A Comparative Study of The Consistent and Simplified Finite Element Analyses of Eigenvalue Problems

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SUMMARY

Classical displacement method of the finite element analysis of eigenvalue problems requires the use of consistent and conforming elements. However, simpler approaches, based on relaxing the condition of consistency of the element descriptions, such as lumped inertia force method and others are also found to yield satisfactory results. In this paper we make a comparative study of the consistent and simplified approaches with reference to four representative problems. In the simplified approach studied in this paper, the contribution of straining modes in the derivation of the mass and geometric stiffness matrices is neglected and this simplifies their derivation substantially. The results indicate that this simplification introduces only small errors in the eigenvalues.

NOTATION :

- E Young's modulus
- I moment of inertia
- I_0 moment of inertia of a tapered beam or column at the root section
- k frequency parameter of the beam defined as $k = \frac{m\omega^2 L^4}{E I_0}$
- l length of the beam element
- L length of the beam or column
- m mass per unit length of the beam
- P axial load on the column
- P_{cr} critical load of the column
- $P(t)$, P_t, P_s see equation (12)
- TR taper ratio of the tapered beam or column, defined as $TR = \frac{d}{D}$ (see Figs. (1) and (2))
- λ critical load parameter of the column defined as $\lambda = \frac{P_{cr} L^2}{E I_0}$
- ω circular frequency of the beam
- ω_1 fundamental frequency of the beam
- Ω frequency of the periodic axial force on the column
- Ω_1, Ω_2 frequency bounds, of the axial periodic force on the column, between which the column is unstable.

INTRODUCTION

With the advent of high speed digital computers, matrix methods of structural analysis received considerable attention In 1965, Archer[1] proposed a consistent formulation for these problems which has been very widely adopted for many interesting problems. An extensive exposition on the topic can be seen in Ref. (2) However some of the formulations, which were not based on consistent element descriptions such as the lumped mass method, lumped inertia force method[3] are also found to yield accurate results. A similar formulation for stability problem can be seen in Ref. (4). The purpose of this paper is to systematically study the implications of the use of simplifying assumptions in the element representations. In this paper, four representative problems have been studied by using the consistent as well as simplified approaches. Comparison of the results indicate that the use of simplified mass and geometric-stiffness matrices also yield results to comparable degree of accuracy as the consistent formulation.

CONSISTENT AND SIMPLIFIED APPROACHES

The displacement formulation of the free vibration problem, includes the element stiffness and mass matrices. In the consistent approach, the same kinematic descriptions of the elements are used to generate both stiffness and mass matrices. Whereas in the simplified approach, the mass matrix is simplified by considering only the rigid body modes;

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the stiffness matrix is not altered. Similarly in the stability problem, the consistent approach uses elastic stiffness and geometric stiffness matrices derived from the same element displacement distributions whereas in the simplified approach a simpler geometric stiffness matrix based on the rigid body modes of the element is used and the elastic stiffness matrix remains unaltered. For the sake of completeness we present here certain important details in the consistent and simplified formulations of vibrations and stability of beams.

CONSISTENT APPROACH :

A displacement distribution 'w' over each beam element appropriate for studies of the behaviour of the beams can be taken as

$$w(x)=a_0+a_1x+a_2x^2+a_3x^3 \quad \dots (1)$$

The four free constants a_0, a_1, a_2 and a_3 in Eq. (1) can be obtained in terms of the end conditions of the element namely the two lateral displacements v_1, v_2 and the two rotations v_3, v_4.

The strain energy 'U' in the element is given by

$$U=\frac{1}{2}\int_0^l EI\left(\frac{d^2\omega}{dx^2}\right)^2 dx \quad \dots (2)$$

The stiffness coefficient k_{ij} can be obtained from

$$k_{ij}=\frac{\partial^2 U}{\partial v_j \, \partial v_i} \quad \dots (3)$$

where i and j take values from 1 to 4. Similarly the elements of the mass matrix m_{ij} and the elements of the geometric stiffness matrix g_{ij} can be obtained from the expressions

$$m_{ij} = \frac{\partial^2 T}{\partial v_j \, \partial v_i} \quad \text{and} \quad g_{ij} = \frac{\partial^2 W}{\partial v_j \, \partial v_i} \quad \dots (4)$$

where 'T' is the kinetic energy, given by

$$T=\frac{1}{2}\omega^2\int_0^l m\, w^2\, dx \quad \dots (5)$$

and 'W' is the work done, given by

$$W = \frac{P}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx \quad \dots (6)$$

The assembled stiffness matrix [K], mass matrix [M] and geometric stiffness matrix [G] are obtained from

$$[K] = [a]^T [k] [a]$$
$$[M] = [a]^T [m] [a]$$
$$[G] = [a]^T [g] [a] \quad \dots (7)$$

where [a] is the displacement transformation matrix.

