ek$. Figures 2 and 3 explain the result of this Lemma when $\alpha = 1.5$ and $\alpha = 4$ respectively.

Proposition 2 *If k is sufficiently large (small) compared to δ , then there is an inverse (direct) relationship between p^* and δ .*

If the employer's cost of providing the program is sufficiently small (k large relative to δ) then $a^* > 1$. Hence as δ increases, the employee's aversion to effort increases and the employer provides fewer development programs. In the same way, if the program is sufficiently expensive (k small relative to δ) then $a^* < 1$. Hence as δ decreases, the employee's aversion to effort increases and the employer provides fewer programs. Combining Propositions 1 and 2 we conclude that when effort is observable, the employer always rewards decreasing aversion to effort with more training and development programs.

4 Incomplete Information

In this case, the employer can only observe output and not the employee's effort. She chooses θ (the employee's share of output) and p (program) to maximize her payoff. Under the piece rate contract, the employee chooses effort a to maximize, with respect to a , his expected

utility, U^A , given by

$$\begin{aligned} U^A(a) &= 2\sqrt{\theta} \int_0^\infty x^{\frac{1}{2}} \frac{1}{a^p \Gamma(p)} x^{p-1} e^{-x/a} dx - a^\delta \\ &= 2\sqrt{\theta} \left(a^{\frac{1}{2}}\right) \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p)} - a^\delta. \end{aligned} \quad (4.1)$$

Then the employer will choose θ and p to maximize her expected payoff.

Parts of the following result on optimal effort \hat{a} (given p) and the employee's optimal share of output θ^* is also available in Bose, Pal and Sappington (2009).

Lemma 4 (i) U^A is maximized at

$$\hat{a} = \left[\frac{\sqrt{\theta}}{\delta} \left(\frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p)} \right) \right]^{\frac{2}{2\delta-1}}. \quad (4.2)$$

(ii) The agent's participation constraint is satisfied at \hat{a} . That is, $U^A(\hat{a}) > 0$.

(iii) The employee's optimal effort \hat{a} is positively related to p .

(iv) The employer's expected payoff is maximized at $\theta^* = \frac{1}{2\delta}$.

See Subsection 6.5 of Appendix for a proof of Lemma 4.

Henceforth, whenever θ appears in an expression, it is understood that it stands for the above optimum value $\frac{1}{2\delta}$.

We now focus on optimizing L (employer's profit when employee gives effort \hat{a}) with respect to p . Recall

$$L = \int_0^\infty [(1-\theta)x]f(x|\hat{a}) dx - \frac{p^\alpha}{\alpha k} \quad (4.3)$$

$$= (1-\theta)\hat{a}p - \frac{p^\alpha}{\alpha k}. \quad (4.4)$$

Define

$$H(p) = \log \Gamma\left(p + \frac{1}{2}\right) - \log \Gamma(p).$$

With the above notation, we get

$$\frac{\partial L}{\partial p} = (1-\theta)\hat{a} + (1-\theta)p\hat{a} \frac{2}{2\delta-1} H'(p) - \frac{p^{\alpha-1}}{k}.$$

Hence, the first order condition for maximizing L with respect to p is given by

$$\frac{p^{\alpha-1}}{k} = (1-\theta)\hat{a} \left[1 + \frac{2p}{2\delta-1} H'(p) \right]. \quad (4.5)$$

Using $\theta = \frac{1}{2\delta}$, this may also be written in the alternative forms

$$\frac{p^{\alpha-1}}{k} = (1 - \theta)\widehat{a}\left[1 + \frac{2p}{2\delta - 1}H'(p)\right] \quad (4.6)$$

$$= \frac{1}{2\delta}[2\delta - 1 + 2pH'(p)]\widehat{a} \quad (4.7)$$

$$= \frac{1}{2^{\frac{2}{2\delta-1}}} [2\delta - 1 + 2pH'(p)] \left[\frac{\Gamma(p + \frac{1}{2})}{\Gamma p}\right]^{\frac{2}{2\delta-1}} \delta^{-\frac{2(\delta+1)}{2\delta-1}}. \quad (4.8)$$

Lemma 5 proves the existence of a unique interior optimum value of p^* which maximizes L . See Subsection 6.6 for proof of Lemma 5.

Lemma 5 *If $\alpha > 1 + \frac{8\delta}{4\delta^2-1}$, then L attains a unique maximum at a finite $p = p^*$ and $L(p^*) > 0$.*

Remark 1 *As we shall see in the proof of Lemma 5, when $\alpha < 1 + \frac{1}{2\delta(2\delta-1)}$, L attains a global minimum at a finite p and its global maximum is attained only at $p = \infty$. Thus this case is uninteresting and out of our consideration.*

To investigate what happens to the optimum solution p^* as δ varies, using (4.8) (and **writing p for p^* to ease notation**),

$$\begin{aligned} (\alpha - 1) \log p - \log k &= -\frac{2\delta}{2\delta - 1} \log 2 - \frac{2(\delta + 1)}{2\delta - 1} \log \delta + \frac{2}{2\delta - 1} H(p) \\ &\quad + \log(2\delta - 1 + 2pH'(p)) \\ &= \left[-\frac{2\delta - 1}{2\delta - 1} \log 2 - \frac{\log 2}{2\delta - 1}\right] + \left[-\frac{2\delta - 1}{2\delta - 1} \log 2 - \frac{3}{2\delta - 1} \log \delta\right] \\ &\quad + \frac{2}{2\delta - 1} H(p) + \log(2\delta - 1 + 2pH'(p)) \\ &= -\log 2 - \frac{\log 2}{2\delta - 1} - \log \delta - \frac{3}{2\delta - 1} \log \delta \\ &\quad + \frac{2}{(2\delta - 1)} H(p) + \log(2\delta - 1 + 2pH'(p)). \end{aligned}$$

Differentiating the above first order equation $\frac{\partial L}{\partial p} = 0$ with respect to δ ,

$$\begin{aligned} \frac{(\alpha - 1) \partial p}{p \partial \delta} &= \frac{2 \log 2}{(2\delta - 1)^2} - \frac{1}{\delta} - \frac{3}{(2\delta - 1)\delta} + \frac{6}{(2\delta - 1)^2} \log \delta \\ &\quad - \frac{4}{(2\delta - 1)^2} H(p) + \frac{2}{(2\delta - 1)} H'(p) \frac{\partial p}{\partial \delta} \\ &\quad + \frac{1}{2\delta - 1 + 2pH'(p)} [2 + 2(H'(p) + pH''(p))] \frac{\partial p}{\partial \delta} \end{aligned}$$

which may be written as

$$C_1 \frac{\partial p}{\partial \delta} = C_2, \quad \text{say,}$$

where

$$C_1 = \frac{\alpha(2\delta - 1) - (2\delta - 1 + 2pH'(p))}{p(2\delta - 1)} - \frac{2(H'(p) + pH''(p))}{2\delta - 1 + 2pH'(p)}, \quad (4.9)$$

$$C_2 = \left[\frac{2 \log 2}{(2\delta - 1)^2} - \frac{1}{\delta} - \frac{3}{\delta(2\delta - 1)} + \frac{6}{(2\delta - 1)^2} \log \delta \right] \quad (4.10)$$

$$+ \left[-\frac{4}{(2\delta - 1)^2} H(p) + \frac{2}{2\delta - 1 + 2pH'(p)} \right]. \quad (4.11)$$

Lemma 6 and 7 provide conditions that determine the signs of C_1 and C_2 . That in turn determines the direction of the relationship between the employee type and the program offered in the optimal contract. The proofs are given in Subsection 6.7 and 6.8 in the Appendix. Since we are dealing with highly nonlinear equations and with implicit solutions, it is inevitable that the parameter conditions for the sign of the derivative turns out to be messy. The proof of 7 is technically the most difficult among all the arguments of this paper.

Lemma 6 *If $\alpha > 1 + \frac{10\delta-1}{2\delta(2\delta-1)}$, then $C_1 > 0$.*

Lemma 7 *Suppose $\delta > 1/2$ and $\alpha > 1 + \frac{8\delta}{4\delta^2-1}$.*

(i) *If $k \geq \frac{(2\delta)^{\frac{3}{2\delta-1}}}{\pi^{\frac{1}{2\delta-1}}}$ then $p^* \geq 1$, $a^* \geq 1$ and $C_2 < 0$.*

(ii) *If $k \leq \frac{2\delta}{2\delta+1} \min \left\{ \frac{(2\delta)^{\frac{3}{2\delta-1}}}{(\pi e^4)^{\frac{1}{2\delta-1}}}, \frac{(2\delta)^{3(\alpha-1)}}{(\pi e^4)^{(\alpha-1)}} \right\}$ then $p^* \leq 1$, $a^* \leq 1$ and $C_2 > 0$.*

It may be noted that $\frac{3}{2\delta-1} > 3(\alpha-1)$. Hence if $\delta \geq 1$, then $(2\delta)^{\frac{3}{2\delta-1}} > (2\delta)^{3(\alpha-1)}$. On the other hand $(\pi e^4)^{\frac{1}{2\delta-1}} < (\pi e^4)^{(\alpha-1)}$. Hence if $\delta > 1$, then a sufficient condition for (ii) to hold is $k < \frac{2\delta}{2\delta+1} \frac{(2\delta)^{\frac{3}{2\delta-1}}}{(\pi e^4)^{(\alpha-1)}}$. Figures 4 and 5 explain the result for this Lemma when $\alpha = 1.2$ and $\alpha = 2.1$ respectively. Clearly, there is subset of the parameter space where the above Lemma does not identify the sign of C_2 . This is due to the fact that p^* is determined only as the solution involving transcendental functions and the derivative of p^* involves the digamma function and its derivative.

