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Transverse Vibrations of Trusses*

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Synopsis

Lumped mass methods have been extensively used to obtain natural frequencies and mode shapes of beams in transverse vibration. Lumped inertia methods have also been used in studying vibration characteristics of beams and girder floors. In this paper characteristics of transverse vibrations of pin jointed trusses are obtained by the use of a rational method of lumping inertia forces. The effective inertia force at each joint is determined by considering the distribution of inertia force in all the members meeting at the joint. Natural frequencies of transverse vibration obtained by this method are compared with those obtained by the method of lumped masses and the discrepancies are found to be considerable particularly at higher frequencies. A method of condensing matrices to obtain natural frequencies is also included.

Introduction

In many cases a continuous structure is idealized into a structure with lumped masses at discrete points. The method developed by Myklested[1] for analysis of transverse vibrations of beams follows this procedure. Although such methods are widely used because of their simplicity, there does not seem to be sufficient justification for such lumping of masses. Recognizing the necessity for a more rational approach Bleich[2] introduced dynamic reactions considering inertia forces in the analysis of vibrations of beam and girder floors. Leckie and Lindberg[3] in a recent paper pointed out the errors in the lumped mass approach.

The dynamic equations of equilibrium in the form of differential equations governing vibration problems involve elastic and inertia forces. Hence it would be more rational to suitably lump these forces for idealizing a continuous system into a discrete system.

However, there is one inherent difficulty in dealing with the inertia forces. These cannot be explicity obtained until the entire problem is solved, although an iterative procedure can be developed to get over this difficulty. By using an approach similar to the L matrix approach developed by Argyris[4] in connection with the static analysis of wing pannels, a direct solution of the problem is possible. This procedure has been used in the analysis of vibration characteristics of the plane pin-jointed trusses.

A pin-jointed truss has many natural frequencies, some of which pertain primarily to the overall vibration of the truss and the others primarily to those of the

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individual members. The frequencies of the individual members are, in general, large compared to the overall frequencies of the truss.

In practice the individual members have negligible flexibility in bending compared to the truss as a whole. In this paper the individual members are assumed to be rigid in bending ; as a consequence a linear inertia loading is present along each member. This inertia loading is replaced by two equivalent inertia forces at the ends of the member. The total inertia forces at each joint is obtained by summing the contribution at that joint of all the members meeting there. The inertia forces thus obtained are incorporated into the dynamic equation of equilibrium to obtain the natural frequencies of the truss. These are compared with those obtained by lumping the masses, at the joints in the conventional way.

Method of obtaining equivalent inertia forces

Considering a typical member AB of the truss, inclined at an angle $\propto$ to the y axis as shown in Fig. 1, the total inertia force of the element, $I_A + I_B$ is obtained as an integral, over the length b_0 of the element, involving the mass per unit length of the member m, and the acceleration $\ddot{x}$ in the x direction at any distance y from A, as

$$I_A + I_B = \int_0^{b_0} \frac{m}{\cos \propto} \ddot{x} \, dy \qquad \ldots(1)$$

The acceleration $\ddot{x}$ is given in terms of $\ddot{x}_A$ and $\ddot{x}_B$, the accelerations at A and B respectively as

$$\ddot{x} = \ddot{x}_A + \frac{y}{b_0} (\ddot{x}_B - \ddot{x}_A)$$

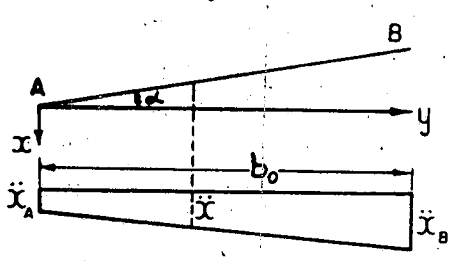

FIG. 1. — A TYPICAL MEMBER OF THE TRUSS.