The order of the element stiffness, mass and geometric stiffness matrices is 4 x 4 in this case. Using the above matrices in the appropriate governing equations, one can compute the free vibration or the stability characteristics of beams.

SIMPLIFIED APPROACH :

In this approach the mass and geometric stiffness matrices are simplified by neglecting x^2 and x^3 terms in Eq. (1). A linear displacement distribution over each element is assumed as

$$w(x)=a_0+a_1 x \quad \dots (8)$$

The free constants a_0 and a_1 in Eq. (8) can be obtained from the two lateral displacements v_1 and v_2. Following the procedure outlined in the previous section, the element mass matrix [m] and geometric stiffness matrix [g] can be obtained. In this case the order of the element mass and geometric stiffness matrices is 2×2. For use in the simplified aproach a 2×2 element stiffness matrix in the directions v_1 and v_2 can be obtained by reducing the consistent stiffness matrix, treating the displacement v_3 and v_4 as kinematic redundancies.

RESULTS AND DISCUSSION

Vibration of Tapered Beams

Natural frequencies of a tapered cantilever beam of rectangular cross-section with linear depth taper (see Fig. 1.) have been obtained by solving

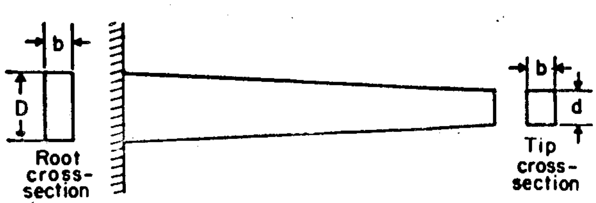

Fig. 1 A Tapered Cantilever Beam

the governing equation.

$$[K]\{v\}-\omega^2[M]\{v\}=0 \quad \dots (9)$$

and using appropriate sitffness and mass matrices.

Table (1) gives the frequency parameter of tapered cantilever beams for the first and second modes. From the table it can be seen that the difference between the frequency parameters obtained by using simplified and consistent approaches is less than 0.15% for TR=1.0 and less than 0.4% when TR=0.4 for the first mode and about 2% for both taper ratios for the second mode, when the order of the dynamical matrix is 10×10. It may be noted here that ten elements in the simplified approach and five elements in the consistent approach give the dynamical matrix of the same order 10×10.

Stability of Tapered Columns

The critical loads of tapered cantiliver colums of circular cross-section Fig. 2 are obtained by solving the governing equation

$$[K]\{v\}-\lambda[G]\{v\}=0 \quad \dots (10)$$

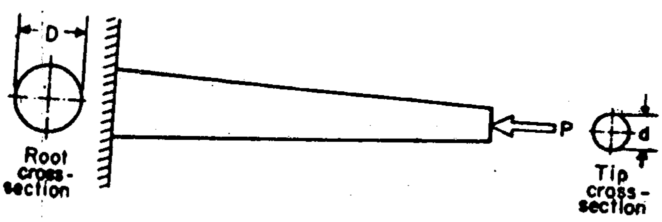

Fig. 2 A Tapered Cantilever Column

and using both the consistent and simplified approaches. Table (2) gives the critical load parameter of tapered cantilever columns for the first and second modes. From the table it can be noted that the difference between the simplified and consistent element is about 0.2% for TR=1 0 and less than 0.4% for TR=0.4 for the first mode and less than 1 8% for TR=1.0 and less than 2.3% for TR=0.4 for the second mode, when the order of dynamical matrix is 10×10.