Using the above results, we now have the relationships between employee type δ , the optimal program p and the employer's cost of providing the program. These are summarized in Proposition 3.

Proposition 3 *There is an inverse relationship between the optimal quantity of program and increasing aversion to effort for both a relatively lazy and a relatively sincere employee. This holds regardless of whether the program is relatively cheap or expensive.*

We see that as the employee exhibits an increasing aversion to effort, (that is, as δ increases for $a^* \geq 1$ and decreases for $a^* \leq 1$), the employer uses less of the instrument that she controls directly (the quantity of the development program, p). This is because while she cannot

observe actual effort, she knows and uses her information about an employee's aversion toward effort. Consequently, she punishes increasing aversion to effort with fewer programs. Lemma 7 also proves that when the program is relatively cheap (k is relatively large) then the employer will provide more of it ($p^* \geq 1$) in an optimal contract, while choosing to provide less when it is relatively expensive (if k is relatively small then $p^* \leq 1$). While this latter result by itself may not be especially surprising, what is interesting is that this is true regardless of the employee's aversion to effort.

5 Conclusion

Economists have analyzed a variety of optimal incentive contracts that are aimed at eliciting optimal effort from employees under different work conditions. At the same time, managers have introduced a variety of human resource management practices to motivate and train employees. In a moral hazard framework, this work studies how an employer will choose the quantity and allocation of training and development programs when employees have different aversion to effort. While the empirical literature on such programs is quite unequivocal about its benefits, the theoretical relationship with employee attitudes to effort has not been studied in any systematic way. One of the challenges was to derive consistent conditions between and across the fairly large number of parameters and the endogenous variables in the model. To the extent feasible, we provide a complete and consistent framework to demonstrate this relationship. We discuss a complete and an incomplete information framework and show that the unobservability of effort does not change the comparative statics analysis in any qualitative way. One may think of extending the model in different directions. The rather more obvious ones include different formulations of the wage contract, which we believe will become extremely complex. What may be a potentially interesting testable hypothesis is whether and to what extent intrinsic employee aversion to effort might be influenced by training programs.

6 Appendix

6.1 Arguments for Lemma 1

For fixed p , the maximum value of π with respect to a will be denoted by $\pi^*(p)$ and the value of a at which this is attained will be denoted by \hat{a} .

Proof of Lemma 1. Fixing p , and taking derivatives with respect to a ,

$$\frac{\partial \pi}{\partial a} = p - \frac{2\delta}{4} a^{2\delta-1}, \quad (6.1)$$

$$\frac{\partial^2 \pi}{\partial a^2} = -\frac{2\delta(2\delta-1)}{4} a^{2\delta-2} < 0.$$

Hence using (6.1) for fixed p , the global maximum of π with respect to a (when a is unrestricted) is obtained at

$$a^{2\delta-1} = \frac{2p}{\delta} \quad (6.2)$$

and the corresponding value of π equals

$$\pi^*(p) = p \left(\frac{2p}{\delta} \right)^{\frac{1}{2\delta-1}} - \frac{1}{4} \left(\frac{2p}{\delta} \right)^{\frac{2\delta}{2\delta-1}} - \frac{p^\alpha}{\alpha k} \quad (6.3)$$

$$= p^{\frac{2\delta}{2\delta-1}} \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} \left[1 - \frac{1}{2\delta} \right] - \frac{p^\alpha}{\alpha k}. \quad (6.4)$$

Now we maximize with respect to p . Taking first derivative,

$$\frac{\partial \pi^*(p)}{\partial p} = \frac{2\delta}{2\delta-1} p^{\frac{1}{2\delta-1}} \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} \left[1 - \frac{1}{2\delta} \right] - \frac{p^{\alpha-1}}{k} \quad (6.5)$$

$$= p^{\frac{1}{2\delta-1}} \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} - \frac{p^{\alpha-1}}{k}. \quad (6.6)$$

Solving for $\frac{\partial \pi^*(p)}{\partial p} = 0$ yields p^* as given in (3.1). Note that as $p \rightarrow 0$, we have $\pi^*(p) \rightarrow 0$. Further, since $\alpha > \frac{2\delta}{2\delta-1}$, as $p \rightarrow \infty$ we have $\pi^*(p) \rightarrow -\infty$. Thus p^* indeed gives the global maximum of $\pi^*(p)$. Now a^* can be obtained by going back to (6.2). The maximum value π^* , using (6.6) is obtained as

$$\begin{aligned} \pi^* &= (p^*)^{\frac{2\delta}{2\delta-1}} \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} \left[1 - \frac{1}{2\delta} \right] - \frac{(p^*)^\alpha}{\alpha k} \\ &= (p^*)^{\frac{2\delta}{2\delta-1}} \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} \left[1 - \frac{1}{2\delta} \right] - \frac{p^* (p^*)^{\frac{1}{2\delta-1}} \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}}}{\alpha k} \\ &= (p^*)^{\frac{2\delta}{2\delta-1}} \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} \left[1 - \frac{1}{2\delta} - \frac{1}{\alpha} \right]. \end{aligned}$$

This proves (i).

(ii) To prove (ii), recall (6.4). If $\alpha < \frac{2\delta}{2\delta-1}$, then it is easy to see that as $p \rightarrow \infty$, we have $\pi^*(p) \rightarrow \infty$ and (ii) follows immediately.

(iii) To prove (iii), observe that when $\alpha = \frac{2\delta}{2\delta-1}$, using (6.4),

$$\begin{aligned} \pi^*(p) &= p^{\frac{2\delta}{2\delta-1}} \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} \left[1 - \frac{1}{2\delta} \right] - \frac{p^\alpha}{\alpha k} \\ &= p^\alpha \left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} \frac{1}{\alpha} - \frac{p^\alpha}{\alpha k} \\ &= \frac{p^\alpha}{\alpha} \left[\left(\frac{2}{\delta} \right)^{\frac{1}{2\delta-1}} - \frac{1}{k} \right] \end{aligned}$$

and then the two claims in (a) and (b) follows easily. ■

6.2 Arguments for Lemma 2

Let

$$f_1(\delta) = 1 + 2\delta [\log \delta - 1 - \log 2]. \quad (6.7)$$

$$f_2(\delta) = 2\delta [\log \delta - \log 2 + \log k - 1]. \quad (6.8)$$

The behavior of these two functions, given below in Lemma 8, will be needed in the proof of Lemma 2.

Lemma 8 (i) For $\delta > 2$, there is a unique δ_0 , $4.9 < \delta_0 < 4.95$ such that $f_1(\delta_0) = 0$, $f_1(\delta) < 0$ for $\delta < \delta_0$ and $f_1(\delta) > 0$ for $\delta_0 < \delta$.

(ii) $f_2(\delta) > 0$ if and only if $\delta > \frac{2e}{k}$.

Proof of Lemma 8. (i) It is easy to check that

$$\begin{aligned} f_1'(\delta) &= 2 [\log \delta - 1 - \log 2] + \frac{2\delta}{\delta} \\ &= 2(\log \delta - \log 2) > 0. \end{aligned}$$

Thus $f_1(\cdot)$ is increasing. It is easily checked that

$$\begin{aligned} f_1(4.9) &= 1 + 9.75(1.58923 - 1 - 0.69314) = -0.013 < 0 \\ f_1(4.95) &= 1 + 9.90(1.59939 - 1 - 0.69314) = 0.07 > 0. \end{aligned}$$

This proves (i).

(ii) This part is trivial once it is observed that $\log \delta - \log 2 + \log k - 1 = \log\left(\frac{k\delta}{2e}\right)$. ■

Proof of Lemma 2. Let $\alpha > 1 + \frac{1}{2\delta-1}$. Using the value of a^* given in (3.1), and taking logarithm, we get,

$$\log a^* = \frac{1}{2\delta(\alpha-1) - \alpha} \log k + \frac{\alpha-1}{2\delta(\alpha-1) - \alpha} [\log 2 - \log \delta].$$

To see if this is increasing or decreasing in δ , taking derivative,

$$\frac{\partial \log a^*}{\partial \delta} = \frac{-2(\alpha-1) \log k}{[2\delta(\alpha-1) - \alpha]^2} + \frac{-2(\alpha-1)^2 [\log 2 - \log \delta]}{[2\delta(\alpha-1) - \alpha]^2} + \frac{\alpha-1}{2\delta(\alpha-1) - \alpha} \left(-\frac{1}{\delta}\right) \quad (6.9)$$

$$= \frac{\alpha-1}{\delta [2\delta(\alpha-1) - \alpha]^2} [-2\delta \log k - 2\delta(\alpha-1)(\log 2 - \log \delta) - 2\delta(\alpha-1) + \alpha] \quad (6.10)$$

$$= \frac{\alpha-1}{\delta [2\delta(\alpha-1) - \alpha]^2} [2\delta\{-\log k - (\alpha-1)(\log 2 - \log \delta) - (\alpha-1)\} + \alpha] \quad (6.11)$$

$$= \frac{\alpha-1}{\delta [2\delta(\alpha-1) - \alpha]^2} [2\delta\{\log \delta^{\alpha-1} - \log(ke^{\alpha-1}2^{\alpha-1})\} + \alpha]. \quad (6.12)$$

Since $\alpha > 1$, the first factor above is positive.