Taking moments about B and A, one obtains I_A and I_B, the equivalent inertia forces at A and B respectively, as

$$I_A = \frac{1}{b_0} \int_0^{b_0} \frac{m}{\cos \propto} \ddot{x} (b_0 - y) \, dy \ldots(2)$$

$$I_B = \frac{1}{b_0} \int_0^{b_0} \frac{m}{\cos \propto} \ddot{x} \, y \, dy \qquad \ldots(3)$$

When m is constant, Eqs. 2 and 3 can be rewritten in terms of the total mass M_{AB} of the member AB and the end acceleration as

$$I_A = M_{AB} \left[\frac{\ddot{x}_A}{3} + \frac{\ddot{x}_B}{6} \right] \qquad \ldots(2a)$$

$$I_B = M_{AB} \left[\frac{\ddot{x}_B}{3} + \frac{\ddot{x}_A}{6} \right] \qquad \ldots(3a)$$

Method of analysis

Examination of Eqs. 2a and 3a reveals that the equivalent inertia force at a joint A depends not only on the acceleration at that point but also on the accelerations at its neighbouring points. This fact is completely missed in conventional lumped mass methods. Here it is convenient to introduce the concept of inertia coefficients defined as follows. The inertia coefficient m_{ij} is the equivalent inertia force in the direction i due to unit acceleration in the direction j. The contribution of the member (ij) to the inertia coefficient m_{ji} is denoted by $m_{ij}^{(ij)}$. (It is obvious that in the superscript (ij), i and j cannot be equal). Using this notation Eqs. 2a and 3a can be written as

$$I_A = m_{AA}^{(AB)} \ddot{x}_A + m_{AB}^{(AB)} \ddot{x}_B \qquad \ldots(4)$$

and

$$I_B = m_{BB}^{(AB)} \ddot{x}_B + m_{BA}^{(AB)} \ddot{x}_A \qquad \ldots(5)$$

where the inertia coefficients are given by,

$$\left.\begin{aligned} m_{AA}^{(AB)} &= m_{BB}^{(AB)} = \frac{M_{AB}}{3} \\ \text{and} \quad m_{AB}^{(AB)} &= m_{BA}^{(AB)} = \frac{M_{AB}}{6} \end{aligned}\right\} \qquad \ldots(6)$$

Using Eqs. 4 and 5, the inertia force at any joint i can be written as

$$I_i = \left(\sum_j m_{ii}^{(ij)} \right) \ddot{x}_i + \sum_j m_{ij}^{(ij)} \ddot{x}_j \quad ...(7)$$

Let

$$\sum_j m_{ii}^{(ij)} = m_{ii} \; ; \; m_{ij}^{(ij)} = m_{ij}$$

Then Eq. 7 can be written as

$$I_i = m_{ii} \ddot{x}_i + \sum_j m_{ij} \ddot{x}_j$$

which can be put in the matrix form

$$\{I_i\} = [m_{ij}] \{\ddot{x}_j\} \quad ...(8)$$

It may be emphasized here that $m_{ij} = m_{ji}$ and exists only when i and j are neighbouring joints and $m_{ii} = \sum_j m_{ii}^{(ij)}$ summed over all the members (ij) meeting at i. Assuming sinusoidal vibration of natural frequency ω, Eq. 8 may be written in terms of x_j, the x displacement at the joint j, as

$$\{I_i\} = - \omega^2 [m_{ij}] \{x_j\} \quad ...(9)$$

The dynamical equation of equilibrium[5] in terms of the stiffness matrix K_{ij} and the inertia forces I_i is

$$[K_{ij}] \{x_j\} + \{I_i\} = 0$$

Substituting for I_i from Eq. 9, pre-multiplying by the flexibility matrix $[F_{ij}]$ and introducing I for unit matrix, one obtains

$$[I - \omega^2 (F_{ij}) \; (m_{ij})] \; \{x_j\} = 0 \quad .. (10)$$

from which the natural frequencies and mode shapes can be computed.

