Torsional Vibrations of Multi-Cell Tubes

by

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Summary

This paper presents torsional vibration analysis of multi-cell tubes. The main difference between the single and multi-cell tube analysis lies in the determination of the cross-sectional constants. Derivation of cross-sectional constants, involved in the governing equations, is illustrated by considering a four-cell tube. The improvement, in the estimation of the natural frequency by the present proposal, is brought out by comparing the theoretical value with the experimental observation in a specific case.

1. Introduction

Aeronautical engineers have been interested in natural vibrations of airplanes ever since they flew. With the advent of higher speeds and low aspect ratio wings, the problem of aircraft vibrations has assumed increased importance. Conventional aircraft wings are tubular in construction; normally they have more than one cell. Shell theories are too complicated to use whereas beam theories are inadequate owing to substantial influence of secondary effects such as transverse shear, shear lag and longitudinal inertia.

Over the years considerable work has been done to adapt the beam theory suitably for the analysis of tubes by incorporating transverse shear and shear lag. Most of the previous workers dealt with rectangular cross-section only and the influence of shear lag was not adequately considered. In a recent paper, Krishna Murty and Joga Rao[1] proposed a generalised theory for vibrations of cylindrical tubes with arbitrary cross-section; simpler governing equations to various orders of approximation are also presented in the same reference. Application of this theory to torsional oscillations of some doubly symmetric tubes can be seen in Ref. 2.

Often aircraft wings are of multi-cell construction. Notwithstanding the intense research activity on the dynamic behaviour of single cell tubes, natural vibrations of multi-cell tubes do not seem to have received adequate consideration. In this paper, we adapt the single cell tube theory proposed in Ref. 1, to natural vibrations of multi-cell tubes. The main difference in this adaptation lies in the determination of cross-sectional constants. In order to bring out the improvement effected by the

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present proposal, in the estimation of natural frequencies, a cantilever four-cell tube is fabricated. The fundamental frequency as estimated by the present theory is compared with the experimentally observed frequency and the agreement is found to be good.

Notation

A		area enclosed in each cell
a		width of each cell
$B_{\theta\theta}$, $b_{\theta\theta}$, $\bar{b}_{\theta\theta}$		cross-sectional constants defined by equations (5)
b		half the depth of the tube
D_o, D_1, D_2		defined by equations (10) and (13)
E		Young's modulus
G		shear modulus
I_p		polar moment of inertia
J_B		Bredt-Batho torsion constant for multi-cell tube
k^2	=	E/G
k_s^2	=	$\omega^2 \rho L^2/G$
k_w^2	=	k_s^2 / k^2
$k_{\theta_B}^2$	=	$k_s^2 I_p / J_B$
k_θ^2	=	$k_s^2 I_p / S_{\theta\theta}$
L		length of the tube
p		perpendicular distance of the tangent at any point on the periphery of the tube from the origin
q_M		shear flow in M^{th} cell
r		radial distance of any point on the tube walls from the origin
$S_{\theta\theta}$, $\bar{S}_{\theta\theta}$		cross-sectional constants defined by equations (5)
s		peripheral coordinate
t		thickness of the tube walls
T		torque applied
w		warping displacement
$\bar{w}_1$		variation of warping displacement along s direction
z		axial coordinate
Δ_1, Δ_{12}		defined by equations (A8)
$\delta(s-a)$		Dirac-delta function
θ		rotational displacement
λ		defined by equation (18)
λ_1, λ_2		roots of equation (17)
μ		defined by equation (18)
ν		defined by equation (20)

ϕ variation of warping displacement along z direction

ω natural frequency rad./sec.

2. Governing Equations

The main assumption in this paper is that the tube is maintained by a closely spaced system of rigid and massless diaphragms, which are rigid in their own planes, but offer no resistance to out of plane movements. Besides this, the tube is considered to be cylindrical with thin walls and the Poisson's ratio effect is ignored.