(i) It is easy to see that if $\delta > 2ek^{\frac{1}{\alpha-1}}$, then the term within $\{ \}$ in the numerator is also positive. Hence in this case, a^* is an increasing function of δ .

(ii) The term within $[]$ in (6.12) may be written as $\alpha f_1(\delta) - f_2(\delta)$.

First assume that $\frac{2e}{k} < \delta \leq \delta_0$. Note that from Lemma 8 (ii), $f_2(\delta) > 0$ since $\frac{2e}{k} < \delta$. Since $\delta < \delta_0$, again from Lemma 8 (i), $f_1(\delta) \leq 0$. Hence $\alpha f_1(\delta) - f_2(\delta) < 0$ and (ii) follows in this case.

Now suppose $\delta > \delta_0$. Note that

$$\delta \leq 2k^{\frac{1}{\alpha-1}} \Rightarrow \log \delta \leq \log 2 + \frac{1}{\alpha-1} \log k \quad (6.13)$$

$$\Rightarrow \alpha - 1 \leq \frac{\log k}{\log \delta - \log 2}. \quad (6.14)$$

Further, since $\alpha > 1 + \frac{1}{2\delta-1}$, we have $\frac{1}{\alpha-1} < 2\delta - 1$. Hence

$$\delta \leq 2k^{\frac{1}{\alpha-1}} \Rightarrow \delta \leq 2k^{2\delta-1} \Rightarrow (1 - 2\delta) \log k + \log \delta - \log 2 < 0.$$

Using these,

$$\alpha f_1(\delta) - f_2(\delta) = \alpha f_1(\delta) - [-1 + 2\delta \log k + f_1(\delta)] \quad (6.15)$$

$$= (\alpha - 1)[1 + 2\delta(\log \delta - 1 - \log 2)] + 1 - 2\delta \log k \quad (6.16)$$

$$\leq \frac{\log k}{\log \delta - \log 2} [1 + 2\delta(\log \delta - 1 - \log 2)] + 1 - 2\delta \log k \quad (6.17)$$

$$= \frac{\log k [1 + 2\delta(\log \delta - 1 - \log 2)] + (\log \delta - \log 2)(1 - 2\delta \log k)}{\log \delta - \log 2} \quad (6.18)$$

$$= \frac{(1 - 2\delta) \log k + \log \delta - \log 2}{\log \delta - \log 2} < 0. \quad (6.19)$$

This proves (ii). ■

6.3 Arguments for Lemma 3

Proof of Lemma 3. Recall from (3.1) that

$$p^* = \left(\frac{2}{\delta}\right)^{\frac{1}{2\delta(\alpha-1)-\alpha}} (k)^{\frac{2\delta-1}{2\delta(\alpha-1)-\alpha}} = k^{\frac{1}{\alpha-1}} \left[\frac{2}{\delta} k^{\frac{1}{\alpha-1}}\right]^{\frac{1}{2\delta(\alpha-1)-\alpha}}. \quad (6.20)$$

Hence to study the behavior of p^* as δ varies, it is enough to study $K = K(\delta)$ given by

$$K = \frac{1}{2\delta(\alpha-1)-\alpha} \left[\log(2k^{\frac{1}{\alpha-1}}) - \log \delta \right].$$

Note that

$$\frac{\partial K}{\partial \delta} = \frac{[\log 2k^{\frac{1}{\alpha-1}} - \log \delta](-2(\alpha-1))}{(2\delta(\alpha-1) - \alpha)^2} + \frac{1}{2\delta(\alpha-1) - \alpha} \left(-\frac{1}{\delta}\right) \quad (6.21)$$

$$= \frac{-1}{(2\delta(\alpha-1) - \alpha)} \left[\frac{1}{\delta} + \frac{2(\alpha-1)[\log(2k^{\frac{1}{\alpha-1}}) - \log \delta]}{(2\delta(\alpha-1) - \alpha)} \right] \quad (6.22)$$

$$= \frac{-1}{2(\alpha-1)} \left(\frac{1}{\delta - \frac{\alpha}{2(\alpha-1)}} \right) \left[\frac{1}{\delta} + \frac{\log(2k^{\frac{1}{\alpha-1}}) - \log \delta}{\delta - \frac{\alpha}{2(\alpha-1)}} \right] \quad (6.23)$$

$$= \frac{-1}{2\delta(\alpha-1)} \left(\frac{1}{\delta - \frac{\alpha}{2(\alpha-1)}} \right)^2 \left[\delta \left(1 + \log(2k^{\frac{1}{\alpha-1}}) - \log \delta \right) - \frac{\alpha}{2(\alpha-1)} \right] \quad (6.24)$$

$$= \frac{-1}{2\delta(\alpha-1)} \left(\frac{1}{\delta - \frac{\alpha}{2(\alpha-1)}} \right)^2 \left[\delta \log \left(\frac{2ek^{\frac{1}{\alpha-1}}}{\delta} \right) - \frac{\alpha}{2(\alpha-1)} \right]. \quad (6.25)$$

The result now follows easily from (6.25). We omit the algebraic details. ■

6.4 Properties of the functions Γ and H

Define the well known digamma function $D(\cdot)$ and a related function $H(\cdot)$ as

$$D(p) = \log \Gamma(p), \quad H(p) = \log \Gamma\left(p + \frac{1}{2}\right) - \log \Gamma(p) = D\left(p + \frac{1}{2}\right) - D(p).$$

See Abramowitz and Stegun (1972, pp. 258-259) for some of the properties of $D(\cdot)$ that we state and use below. The function $H(\cdot)$ will play an important role in our analysis. For instance, we can rewrite \hat{a} as

$$\hat{a} = \left(\frac{\sqrt{\theta}}{\delta} \right)^{\frac{2}{2\delta-1}} e^{\frac{2}{2\delta-1}H(p)}.$$

Lemma 9 (i) For all $p > 0$,

$$1 < 2pH'(p) < 2. \quad (6.26)$$

(ii) For all $p > 0$,

$$H''(p) > -\frac{1}{p^2}. \quad (6.27)$$

(iii) As $p \rightarrow \infty$,

$$\frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p)} \approx p^{1/2}. \quad (6.28)$$

(iv) As $p \rightarrow 0$,

$$\frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p)} \approx \sqrt{\pi p}. \quad (6.29)$$

(v) As a consequence of (iii) and (iv),

$$\lim_{p \rightarrow 0} \frac{H(p)}{\log p} = 1, \quad \lim_{p \rightarrow \infty} \frac{2H(p)}{\log p} = 1. \quad (6.30)$$

(vi) For all $0 < p_0 \leq 1 \leq p_1 < \infty$,

$$\frac{\sqrt{\pi}}{2} p_0 \leq \frac{\Gamma(p_0 + 1/2)}{\Gamma(p_0)} \leq \frac{\sqrt{\pi}}{2} p_0^{1/2} \leq \frac{\sqrt{\pi}}{2} p_1^{1/2} \leq \frac{\Gamma(p_1 + 1/2)}{\Gamma(p_1)} \leq \frac{\sqrt{\pi}}{2} p_1. \quad (6.31)$$

Proof of Lemma 9. We use the following known facts for digamma functions. For every $p > 0$,

$$D'(p) = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-pt}}{1 - e^{-t}} \right] dt. \quad (6.32)$$

$$D''(p) = \sum_{n=0}^{\infty} \frac{1}{(p+n)^2} > 0. \quad (6.33)$$

(i) From (6.32), it follows that

$$H'(p) = D'(p + \frac{1}{2}) - D'(p) \quad (6.34)$$

$$= \int_0^\infty \left[\frac{e^{-pt} - e^{-(p+\frac{1}{2})t}}{1 - e^{-t}} \right] dt \quad (6.35)$$

$$= \int_0^\infty e^{-pt} \left[\frac{1 - e^{-\frac{t}{2}}}{1 - e^{-t}} \right] dt \quad (6.36)$$

$$= \int_0^\infty e^{-pt} \left[\frac{1}{1 + e^{-\frac{t}{2}}} \right] dt. \quad (6.37)$$