Condensation of matrices

In many types of trusses when transverse displacements of adjacent joints are nearly equal, the size of the matrix in equation (10) can be reduced without much loss of accuracy[6].

Consider as an example, the case of a truss in which

$$x_k \approx x_l \; ; \; x_m \approx x_n \; ; \; ... \text{ etc.} \quad (11)$$

Then elements equal to the number of approximate equalities can be omitted from the $\{x_j\}$ matrix and the new matrix designated as $\{x_j^*\}$ can be related to the original matrix $\{x_j\}$ by means of the transformation matrix [b] as

$$\{x_j^*\} = [b] \{x_j\} \quad ...(12)$$

The flexibility matrix corresponding to the starred system

$[F_{ij}^*]$ is

$$[F_{ij}^*] = [b] \, [F_{ij}] \, [b'] \quad ...(13)$$

Considering the transformation matrix [B] which transforms $\{x_j^*\}$ into $\{x_j\}$ as

$$\{x_j\} = [B] \, \{x_j^*\} \quad ...(14)$$

one gets the $[m_{ij}]$ matrix in the starred system designated by $[m_{ij}^*]$ as

$$[m_{ij}^*] = [B'] \, [m_{ij}] \, [B] \quad ...(15)$$

The governing equations of dynamical equilibrium in the starred system corresponding equation (10) in original system is

$$[I - \omega^2 \, (F_{ij}^*) \; (m_{ij}^*)] \{x_j^*\} = 0 \quad ...(16)$$

Illustrative numerical examples

The approach developed in this paper is applied to two simple examples. The results obtained are compared with those obtained by the lumped mass approach.

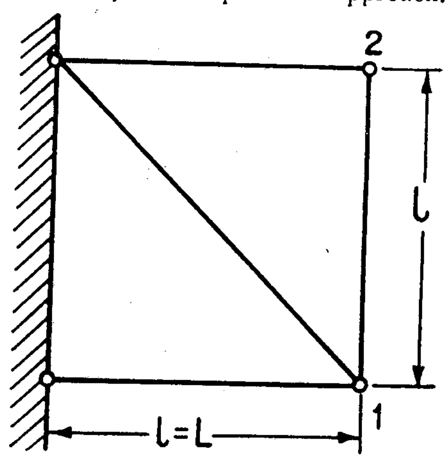

Fig. 2. — A Pin Jointed Truss with one Bay.

Example 1

Consider a single bay truss as shown in Fig. 1. The length, area of cross-section, Young's modulus of i^{th} member are designated by l_i, A_i and E_i respectively. The length l has been indicated in Fig. 2. Considering a unit load applied in the vertical direction at joint 1, one obtains F_{11} in terms of $\bar{N}_{i1}$'s, $\bar{N}_{ij}$ being the force in the member i due to unit load in the direction j, as

$$F_{11} = \sum_i \frac{\bar{N}^2_{i1}\, l_i}{A_i E_i} = \frac{l}{AE}(1+2\sqrt{2})$$

Similarly considering a unit load applied in the vertical direction at joint 2,

$$F_{12} = \sum_i \frac{\bar{N}_{i1}\, \bar{N}_{i2}\, l_i}{A_i E_i} = \frac{l}{AE}(1+2\sqrt{2})$$

and

$$F_{22} = \sum_i \frac{N^2_{i2}\, l_i}{A_i E_i} = \frac{l}{AE}(2+2\sqrt{2})$$

leading to

$$[F_{ij}] = \frac{l}{AE}\begin{Bmatrix} 1+2\sqrt{2} & 1+2\sqrt{2} \\ 1+2\sqrt{2} & 2+2\sqrt{2} \end{Bmatrix}$$

Considering the approach of lumping of inertias,

$$m_{11} = \sum_j m^{(ij)}_{11} = \frac{ml}{3}(2+\sqrt{2})$$

$$m_{12} = \frac{ml}{6} = m_{21}$$

$$m_{22} = \sum_j m^{(2j)}_{22} = \frac{2ml}{3}$$

$$[m_{ij}] = \frac{ml}{3}\begin{Bmatrix} 2+\sqrt{2} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{Bmatrix}$$