In this paper, we consider doubly symmetric multi-cell tubes and so there is no coupling between flexural and torsional oscillations. The well-known equations for torsional oscillations of uniform shafts can be adapted to multi-cell tubes as*

$$\frac{d^2\theta}{dz^2} + k^2_{\theta B}\,\theta = 0 \tag{1}$$

The expression for $k^2_{\theta B}$ contains J_B, the Bredt-Batho torsion constant for multi-cell tube. Derivation of J_B for a four-cell tube is presented in the appendix.

The use of equation (1) implies neglect of axial constraint stresses. The influence of axial constraint stress on the natural frequency becomes important when the tube is short[2] and can be considered by adapting the non-dimensional first order approximation equation (discussed in detail in Ref. 2) as**

$$S_{\theta\theta}\,\frac{d^2\,\theta}{d\,z^2} - b_{\theta\theta}\,\frac{d\,\phi}{d\,z} + k^2_s\,I_P\,\theta = 0$$

and

$$k^2\,B_{\theta\theta}\,\frac{d^2\,\phi}{d\,z^2} + \left(\bar{S}_{\theta\theta}\,\frac{d\theta}{dz} - \bar{b}_{\theta\theta}\,\phi\right) + k^2_s\,B_{\theta\theta}\,\theta = 0 \tag{2}$$

In the derivation of these equations the expression for warp is assumed to be

$$w\,(z,\,s) = -\,\bar{w}_1\,(s)\,\phi\,(z) \tag{3}$$

where $\bar{w}_1$ has to be evaluated from the equation

$$\frac{d^2\bar{w}_1}{ds^2} = \frac{dp}{ds} \tag{4}$$

The deviation of $\bar{w}_1$ involves an effective use of continuity of warping displacement and the condition of zero net axial force. The expression for $\bar{w}_1$ in the case of four-cell tube is deduced in the next section. The cross-sectional constants involved in equations (2) are defined by

$$S_{\theta\theta} = \int_C p^2\,t\,ds$$

*All the symbols in this paper, except those in the appendix, are in non-dimensional form; to get non-dimensional quantities all the quantities, whose dimension is length, are divided by the length of the tube.

**These equations can also be seen in section 1.12 of Ref. 1.

$$B_{\theta\theta} = \int_c \bar{w}_1^2 \, t \, ds$$

$$\bar{b}_{\theta\theta} = \int_c \left(\frac{d\bar{w}_1}{ds} \right)^2 t \, ds$$

$$b_{\theta\theta} = \bar{S}_{\theta\theta} = \int_c p \frac{d\bar{w}_1}{ds} t \, ds$$

$$I_p = \int_c r^2 \, t \, ds \tag{5}$$

where $\int_c$ denotes integration over all the tube walls in the cross-section. The boundary condition statement in this mathematical model can be, at each end

either $\theta = 0$ or $S_{\theta\theta} \frac{d\theta}{dz} - b_{\theta\theta}\phi = 0$

either $\phi = 0$ or $\frac{d\phi}{dz} = 0$ (6)

3. Cross-Sectional Constants

The first step in the determination of cross-sectional constants is the derivation of the warp function $\bar{w}_1$. Since the cross-section of the cantilever four-cell tube under consideration (see Fig. 1) is doubly symmetric it is sufficient to consider only