Hence (i) follows by observing that for all $t > 0$,

$$\frac{1}{2} \leq \frac{1}{1 + e^{-\frac{t}{2}}} \leq 1 \quad \text{and} \quad \int_0^\infty e^{-pt} dt = \frac{1}{p}.$$

(ii) To prove (ii), using (6.33),

$$\begin{aligned} H''(p) &= D''(p + \frac{1}{2}) - D''(p) < 0 \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{(p + \frac{1}{2} + n)^2} - \frac{1}{(p + n)^2} \right] \\ &= -\frac{1}{p^2} + \sum_{n=0}^{\infty} \left[\frac{1}{(p + \frac{1}{2} + n)^2} - \frac{1}{(p + n + 1)^2} \right] > -\frac{1}{p^2}. \end{aligned}$$

(iii)-(v) Once (iii) and (iv) are proved, (v) follows trivially. To prove (iii) and (iv), by using Stirling's approximation, $\Gamma(x+1) \approx \sqrt{(2\pi)} e^{-x} x^{x+\frac{1}{2}}$, as $x \rightarrow \infty$, it is immediately verified that as $p \rightarrow \infty$,

$$\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \approx e^{1/2} \frac{(p - \frac{1}{2})^p}{(p - 1)^{p - \frac{1}{2}}} \approx p^{1/2}.$$

To prove (iv) note that $\Gamma(x)$ is a continuous function of x . Further, $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Hence, as $p \rightarrow 0$, Part (v) follows immediately from this.

(vi) Using part (i),

$$\int_{p_0}^1 \frac{1}{2p} dp \leq \int_{p_0}^1 H'(p) dp \leq \int_{p_0}^1 \frac{1}{p} dp$$

$\Rightarrow H(1) + \log p_0 \leq H(p_0) \leq H(1) + \frac{1}{2} \log p_0 \Rightarrow p_0 e^{H(1)} \leq e^{H(p_0)} \leq p_0^{1/2} e^{H(1)}$. Note that

$$e^{H(1)} = \frac{\Gamma(3/2)}{\Gamma(1)} = \frac{\sqrt{\pi}}{2}.$$

This proves the first two inequalities of 6.31. The proof of the other three inequalities is similar and is omitted. ■

6.5 Arguments for Lemma 4

Proof of Lemma 4. (i) Taking a derivative, the first order condition for this maximization is

$$(U^A(a))' = 2\sqrt{\theta} \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right) \frac{1}{2\sqrt{a}} - \delta a^{\delta-1} = 0.$$

This implies

$$\hat{a} = \left[\frac{\sqrt{\theta}}{\delta} \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right) \right]^{\frac{2}{2\delta-1}}. \quad (6.38)$$

Since $U^A(a) \rightarrow -\infty$, as $a \rightarrow \infty$ and $U^A(a) \rightarrow 0$ as $a \rightarrow 0$, the above indeed yields the global maximum.

(ii) It then follows using (6.38) that

$$\begin{aligned} U^A(\hat{a}) &= 2\sqrt{\theta} \left(\hat{a}^{\frac{1}{2}} \right) \frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} - \hat{a}^\delta \\ &= (2\delta - 1)\hat{a}^\delta > 0. \end{aligned}$$

(iii) It is easily checked that

$$\frac{\partial \log \hat{a}}{\partial p} = \frac{2}{2\delta - 1} H'(p) > 0.$$

(iv) The employer will choose θ and p to maximize her expected payoff, given by

$$L = \int_0^\infty [(1 - \theta)x] f(x|\hat{a}) dx - \frac{p^\alpha}{\alpha k} \quad (6.39)$$

$$= (1 - \theta)\hat{a}p - \frac{p^\alpha}{\alpha k}. \quad (6.40)$$

Note that

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= -\widehat{a}p + (1 - \theta)p\widehat{a}\frac{1}{(2\delta - 1)\theta} \\ &= \widehat{a}p\left[-1 + \frac{1}{2\delta - 1}\left(\frac{1}{\theta} - 1\right)\right].\end{aligned}$$

Hence,

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{1}{2\delta - 1} \left(\frac{1}{\theta} - 1 \right) = 1 \Rightarrow \theta^* = \frac{1}{2\delta}.$$

Further,

$$\begin{aligned}\frac{\partial^2 L}{\partial \theta^2} &= \frac{\widehat{a}p}{(2\delta - 1)\theta} \left[-1 + \frac{1}{2\delta - 1}\left(\frac{1}{\theta} - 1\right)\right] + \widehat{a}p \left[-\frac{1}{(2\delta - 1)\theta^2}\right] \\ &= \frac{\widehat{a}p}{(2\delta - 1)\theta} \left[-1 - \frac{1}{2\delta - 1}\right] < 0.\end{aligned}$$

Hence $\theta^* = \frac{1}{2\delta}$ is the global maximum. ■

6.6 Arguments for Lemma 5

Taking logarithm on both sides of (4.8), p^* is a solution of the equation

$$h(p) = 0 \quad \text{where} \tag{6.41}$$

$$h(p) = (\alpha - 1) \log p - \log k - \log \widehat{a} - \log \left[1 + \frac{2p}{2\delta - 1} H'(p) \right] \tag{6.42}$$

$$= (\alpha - 1) \log p - \log k - \log g(p, \delta) \quad (\text{say}). \tag{6.43}$$

The nature of this function $h(\cdot)$ as a function of p is crucial to the proof of Lemma 5. This is established in the following lemma. Note that

$$\begin{aligned}h'(p) &= \frac{(\alpha - 1)}{p} - \frac{\partial}{\partial p} \log g(p, \delta) \\ &= \frac{(\alpha - 1)}{p} - \left[\frac{2}{2\delta - 1} H'(p) + \frac{2[H'(p) + pH''(p)]}{2\delta - 1 + 2pH'(p)} \right].\end{aligned}$$

Lemma 10 (i) If $\alpha > 1 + \frac{8\delta}{4\delta^2 - 1}$, then $h'(p) > 0$.

(ii) If $\alpha > 1 + \frac{2}{2\delta - 1}$, then $h(p) \rightarrow -\infty$ as $p \rightarrow 0$.

(iii) If $\alpha > 1 + \frac{1}{2\delta - 1}$, then $h(p) \rightarrow \infty$ as $p \rightarrow \infty$.

Proof of Lemma 10. (i) Note that $h'(p) > 0$ if and only if

$$2pH'(p)[(2\delta - 1)(3 - \alpha) + 2pH'(p)] < (\alpha - 1)(2\delta - 1)^2$$

But since $pH'(p) \leq 1$, a sufficient condition for the above to hold is

$$2pH'(p)[(2\delta - 1)(3 - \alpha) + 2] < (\alpha - 1)(2\delta - 1)^2.$$

Note that if $(2\delta - 1)(3 - \alpha) + 2 < 0$, that is if $\alpha > 3 + \frac{2}{2\delta - 1}$, the above holds trivially.

On the other hand, if $(2\delta - 1)(3 - \alpha) + 2 > 0$, then we need

$$\begin{aligned} 2[(2\delta - 1)(3 - \alpha) + 2] &< (\alpha - 1)(2\delta - 1) + (2\delta - 1)^2 \\ \Leftrightarrow \alpha[(2\delta - 1)^2 + 2(2\delta - 1)] &> 4 + 6(2\delta - 1) + (2\delta - 1)^2 \\ \Leftrightarrow \alpha(4\delta^2 - 1) &> 4\delta^2 + 8\delta - 1 \Leftrightarrow \alpha > 1 + \frac{8\delta}{4\delta^2 - 1}. \end{aligned}$$

Thus we have shown that $h'(p) > 0$ if either (i) $\alpha > 3 + \frac{2}{2\delta - 1}$ or $1 + \frac{8\delta}{4\delta^2 - 1} < \alpha < 3 + \frac{2}{2\delta - 1}$.

Since $1 + \frac{8\delta}{4\delta^2 - 1} < 3 + \frac{2}{2\delta - 1}$, we have $h'(p) > 0$ if $\alpha > 1 + \frac{8\delta}{4\delta^2 - 1}$, establishing (i).

(ii)

$$h(p) = (\alpha - 1) \log p - \log k - [\log 2 + \log(2\delta - 1) - \log \delta + \log \hat{a}] + \log\left(1 + \frac{2p}{2\delta - 1} H(p)\right).$$

First let $p \rightarrow 0$. Then $\log\left(1 + \frac{2p}{2\delta - 1} H(p)\right)$ is bounded. Also using (6.30),

$$\log \hat{a} = c + \frac{2}{2\delta - 1} \log \left(\frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(p)} \right) \approx \frac{2}{2\delta - 1} \log p.$$

So, as $p \rightarrow 0$,

$$h(p) \approx (\alpha - 1) \log p - \frac{2}{2\delta - 1} \log p$$

which tends to $-\infty$ if α is greater than $1 + \frac{2}{2\delta - 1}$. This proves (ii).