For non-trivial values of x_j in Eq. 10, the characteristic determinant is equated to zero, from which the frequency parameter k_T is to be obtained. Thus,

$$\left| I - \frac{k_T^2}{b}\begin{Bmatrix} 13+12\sqrt{2} & 5+10\sqrt{2} \\ 14+2\sqrt{2} & 9+10\sqrt{2} \end{Bmatrix} \right| = 0$$

which yields $k_T = 0{\cdot}343$; $1{\cdot}745$.

The same problem is now solved by lumping the masses of the members at the joints. Let $M^{(ij)}_i$ be the lumped mass at i of the member (ij) and M_{ii} be total lumped mass at i.

then

$$M_{11} = \sum_j M^{(1j)}_1 = \frac{2+\sqrt{2}}{2}\, ml$$

$$M_{22} = \sum_j M^{(2j)}_2 = ml$$

$$[M_{ij}] = ml\begin{Bmatrix} 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{Bmatrix}$$

For non-trivial values of x_j in Eq. 10, using $[M_{ij}]$ in place of $[m_{ij}]$, results in $k_T = 0{\cdot}305$; $1{\cdot}274$.

Now the procedure of condensing the matrices will be demonstrated in lumped inertia approach. Considering the lumped inertia approach, treating $x_2 = x_1$ and considering only the joint 1,

$$x_1^* \left[\tfrac{1}{2}\ \tfrac{1}{2}\right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

corresponding to Eq. 12. Eq. 13 now becomes

$$[F^*_{11}] = \frac{l}{AE}\left[\tfrac{1}{2}\ \tfrac{1}{2}\right]\begin{bmatrix} 1+2\sqrt{2} & 1+2\sqrt{2} \\ 1+2\sqrt{2} & 2+2\sqrt{2} \end{bmatrix}\begin{Bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix}$$

$$= (1{\cdot}25+2\sqrt{2})\,\frac{l}{AE}$$

Eq. 14 now reads

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} x_1^*$$

Eq. 15 yields

$$[m^*_{ij}] = \frac{ml}{3}[1\ \ 1]\begin{bmatrix} 2+\sqrt{2} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \frac{5+\sqrt{2}}{3}\, ml$$

Non-zero value of x_j^*, from equation (16) results in $k_T = 0{\cdot}338$.

By the use of the values of natural frequencies obtained above in Eq. 10 the amplitude ratios at the joints 1 and 2 have been obtained. All the results are summarised in Table 1.

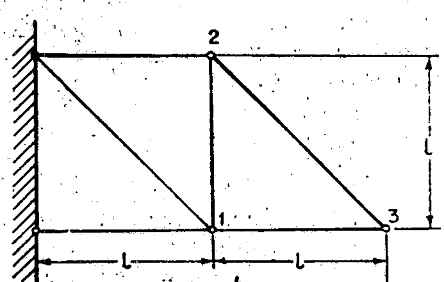

FIG. 3. — A PIN JOINTED TRUSS WITH TWO BAYS.

The numerical results obtained by following a similar procedure as in example 1 are tabulated in Table II.

TABLE I—VALUES OF FREQUENCY PARAMETERS AND AMPLITUDE RATIOS FOR THE TRUSS SHOWN IN FIG. 2.

Mode of vibration	Lumped Inertia Method		Lumped Mass Method		Lumped inertia method by condensation of Matrices Frequency Parameter k_T
	Frequency Parameter k_T	Amplitude Ratios	Frequency Parameter k_T	Amplitude Ratios	
1	0·342	$\frac{x_2}{x_1} = 1{\cdot}106$	0·305	$\frac{x_2}{x_1} = 1{\cdot}103$	0·338
2	1·745	$\frac{x_2}{x_1} = -1{\cdot}463$	1·274	$\frac{x_2}{x_1} = -1{\cdot}573$	—

TABLE II—VALUES OF FREQUENCY PARAMETERS AND AMPLITUDE RATIOS FOR THE TRUSS SHOWN IN FIG. 3.