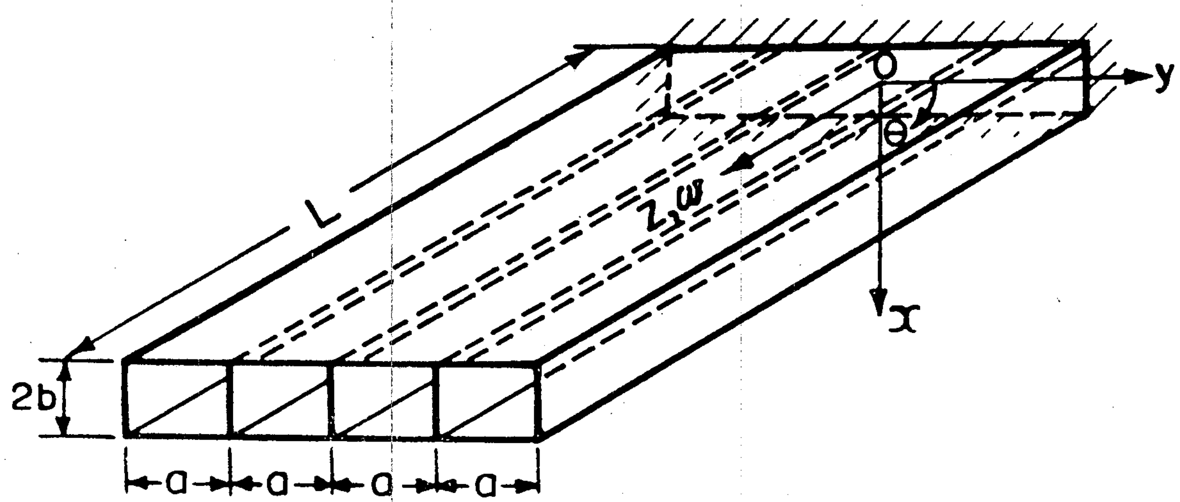

Fig. 1 A Cantilever Four—Cell Tube

one quarter of the tube (regions O′BC and O″AB in Fig. 2) for the derivation of $\bar{w}_1$. The variation of p in the regions O′BC and O″AB can be seen in Figs. (3) and (4). It is obvious from Fig. 3 that the expression for dp/ds in the region O′BC can be written as

$$\frac{dp}{ds} = (a - b)\, \delta\, (s - a) \tag{7}$$

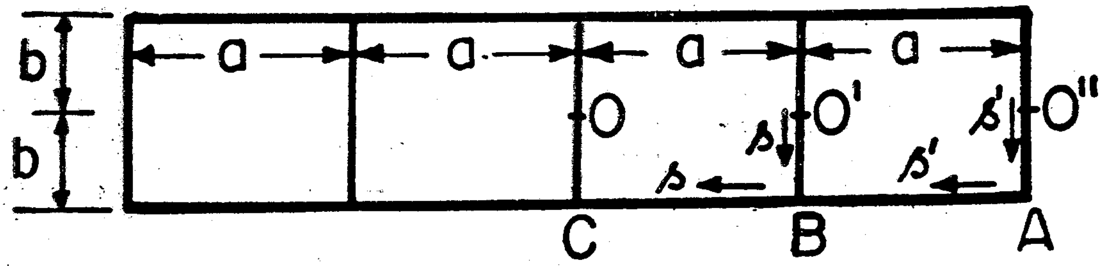

Fig. 2 A Cross—Section of Four—Cell Tube

where δ is Dirac-delta function defined as

$$\delta(s-a)=0 \quad \text{when } s \neq a$$
$$=1 \quad \text{when } s = a \qquad (8)$$

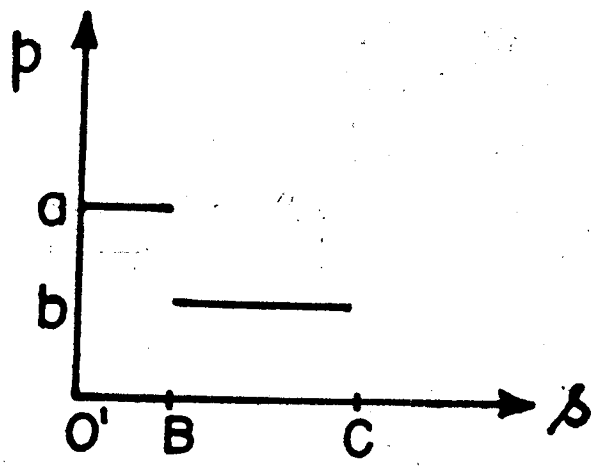

Fig. 3 Variation of p with s in the region O'BC

Using equation (8) in equation (4) and using the conditions that warp is zero at 0′ and C (see Fig. 2), the expression for $\bar{w}_1$ in this region can be deduced as

$$\bar{w}_1 = D_o \frac{s}{b} \quad \text{when } 0 \leq s \leq b$$
$$= D_o \frac{a+b-s}{a} \quad \text{when } b \leq s \leq a+b \qquad (9)$$