VIBRATIONS OF BEAM COLUMNS

The natural frequencies of a beam subjected to an axial force P, depend on the value of P. In this case, effective stiffness '$[K]_{eff}$' of the beam is given by:

$$[K]_{eff}=[K]-\frac{P}{P_{cr}}[G] \quad \dots (11)$$

Using the effective stiffness $[K]_{eff}$ in Eq. (9), the frequency parameter of a beam column can be computed. Table (3) shows the frequency parameter of a tapered cantiliver beam column of rectangular cross-section with linear depth taper for various values of $\frac{P}{P_{cr}}$ when the order of the dynamical matrix is 10×10. The difference in the frequency parameter by using both consistent and simplified approaches is less than 0.25% for TR=1.0, and $\frac{P}{P_{cr}}=0.75$ and it is less than 0.7% for TR=0.4, and $\frac{P}{P_{cr}}=0.75$.

DYNAMIC STABILITY OF TAPERED COLUMNS

when a column is subjected to periodic axial force, there will be two frequencies of the axial force between which the column will be unstable. A detailed study of this phenomenon using consistent elements can be seen in Ref. 5. For a periodic axial load 'P(t)' on a column, the equation governing the motion of the column, in the matrix form is taken to be

$$\left[[K]-(\alpha\pm\frac{1}{2}\beta)P_{cr}[G]-\frac{\Omega^2}{4}[M]\right]\{v\}=0 \quad \dots (12)$$

where $P(t)=P_s+P_t\cos\Omega t$

$P_s=\alpha P_{cr}$; $P_t=\beta P_{cr}$

The solution of the above equation yields two values for the frequency of the applied oscillatory force between which the column is unstable. A tapered cantilever column of rectangular cross-section with linear depth taper has been analysed for various taper ratios and the results are presented for one taper ratio. Table (4) gives the frequency bounds of P (t) on a tapered cantilever column obtained by both consistent and simplified approaches for TR=0.4 and $\alpha=0.8$ for various values of the nondimensional parameter μ, given by $\mu=\beta/2(1-\alpha)$. The difference in the results is about 0.2%.

The agreement between the simplified and the consistent approaches in the case of the dynamic stability problem is better than any other problem considered in this paper. An examination of thi brings out an interesting feature of the simplified approach.

Let us define a parameter $\bar{K}(\bar{\lambda})$ as the ratio of the eigenvalue parameter of the tapered beam (column) to the eigen value parameter of the uniform beam (column), each obtained by using same number of elements. Table (5) shows the comparison of this parameter obtained both by the consistent and simplified approaches Apparently this parameter is closer to the consistent formulation than the results presented in Table 1. This high accuracy is due to the definition of the parameter which introduces compensating errors. This feature may be used with advantage in the analysis of certain structural elements of variable cross-section. By analysing the beam (column) of variable cross-section and a corresponding uniform beam (column) using the simplified approach, the value of $\bar{k}(\bar{\lambda})$ may be obtained accurately employing only a few elements. An accurate estimate of k (λ) may be obtained by multiplying $\bar{k}(\bar{\lambda})$ by the known exact eigenvalue of the corresponding uniform beam (column).

CONCLUSIONS

In this paper a comparative study of the consistent and the simplified approaches to the eigenvalue problems, has been made with reference to four representative problems. Comparison of the results shows that the difference in the results by the consistent and the simplified approaches is small. So the simlified approach based on the neglect of the straining modes in the derivation of the mass and geometric stiffness matrices which simplifies their derivation substantially can be used in situations where the derivation of the consistent mass matrix is rather complicated.

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TABLE 1

Frequency parameter (k) of a tapered cantilever beam of rectangular cross-section

Sl. No.	Order of dynamical matrix	TR = 1.0 Simplified approach (1)	TR = 1.0 Consistent approach (2)	TR = 1.0 % difference $\frac{(1)-(2)}{(1)} \times 100$	TR = 0.4 Simplified approach (3)	TR = 0.4 Consistent approach (4)	TR = 0.4 % difference $\frac{(3)-(4)}{(3)} \times 100$
FIRST MODE							
1	4	12.2404	12.3743	−1.0939	15.1176	15.4891	−2.4574
2	6	12.3109	12.3649	−0.4386	15.3260	15.4813	−1.0133
3	8	12.3340	12.3632	−0.2367	15.3944	15.4797	−0.5541
4	10	12.3444	12.3627	−0.1482	15.4252	15.4791	−0.3494
SECOND MODE							
1	4	544.9516	493.7939	9 3876	333.7231	316.8165	5.0660
2	6	514.0761	488.7132	4.9337	320.9782	307.2946	4.2631
3	8	501.9333	486.6509	3.0447	314.8637	306.3757	2.6958
4	10	496.1198	486.0043	2 0389	311.7571	306.0958	1.8159