Mode of vibration	Lumped Inertia Method		Lumped Mass Method		Lumped inertia method by condensation of Matrices Frequency Parameter k_T
	Frequency Parameter k_T	Amplitude Ratios	Frequency Parameter k_T	Amplitude Ratios	
1	0·392	$\frac{x_2}{x_1} = 1{\cdot}161$	0·385	$\frac{x_2}{x_1} = 1{\cdot}157$	0·399
		$\frac{x_3}{x_1} = 2{\cdot}651$		$\frac{x_3}{x_1} = 1{\cdot}875$	
2	1·397	$\frac{x_2}{x_1} = 0{\cdot}764$	0·991	$\frac{x_2}{x_1} = 0{\cdot}851$	1·381
		$\frac{x_3}{x_1} = -1{\cdot}793$		$\frac{x_3}{x_1} = -1{\cdot}706$	
3	2·636	$\frac{x_2}{x_1} = -0{\cdot}901$	2·130	$\frac{x_2}{x_1} = -1{\cdot}299$	
		$\frac{x_3}{x_1} = -0{\cdot}183$		$\frac{x_3}{x_1} = 0{\cdot}158$	

Conclusions

In long trusses of flexibilities of individual elements in bending are negligible compared to the flexibility of the truss as a whole. Hence for simplicity, in this analysis the flexibilities of individual elements in bending have been neglected. Besides the members are assumed pin-jointed without play and the spanwise displacements and inertias are ignored.

Once these assumptions are made, the analysis presented in the lumped inertia method is exact.

In the lumped mass method half the mass of each member is lumped at its ends. The inertia force at a joint developed by these lumped masses is different from the actual inertia force at that joint and the existence of cross inertias is not brought out. These introduce errors which are considerable at higher frequencies.

The procedure outlined in this paper can be directly used to obtain the natural frequencies of bridges without much error. The presence of redundant members does not present any difficulties. The method can also be adapted to take into account the response of individual members.

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APPENDIX—Notation

The following letter symbols have been adopted for use in this paper :

A_i = area of cross section of i^{th} member ;

b = a matrix appearing in equation (12) ;

b_o = projected length of the member ;

B = a matrix appearing in equation (14) ;

E_i = Young's modulus of i^{th} member;

F_{ij} = displacement at the joint i due to unit load at the joint j ;

F^*_{ij} = displacement at the joint i due to unit load at the joint j in the starred system ;

I = unit matrix ;

I_i = inertia force at i ;

K_{ij} = stiffness matrix ;

k_T = frequency parameter given by $w\sqrt{\frac{mL^2}{AE}}$;

l = length of each bay ;

l_i = length of i^{th} member ;

L = length of the truss ;

m = mass per unit length of the member ;

$m^{(ij)}_{ij}$ = inertia at the joint i due to unit acceleration at the joint j due to member (ij) written as superscript (in superscript i ≠ j) ;

m_{ij} = inertia at the joint i due to unit acceleration at the joint j ;

m^*_{ij} = inertia at the joint i due to unit acceleration at the joint j in the starred system ;

M_{ij} = mass of the member (ij) ;

$M^{(ij)}_i$ = mass at i of the member ij in the lumped mass method ;

M_{ii} = lumped mass at i ;

$\overline{N}_{ij}$ = forces in the member i due to unit load in the direction j ;

x, y = Cartesian coordinates ;

$\ddot{x}$ = acceleration in the x direction ;

x = displacement at the joint i along the x axis ;

$\ddot{x}_i$ = acceleration in the x direction at the joint i ;

x^*_i = displacement at the joint i along the x axis in the starred system ;

(alpha) α = angle between the axis of a member and the y axis ; and

(Omega) ω = natural frequency.

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