where D_o is the value of $\bar{w}_1$ at corner B. which is given by

$$D_o = (\bar{w}_1) \text{ at B} = \frac{ab\,(a-b)}{(a+b)} \qquad (10)$$

Similarly, in the region O′′AB, the expression for $\frac{dp}{ds}$ is (see Fig. 3)

$$\frac{d\,p}{d\,s} = (2a-b)\,\delta\,(s'-b) \qquad (11)$$

From equations (11) and (4) and in conjunction with the conditions of zero warp at O″ and B (see Fig. 2) one gets the expression for $\bar{w}_1$ in this region as

$$\bar{w}_1 = D_1 \frac{s'}{b} \quad \text{when } a \leq s' \leq b$$

$$= D_1 \frac{D_2 + b - s'}{D_2} \quad \text{when } b \leq s' \leq a + b \tag{12}$$

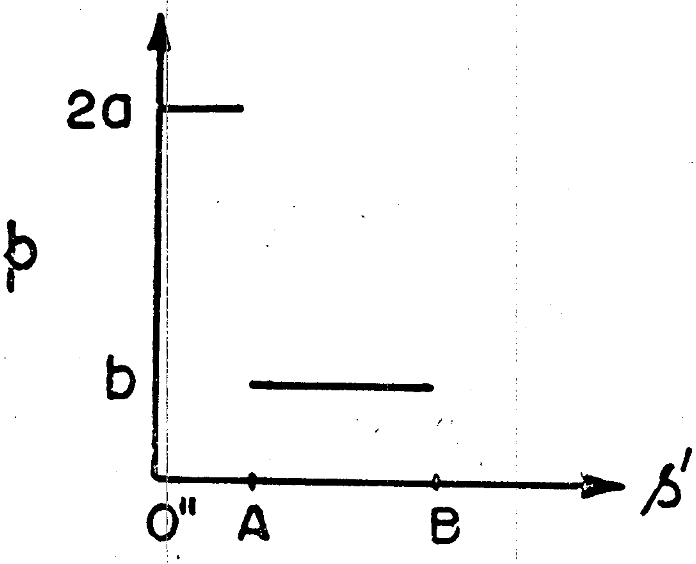

Fig. 4 Variation of p with s′ in the region O″AB

where

$$D_1 = \frac{(2a - b)\, D_2\, b}{b + D_2}$$

and

$$D_2 = \frac{(2a - b)(a + b)\, a + ab\,(a - b)}{(2a - b)(a + b) - a(a - b)} \tag{13}$$

The peripheral distribution of $\bar{w}_1$ can be seen in Fig 5.

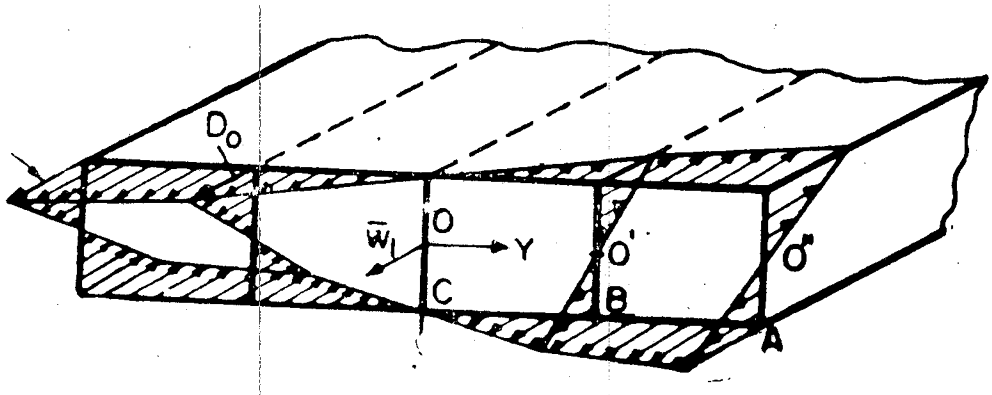

Fig. 5 Distribution of $\bar{w}_1$ in Four—Cell Tube

Having determined $\bar{w}_1$, cross-sectional constants may be determined by substituting $\bar{w}_1$ in equations (5) as