(iii) Similarly, as $p \rightarrow \infty$,

$$h(p) \approx (\alpha - 1) \log p - \frac{2}{2\delta - 1} H(p) \approx \left[(\alpha - 1) - \frac{1}{2\delta - 1} \right] \log p$$

which tends to ∞ when α is greater than $1 + \frac{1}{2\delta - 1}$, proving (iii). ■

Proof of Lemma 5. If $\alpha > 1 + \frac{8\delta}{4\delta^2 - 1}$, then from Lemma 10, (i) $h(p) \rightarrow \infty$ as $p \rightarrow \infty$, (ii) $h(p) \rightarrow -\infty$ as $p \rightarrow 0$ and (iii) $h'(p) > 0$ for all $p > 0$. Hence there exists a unique solution for $h(p) = 0$.

We now show that this solution provides the global maximum. Recall that

$$L = (1 - \theta)\hat{a}p - \frac{p^\alpha}{\alpha k}.$$

Hence using the approximation given in Lemma 9, as $p \rightarrow 0$, it is easy to see that $L \rightarrow 0$.

On the other hand, using again the approximations given in Lemma 9, as $p \rightarrow \infty$, since, if $\alpha > \frac{2\delta}{2\delta-1}$,

$$L \approx c_1 p \cdot p^{\frac{1}{2}(\frac{2}{2\delta-1})} - c_2 p^\alpha \approx c_1 p^{\frac{2\delta}{2\delta-1}} - c_2 p^\alpha \rightarrow -\infty.$$

This shows that the interior solution of $h(p) = 0$ indeed provides the global maximum.

Now we show that the value of L at the optimum is positive. Recall that at the interior optimum

$$\frac{p^\alpha}{k} = (1 - \theta)\hat{a}p\left[1 + \frac{2p}{2\delta - 1}H'(p)\right].$$

Hence

$$\begin{aligned} L &= (1 - \theta)\hat{a}p - \frac{p^\alpha}{\alpha k} \\ &= \frac{p^\alpha}{k\left(1 + \frac{2p}{2\delta-1}H'(p)\right)} - \frac{p^\alpha}{\alpha k} \\ &= \frac{p^\alpha}{\alpha k\left(1 + \frac{2p}{2\delta-1}H'(p)\right)} \left[\alpha - \left(1 + \frac{2p}{2\delta-1}H'(p)\right)\right]. \end{aligned}$$

Now, recalling that $\frac{1}{2} \leq pH'(p) \leq 1$, we get

$$\alpha - 1 - \frac{2p}{2\delta - 1}H'(p) \geq \alpha - 1 - \frac{2}{2\delta - 1} > 0.$$

■

6.7 Arguments for Lemma 6

Proof of Lemma 6. Using (4.9), observe that the numerator of C_1 equals

$$\alpha(2\delta - 1)(2\delta - 1 + 2pH'(p)) - (2\delta - 1 + 2pH'(p))^2 - 2p(2\delta - 1)(H'(p) + pH''(p)). \quad (6.44)$$

Since $\frac{1}{2} \leq pH'(p) \leq 1$, and $-p^2H''(p) > 1$, we get from (6.44),

$$\begin{aligned} C_1 &> \alpha(2\delta - 1)(2\delta - 1 + 1) - (2\delta - 1 + 2pH'(p))^2 - 2(2\delta - 1) - 2p^2(2\delta - 1)H''(p) \\ &> \alpha(2\delta - 1)2\delta - (2\delta + 1)^2 - 2(2\delta - 1) \\ &= \alpha(2\delta - 1)2\delta - (4\delta^2 + 8\delta - 1) \\ &> 0 \text{ if } \alpha > 1 + \frac{10\delta - 1}{2\delta(2\delta - 1)}. \end{aligned}$$

This proves the Lemma. ■

6.8 Arguments for Lemma 7

The following Lemma provides a bound for \hat{a} in terms of k . It is used crucially in the proof of Lemma 7.

Lemma 11 p^* satisfies

$$k\widehat{a} \leq p^{\alpha-1} \leq k\widehat{a}\left(1 + \frac{1}{2\delta}\right) \quad (6.45)$$

$$\leq \frac{5}{4}k\widehat{a} \quad \text{for all } \delta \geq 2. \quad (6.46)$$

Proof of Lemma 11. Recall that from Lemma 9 of Section 6.4, $pH'(p) \leq 1$. Hence,

$$p^{\alpha-1} = k\frac{\widehat{a}}{2\delta}[2\delta - 1 + 2pH'(p)] \quad (6.47)$$

$$\leq k\frac{\widehat{a}}{2\delta}[2\delta - 1 + 2] \quad (6.48)$$

$$\leq k\widehat{a}\left[1 + \frac{1}{2\delta}\right] \quad (6.49)$$

$$\leq k\widehat{a}\left[1 + \frac{1}{4}\right] \quad \text{since } \delta \geq 2. \quad (6.50)$$

The left side follows similarly by using $2pH'(p) \geq 1$. ■

Define

$$A(\delta) = \frac{2 \log 2}{(2\delta - 1)^2} - \frac{1}{\delta} - \frac{3}{\delta(2\delta - 1)} + \frac{6}{(2\delta - 1)^2} \log \delta.$$

$$B_1(\delta) = A(\delta) + \frac{1}{\delta}, \quad B_2(\delta) = A(\delta) + \frac{2}{2\delta + 1}, \quad k^* = \frac{\sqrt{\pi}}{2^{3/2}\delta^{3/2}} k^{\frac{2\delta-1}{2}}.$$

It can be checked that

$$e^{\frac{(2\delta-1)^2}{4}B_1(\delta)} = \sqrt{2}\delta^{3/2}e^{\frac{3}{4\delta}-\frac{3}{2}}.$$

Lemma 7 follows from the following stronger result. We omit the algebraic details of that but prove the following result completely.

Lemma 12 Suppose $\delta > 1/2$ and $\alpha > 1 + \frac{8\delta}{4\delta^2-1}$.

(i) If $\delta \leq \frac{\pi^{\frac{1}{3}}}{2} k^{\frac{(2\delta-1)}{3}}$ then $p^* \geq (k^*)^{\frac{2}{(\alpha-1)(2\delta-1)-1}} = K_1^* \geq 1$.

Further,

(a) if $\frac{\sqrt{\pi}}{2} K_1^{*1/2} > \sqrt{2}\delta^{3/2}e^{\frac{3}{4\delta}-\frac{3}{2}}$, then $C_2 < 0$. In particular, if $\delta \leq \frac{\pi^{\frac{1}{3}}}{2} k^{\frac{1}{3(\alpha-1)}} < \frac{\pi^{\frac{1}{3}}}{2} k^{\frac{2\delta-1}{3}}$ then the above condition holds.

(b) if $\delta \leq \frac{\pi^{\frac{1}{3}}}{2} k^{\frac{1}{3(\alpha-1)}}$ then $a^* \geq 1$.

(ii) If $\delta \geq \frac{\pi^{\frac{1}{3}}}{2} \left(\left(1 + \frac{1}{2\delta}\right) k \right)^{\frac{(2\delta-1)}{3}}$ then $p^* \leq \left[\left(1 + \frac{1}{2\delta}\right)^{\frac{2\delta-1}{2}} (k^*) \right]^{\frac{2}{(\alpha-1)(2\delta-1)-1}} = K_2^* \leq 1$.

Further,

(a) if $\frac{\sqrt{\pi}}{2} (K_2^*)^{1/2} < \sqrt{2}\delta^{3/2}e^{\frac{3}{4\delta}-\frac{3}{2}} e^{\frac{-(2\delta-1)^2}{8\delta(2\delta+1)}}$, then $C_2 > 0$. In particular, if $\delta \geq \frac{(\pi e^4)^{\frac{1}{3}}}{2} \left(\left(1 + \frac{1}{2\delta}\right) k \right)^{\frac{1}{3(\alpha-1)}}$ then the above condition holds.

(b) if $\delta \geq \frac{\pi^{\frac{1}{3}}}{2} \max\left\{ \left(\left(1 + \frac{1}{2\delta}\right) k \right)^{\frac{(2\delta-1)}{3}}, \left(\left(1 + \frac{1}{2\delta}\right) k \right)^{\frac{1}{3(\alpha-1)}} \right\}$ then $a^* \leq 1$.