TABLE 2

Critical load parameter (λ) of a tapered cantilever column of circular cross-section

Sl. No.	Order of dynamical matrix	TR = 1.0 Simplified approach (1)	TR = 1.0 Consistent approach (2)	TR = 1.0 % difference $\frac{(1)-(2)}{(1)} \times 100$	TR = 0.4 Simplified approach (3)	TR = 0.4 Consistent approach (4)	TR = 0.4 % difference $\frac{(3)-(4)}{(3)} \times 100$
FIRST MODE							
1	4	2.4993	2.4687	1.2243	0.7775	0.7671	1.3376
2	6	2.4815	2.4676	0.5601	0.7658	0.7599	0.7704
3	8	2.4753	2.4675	0.3151	0.7618	0.7578	0.5251
4	10	2.4725	2.4674	0.2063	0.7599	0 7571	0.3685
SECOND MODE							
1	4	24 8721	22.9462	7.7432	5.0492	3.8591	23.5701
2	6	23.3699	22.3736	4.2632	4.4280	4.2172	4.7606
3	8	22.8558	22 2621	2 5976	4.2394	4.1237	2.7292
4	10	22.6205	22.2299	1.7268	4 1552	4.0619	2.2454

TABLE 3

Frequency parameter (k) of a tapered cantilever beam column of rectangular cross-section — First mode

Sl. No.	$\frac{P}{P_{cr}}$	TR = 1.0 Simplified approach (1)	TR = 1.0 Consistent approach (2)	TR = 1.0 % difference $\frac{(1)-(2)}{(1)} \times 100$	TR = 0.4 Simplified approach (3)	TR = 0.4 Consistent approach (4)	TR = 0.4 % difference $\frac{(3)-(4)}{(3)} \times 100$
1	−0.50	17.9019	17.9231	−0.1184	21.2937	21.3403	−0.2188
2	−0.25	15.1647	15.1847	−0.1319	18.4588	18.5100	−0.2774
3	0	12.3444	12 3627	−0.1482	15.4252	15.4791	−0.3494
4	.25	9.4307	9.4466	−0.1686	12.1432	12.1967	−0.4406
5	0.50	6.4118	6.4240	−0.1903	8.5459	8.5932	−0 5535
6	0.75	3.2737	3 2808	−0.2169	4.5410	4.5726	−0.6959
7	1.00	0	0	—	0	0	—

The order of the dynamical matrix in both consistent and simplified approaches is 10 x 10

TABLE 4

Frequency bounds of the applied oscillating load for the dynamic stability of a tapered cantilever column of rectangular cross-section

TR = 0.4 & α = 0.8

Sl. No.	μ	$FR_1 = \frac{\Omega_1}{\omega_1}$ Simplified approach (1)	Consistent approach (2)	% difference $\frac{(1)-(2)}{(1)} \times 100$	$FR_2 = \frac{\Omega_2}{\omega_1}$ Simplified approach (3)	consistent approach (4)	% difference $\frac{(3)-(4)}{(3)} \times 100$
1	0	0.9770	0.9788	−0.1842	0.9770	0.9788	−0.1842
2	0.1	0.9293	0.9311	−0.1937	1.0219	1.0238	−0.1859
3	0.2	0.8785	0.8803	−0.2049	1.0646	1.0665	−0.1785
4	0.3	0.8240	0.8257	−0.2063	1.1052	1.1071	−0.1719
5	0.4	0.7650	0.7666	−0.2092	1.1440	1.1459	−0.1661
6	0.5	0.7002	0.7018	−0.2285	1.1812	1.1830	−0.1524

The order of the dynamical matrix in both consistent and simplified approaches is 10 x 10.

TABLE 5

Parameter $\bar{k}$ of a tapered cantilever beam of rectangular cross-section

Sl. No.	Order of dynamical matrix	TR = 0.4 Simplified approach (1)	Consistent approach (2)	% difference $\frac{(1)-(2)}{(1)} \times 100$
FIRST MODE				
1	4	1.2350	1.2517	−1.3522
2	6	1.2449	1.2520	−0.5703
3	8	1.2481	1.2521	−0.3205
4	10	1.2495	1.2521	−0.2081
SECOND MODE				
1	4	0.6124	0.6416	−4.7681
2	6	0.6244	0.6288	−0.7047
3	8	0.6273	0.6296	−0.3666
4	10	0.6284	0.6298	−0.2228