$$B_{\theta\theta} = \tfrac{4}{3}\, t \left[D_o^2 (a + b) + D_1^2 \left\{ a + D_2 - \frac{(D_2 - a)^3}{D_2^2} \right\} \right]$$

$$\bar{b}_{\theta\theta} = 4t \left[D_0^2 \frac{a + b}{a\, b} + D_1^2 \left(\frac{D_2^2 + ab}{bD_2^2} \right) \right]$$

$$b_{\theta\theta} = \bar{S}_{\theta\theta} = 4t \left[D_o (a - b) + D_1 \left(2a - \frac{ab}{D_2} \right) \right]$$

$$S_{\theta\theta} = 4\, abt\, (2b + 5a) \tag{14}$$

and the polar moment of inertia becomes

$$I_p = \tfrac{1}{3}\, t\, [32a^3 + 24ab^2 + 60a^2b + 10b^3] \tag{15}$$

4. Cantilever Four-cell Tube

The boundary conditions, in the case of the cantilever tube shown in Fig. 1 are

$$\text{at } z = 0 \qquad \theta = \phi = 0$$

and

$$\text{at } z = 1 \qquad S_{\theta\theta} \frac{d\,\theta}{d\,z} - b_{\theta\theta}\, \phi = \frac{d\,\phi}{dz} = 0 \tag{16}$$

The solution of equations (2) in conjunction with above boundary conditions is presented below. Let $-\lambda_1^2$ and λ_2^2 be the roots of the quadratic equation

$$\xi^2 + \left(k_\theta^2 + k_w^2 - \lambda^2 \right) \xi + k_\theta^2 \left(k_w^2 - \mu^2 \right) = 0 \tag{17}$$

where

$$\mu^2 = \frac{\bar{b}_{\theta\theta}}{k^2 B_{\theta\theta}}$$

$$\lambda^2 = \frac{1}{k^2 B_{\theta\theta} S_{\theta\theta}} \left(S_{\theta\theta}\, \bar{b}_{\theta\theta} - b_{\theta\theta}\, \bar{S}_{\theta\theta} \right) \tag{18}$$

Case 1. $\lambda_2^2 > 0$; the characteristic equation from which the frequency spectrum can be computed is

$$R_1 \lambda_1 (\lambda_1 + R_1 \upsilon^2) - R_1 \lambda_2 (- \lambda_2 + R_2 \upsilon^2)$$
$$+ \cos \lambda_1 \cosh \lambda_2 \left[R_1^2 \lambda_1 \left(\frac{\lambda_2}{R_2} - \upsilon^2 \right) - R_2 \lambda_2 (\lambda_1 + R_1 \upsilon^2) \right]$$
$$+ \sin \lambda_1 \sinh \lambda_2 \left[R_1 \lambda_2 (\lambda_1 + R_1 \upsilon^2) - R_1 \lambda_1 (-\lambda_2 + R_2 \upsilon^2) \right] \tag{19}$$

Note that υ^2 is defined as

$$\upsilon^2 = \frac{b_{\theta\theta}}{S_{\theta\theta}} \tag{20}$$

The mode shape subject to the condition of unit free end rotation is

$$\theta = R_4 \left[\frac{R_2 R_3}{R_1} \sin \lambda_1 z - \cos \lambda_1 z + R_3 \sinh \lambda_2 z + \cosh \lambda_2 z \right]$$

$$w = - \bar{w}_1 R_4 \left[- R_1 \sin \lambda_1 z - R_2 R_3 \cos \lambda_1 z + R_2 \sinh \lambda_2 z + R_2 R_3 \cosh \lambda_2 z \right] \qquad (21)$$

where

$$R_1 = \frac{- \lambda_1^2 + k_\theta^2}{\lambda_1 \upsilon^2}$$

$$R_2 = \frac{\lambda_2^2 + k_\theta^2}{\lambda_2 \nu^2}$$

$$R_3 = \frac{- R_2 \lambda_2 \cosh \lambda_2 + R_1 \lambda_1 \cosh \lambda_1}{R_2 (\lambda_1 \sin \lambda_1 + \lambda_2 \sin \lambda_2)}$$

$$R_4 = \frac{1}{\frac{R_2}{R_1} R_3 \sin \lambda_1 - \cos \lambda_1 + R_3 \sinh \lambda_2 + \cosh \lambda_2} \qquad (22)$$