Proof of Lemma 12. First note that

$$C_2 = A(\delta) - \frac{4}{(2\delta - 1)^2}H(p) + \frac{2}{2\delta - 1 + 2pH'(p)}.$$

From Lemma 9 of Section 6.4, $2pH'(p) > 1$. Using this,

$$C_2 \leq A(\delta) - \frac{4}{(2\delta - 1)^2}H(p) + \frac{2}{2\delta - 1 + 1} \quad (6.51)$$

$$\leq [A(\delta) + \frac{1}{\delta}] - \frac{4}{(2\delta - 1)^2}H(p) \quad (6.52)$$

$$= B_1(\delta) - \frac{4}{(2\delta - 1)^2}H(p).$$

Hence $C_2 < 0$ if $B_1(\delta) - \frac{4}{(2\delta - 1)^2}H(p) < 0$. Or, in other words,

$$C_2 < 0 \text{ if } e^{H(p)} = \frac{\Gamma(p + 1/2)}{\Gamma(p)} > e^{\frac{(2\delta - 1)^2}{4}B_1(\delta)}. \quad (6.53)$$

Similarly, using the other part of Lemma 9 (i) which says that $pH'(p) \leq 1$,

$$C_2 \geq B_2(\delta) - \frac{4}{(2\delta - 1)^2}H(p).$$

Hence

$$C_2 > 0 \text{ if } e^{H(p)} = \frac{\Gamma(p + 1/2)}{\Gamma(p)} < e^{\frac{(2\delta - 1)^2}{4}B_2(\delta)}. \quad (6.54)$$

(i) First suppose, if possible $p^* = p < 1$. To ease notation we write p for p^* . Using Lemma 9 (vi) for $p \leq 1$, and noting that $k^* \geq 1$,

$$k\hat{a} \leq p^{\alpha - 1} \quad (6.55)$$

$$\Rightarrow k \left[\frac{1}{2^{1/2}\delta^{3/2}} \frac{\Gamma(p + 1/2)}{\Gamma(p)} \right]^{\frac{2}{2\delta - 1}} \leq p^{\alpha - 1} \quad (6.56)$$

$$\Rightarrow k^{\frac{2\delta - 1}{2}} \frac{1}{2^{1/2}\delta^{3/2}} \frac{\Gamma(p + 1/2)}{\Gamma(p)} \leq p^{\frac{(\alpha - 1)(2\delta - 1)}{2}} \quad (6.57)$$

$$\Rightarrow k^{\frac{2\delta - 1}{2}} \frac{1}{2^{1/2}\delta^{3/2}} \frac{\sqrt{\pi}}{2} p \leq p^{\frac{(\alpha - 1)(2\delta - 1)}{2}} \quad (6.58)$$

$$\Rightarrow k^* \leq p^{\frac{(\alpha - 1)(2\delta - 1) - 2}{2}} \leq 1 \quad (6.59)$$

which is a contradiction. Hence $p^* > 1$. Now since $p > 1$, again using $k\hat{a} \leq p^{\alpha - 1}$ and using Lemma 9 (vi) for $p > 1$,

$$k \left[\frac{1}{2} \frac{\sqrt{\pi} p^{1/2}}{2^{1/2}\delta^{3/2}} \right]^{\frac{2}{2\delta - 1}} \leq p^{\alpha - 1} \quad (6.60)$$

$$\Rightarrow 1 \leq k^* \leq p^{\frac{(\alpha - 1)(2\delta - 1) - 1}{2}}.$$

This proves the first part of (i).

To prove the first part of (i) (a), using the given condition on k^* , and using Lemma 9 part (vi), since $1 \leq K_1^* \leq p$,

$$e^{\frac{(2\delta-1)^2}{4}B_1(\delta)} < \frac{\sqrt{\pi}}{2}(K_1^*)^{1/2} \leq \frac{\Gamma(K_1^* + 1/2)}{\Gamma(K_1^*)} \leq \frac{\Gamma(p + 1/2)}{\Gamma(p)}.$$

Hence recalling (6.53), we conclude that $C_2 < 0$.

We now show the second part of (i) (a). First note that

$$\delta(2\delta)^{1/2} > e^{\frac{(2\delta-1)^2}{4}B_1(\delta)} \quad (6.61)$$

$$\Leftrightarrow \delta(2\delta)^{1/2} > \sqrt{2}\delta^{3/2}e^{\frac{3}{4\delta}-\frac{3}{2}} \quad (6.62)$$

$$\Leftrightarrow 1 > e^{\frac{3}{4\delta}-\frac{3}{2}} \quad (6.63)$$

$$\Leftrightarrow \frac{3}{4\delta} - \frac{3}{2} < 0 \quad (6.64)$$

which holds if $\delta > 1/2$. Hence it is enough to show that $\frac{\sqrt{\pi}}{2}(K_1^*)^{1/2} > \delta(2\delta)^{1/2}$. This upon simplification yields

$$\delta \leq \frac{\pi^{\frac{1}{3}}}{2} k^{\frac{1}{3(\alpha-1)}}$$

proving (i) (a) completely.

To prove (i) (b) note that since $p^* \geq 1$, by Lemma 9 (i)

$$(a^*)^{\frac{2\delta-1}{2}} = \frac{\sqrt{\theta} \Gamma(p^* + 1/2)}{\delta \Gamma(p^*)} \geq \frac{\sqrt{\theta} \Gamma(K_1^* + 1/2)}{\delta \Gamma(K_1^*)} \geq \frac{\sqrt{\theta} \sqrt{\pi}}{\delta} \frac{\sqrt{\pi}}{2} (K_1^*)^{1/2}.$$

Hence $a^* \geq 1$ if $\frac{\pi}{4}K_1^* > 2\delta^3$. Simplification yields $\delta \leq \frac{\pi^{\frac{1}{3}}}{2} k^{\frac{1}{3(\alpha-1)}}$. This proves (i) (b).

(ii) The proof of (ii) is similar. First suppose, if possible $p^* = p > 1$. Define $\hat{\delta} = 1 + \frac{1}{2\delta}$. Using Lemma 9 (vi) for $p > 1$,

$$p^{\alpha-1} \leq \hat{\delta} k \hat{a} \quad (6.65)$$

$$\Rightarrow p^{\alpha-1} \leq \hat{\delta} k \left[\frac{1}{2^{1/2}\delta^{3/2}} \frac{\Gamma(p + 1/2)}{\Gamma(p)} \right]^{\frac{2}{2\delta-1}} \quad (6.66)$$

$$\Rightarrow p^{\frac{(\alpha-1)(2\delta-1)}{2}} \leq \left(\hat{\delta} k \right)^{\frac{2\delta-1}{2}} \frac{1}{2^{1/2}\delta^{3/2}} \frac{\sqrt{\pi} p}{2} \quad (6.67)$$

$$\Rightarrow \left(\hat{\delta} k \right)^{\frac{2\delta-1}{2}} \frac{\sqrt{\pi}}{2} \frac{1}{2^{1/2}\delta^{3/2}} \geq p^{\frac{(\alpha-1)(2\delta-1)}{2} - 1} \quad (6.68)$$

$$\Rightarrow \left(\hat{\delta} \right)^{\frac{2\delta-1}{2}} k^* \geq p^{\frac{(\alpha-1)(2\delta-1)-2}{2}} \geq 1 \quad (6.69)$$

which is a contradiction. Hence $p^* \leq 1$.

Now again using $p^{\alpha-1} \leq \hat{\delta}k\hat{a}$ and Lemma 9 (vi) for $p \leq 1$,

$$p^{\alpha-1} \leq \hat{\delta}k \left[\frac{1}{2^{1/2}\delta^{3/2}} \frac{\Gamma(p+1/2)}{\Gamma(p)} \right]^{\frac{2}{2\delta-1}} \quad (6.70)$$

$$\Rightarrow p^{\alpha-1} \leq \hat{\delta}k \left[\frac{1}{2} \frac{\sqrt{\pi}p^{1/2}}{2^{1/2}\delta^{3/2}} \right]^{\frac{2}{2\delta-1}} \quad (6.71)$$

$$\Rightarrow p^{\frac{(\alpha-1)(2\delta-1)-1}{2}} \leq \left[\hat{\delta}k \right]^{\frac{2\delta-1}{2}} \frac{\sqrt{\pi}}{2^{3/2}\delta^{3/2}} \quad (6.72)$$

$$\Rightarrow p \leq \left[\left(\hat{\delta}k \right)^{\frac{2\delta-1}{2}} \frac{\sqrt{\pi}}{2^{3/2}\delta^{3/2}} \right]^{\frac{2}{(\alpha-1)(2\delta-1)-1}} = K_2^*$$

This proves the first part of (ii).

To prove the first part of (ii) (a), using the given condition on k^* , and using Lemma 9 part (vi), since $p \leq K_2^* \leq 1$,

$$\frac{\Gamma(p+1/2)}{\Gamma(p)} \leq \frac{\Gamma(K_2^*+1/2)}{\Gamma(K_2^*)} \leq \frac{\sqrt{\pi}}{2} (K_2^*)^{1/2} \leq e^{\frac{(2\delta-1)^2}{4} B_2(\delta)}.$$

Hence recalling (6.54), we conclude that $C_2 > 0$.

The proofs of second part of (ii) (a) and of (ii) (b) are similar to the proofs of second part of (i) (a) and (i) (b). We omit the details. ■

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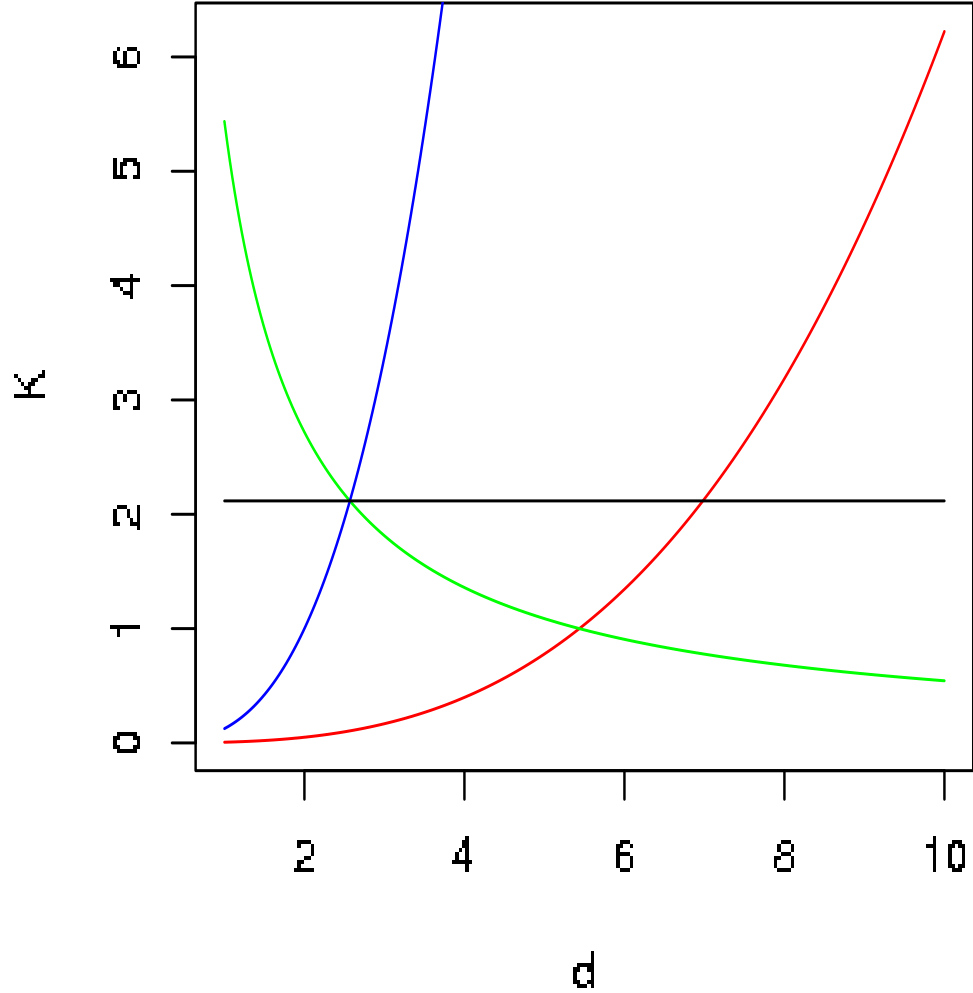


Figure 1: Plot of the four curves in Lemma 2 when $\alpha = 4$ (i) $K(\delta) = \left(\frac{\delta}{2e}\right)^{\alpha-1}$ (red), (ii) $K(\delta) = \frac{2e}{\delta}$ (green), (iii) $K(\delta) = \left(\frac{\delta}{2}\right)^{\alpha-1}$ (blue) and (iv) $K(\delta) = e^{\frac{\alpha-1}{\alpha}}$ (black). Part (i) of Lemma 2 holds in the region below the red curve and part (ii) of the Lemma holds in the region between the green and the blue curves, above the black curve.

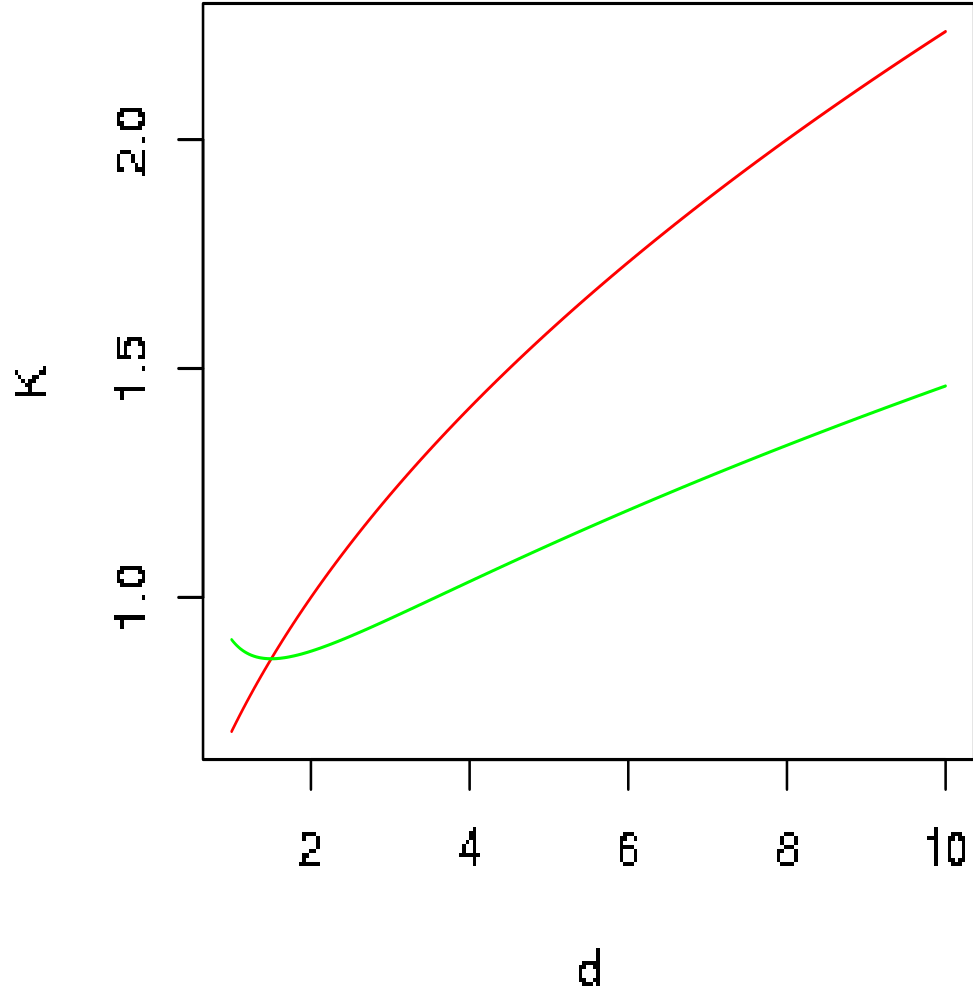


Figure 2: Plot of the two curves in Lemma 3 when $\alpha = 1.5$. (i) $K(\delta) = \left(\frac{\delta}{2}\right)^{\alpha-1}$ (red) and (ii) $K(\delta) = e^{\frac{\alpha}{2\delta}} \left(\frac{\delta}{2e}\right)^{\alpha-1}$ (green). Part (i) of the Lemma holds in the region above the red curve and part (ii) of the Lemma holds in the region below the green curve.

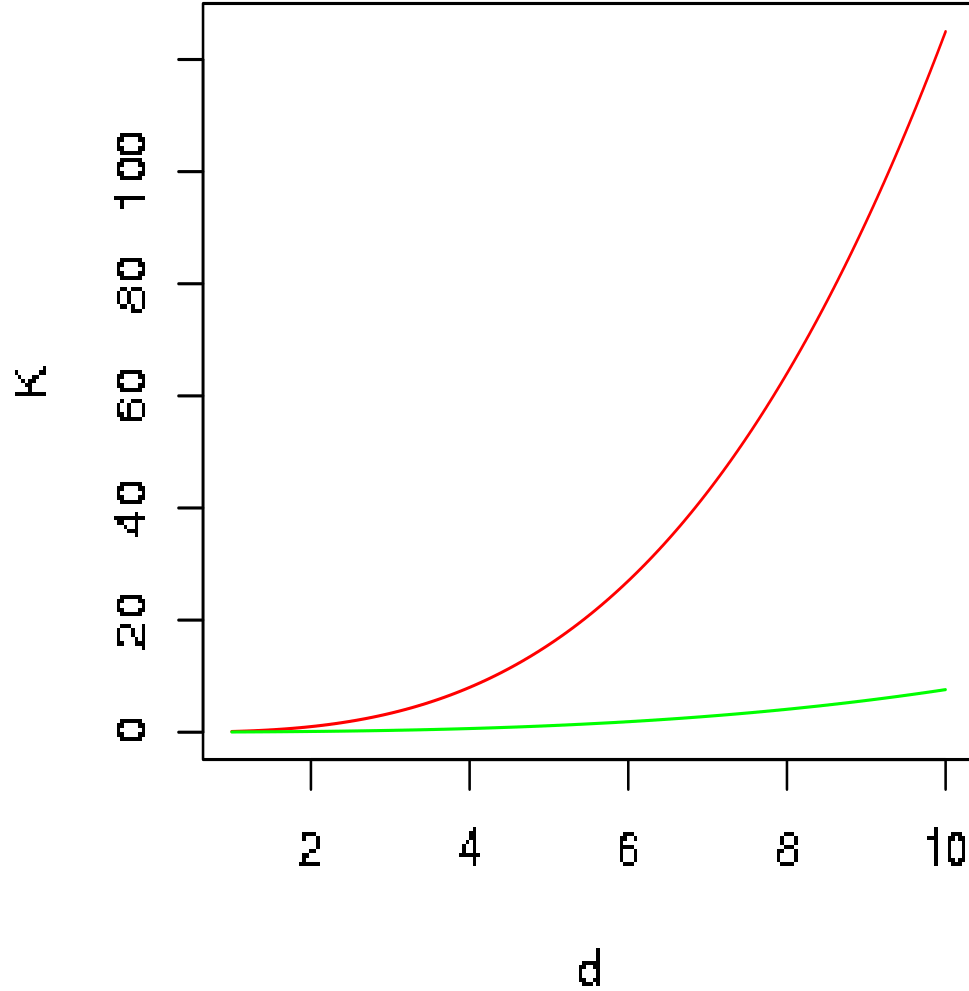


Figure 3: Plot of the two curves in Lemma 3 when $\alpha = 4$. (i) $K(\delta) = \left(\frac{\delta}{2}\right)^{\alpha-1}$ (red) and (ii) $K(\delta) = e^{\frac{\alpha}{2\delta}} \left(\frac{\delta}{2e}\right)^{\alpha-1}$ (green). Part (i) of the Lemma holds in the region above the red curve and part (ii) of the Lemma holds in the region below the green curve.

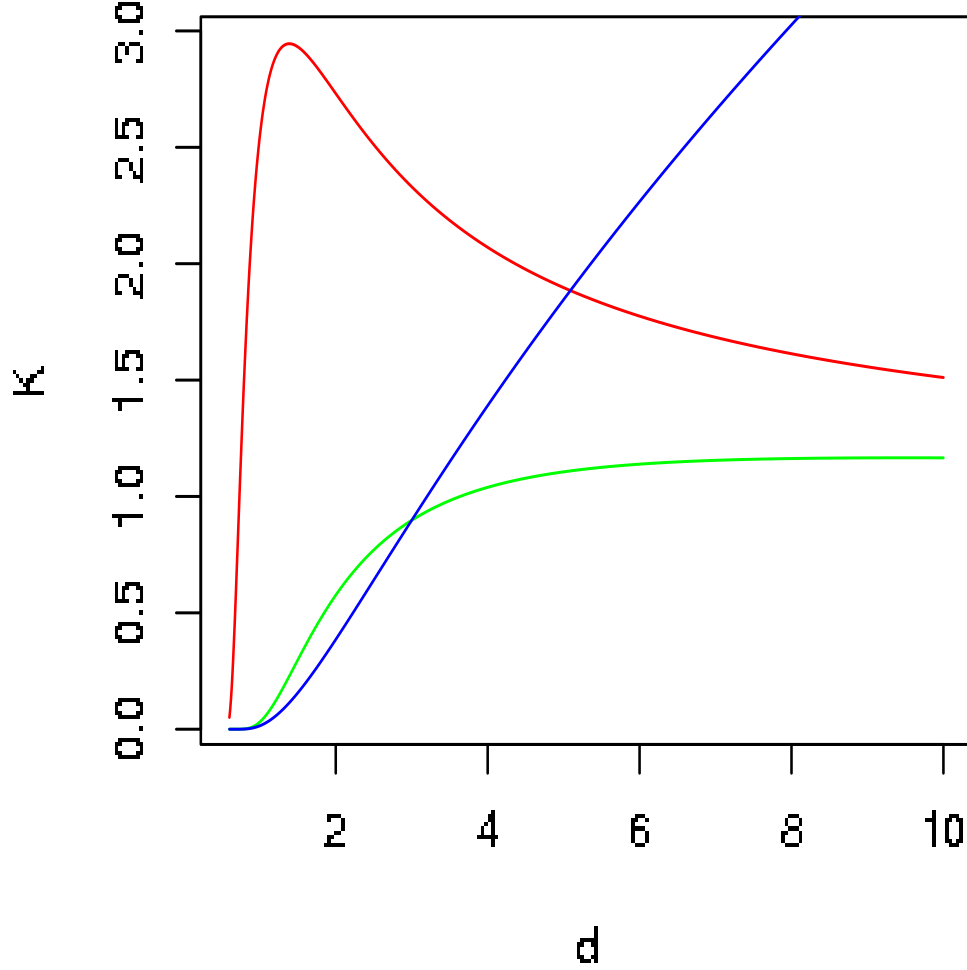


Figure 4: Plot of the three curves in Lemma 7 when $\alpha = 1.2$. (i) $K(\delta) = \frac{(2\delta)^{\frac{3}{2\delta-1}}}{\pi^{\frac{1}{2\delta-1}}}$ (red), (ii) $K(\delta) = \frac{2\delta}{2\delta+1} \left\{ \frac{(2\delta)^{\frac{3}{2\delta-1}}}{(\pi e^4)^{\frac{1}{2\delta-1}}} \right\}$ (green) and (iii) $K(\delta) = \frac{2\delta}{2\delta+1} \left\{ \frac{(2\delta)^{3(\alpha-1)}}{(\pi e^4)^{(\alpha-1)}} \right\}$ (blue). Part (i) of Lemma holds in the region above the red curve. Part (ii) holds in the region in common region below the green and the blue curves.

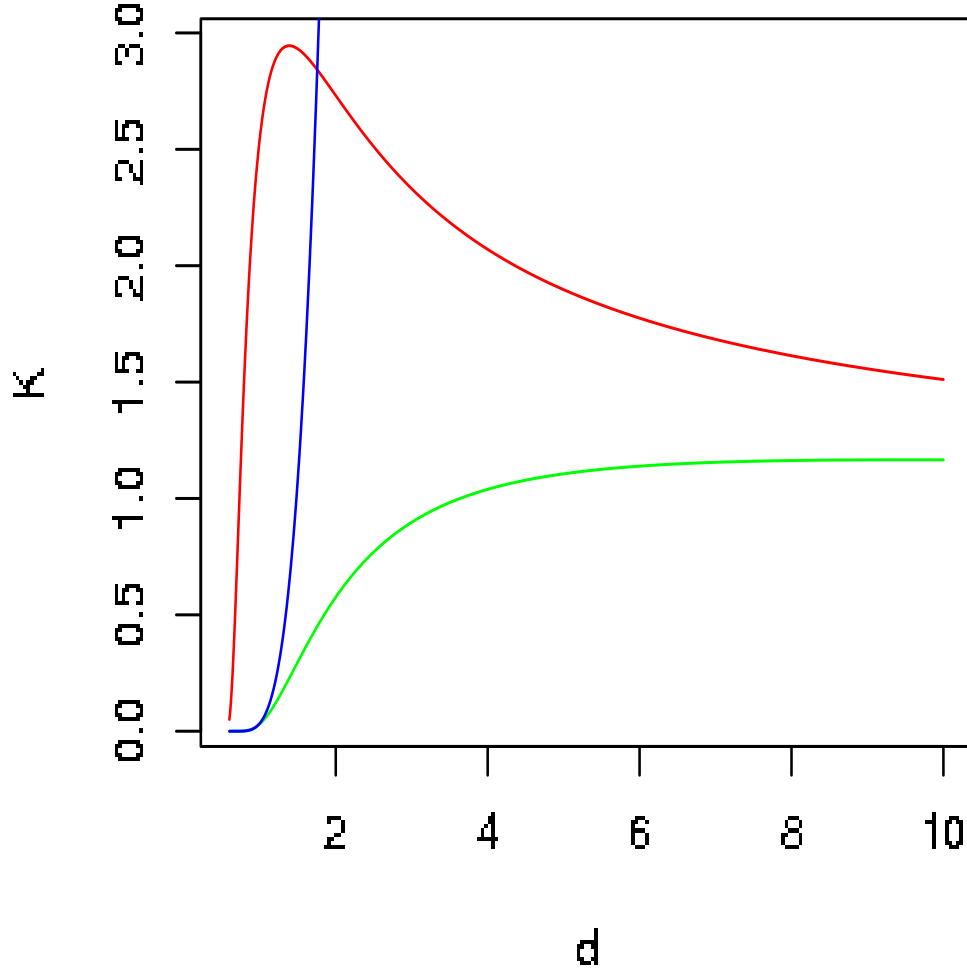


Figure 5: Plot of the three curves in Lemma 7 when $\alpha = 2.1$. (i) $K(\delta) = \frac{(2\delta)^{\frac{3}{2\delta-1}}}{\pi^{\frac{1}{2\delta-1}}}$ (red), (ii) $K(\delta) = \frac{2\delta}{2\delta+1} \left\{ \frac{(2\delta)^{\frac{3}{2\delta-1}}}{(\pi e^4)^{\frac{1}{2\delta-1}}} \right\}$ (green) and (iii) $K(\delta) = \frac{2\delta}{2\delta+1} \left\{ \frac{(2\delta)^{3(\alpha-1)}}{(\pi e^4)^{(\alpha-1)}} \right\}$ (blue). Part (i) of Lemma holds in the region above the red curve. Part (ii) holds in the region in common region below the green and the blue curves.