Case 2. $\lambda_2^2 < 0$; The characteristic equation yielding frequency spectrum is

$$H_1 \lambda_1 (\lambda_1 + H_1 \upsilon^2) + H_1 \lambda_2 (\lambda_2 + H_2 \upsilon^2)$$
$$- \cos \lambda_1 \cos \lambda_2 \left[H_2 \lambda_2 (\lambda_1 + H_1 \upsilon^2) + H_1^2 \lambda_1 \left(\frac{\lambda_2}{H_2} + \upsilon^2 \right) \right]$$
$$- \sin \lambda_1 \sin \lambda_2 [H_1 \lambda_2 (\lambda_1 + H_1 \upsilon^2) + H_1 \lambda_2 (\lambda_2 + H_2 \upsilon^2)] = 0 \qquad (23)$$

and the mode shape subjected to the condition unit tip rotation is

$$\theta = H_4 \left[- \frac{H_2 H_3}{H_1} \sin \lambda_1 z - \cos \lambda_1 z + H_3 \sin \lambda_2 z + \cos \lambda_2 z \right]$$

$$w = - \bar{w}_1 H_4 \left[- H_1 \sin \lambda_1 z + H_2 H_3 \cos \lambda_1 z + H_2 \sin \lambda_2 z - H_2 H_3 \cos \lambda_2 z \right] \qquad (24)$$

where

$$H_1 = \frac{- \lambda_1^2 + k_\theta^2}{\lambda_1 \upsilon^2}$$

$$H_2 = \frac{- \lambda_2^2 + k_\theta^2}{\lambda_2 \upsilon^2}$$

$$H_3 = \frac{H_1 \lambda_1 \cos \lambda_1 - H_2 \lambda_2 \cos \lambda_2}{H_2 (\lambda_2 \sin \lambda_2 - \lambda_1 \sin \lambda_1)}$$

$$H_4 = \frac{1}{- \frac{H_2 H_3}{H_1} \sin \lambda_1 - \cos \lambda_1 + H_3 \sin \lambda_2 + \cos \lambda_2} \qquad (25)$$

5. Results and Discussion

In order to bring out the improvement in the estimation of the natural frequency by the present proposal, a typical cantilever four-cell tube with $L = 21.875''$, $a = 2.0625''$ and $2b = 1.3125''$ (see Fig. 1) is fabricated. The thickness of the tube walls is $0.015625''$. Young's modulus of elasticity is 10^7 psi and $E/G = 2.65$. First two natural frequencies of the above tube were calculated theoretically. The fundamental was also determined experimentally. Table 1 shows the comparison.

Table 1—Comparison of frequencies

Mode	Frequency in cycles per second		
	Elementary theory equation (1)	Present work equation(2)	Experiment
First	388	405	415
Second	1164	1230	—

It can be seen from the above table that, in the case of the fundamental the present theory reduces the discrepancy between theory and experiment, from 4.6% to 2.5%. Although, we are unable to determine second frequency experimentally due to limitations of the experimental equipment, it is apparent from Table 1 that the present theory estimates second frequency more accurately than the elementary theory.

It may be mentioned here that in order to improve the accuracy further one can adapt higher order approximations presented in Ref. 1. Nevertheless, in view of the extra work involved and in view of the fact that the present proposal yields results to a satisfactory degree of accuracy (see also Ref. 2), it may not be worthwhile attempting higher order approximations.

In the tube under consideration the aspect ratio is about 5.25. As reported in Ref. 2 the influence of warping rigidity on the frequency increases with the reduction of the aspect ratio. In the modern wing structures whose aspect ratio is smaller than 5.25, the influence of warping rigidity is more than that noticed in the present example. As such this theory is expected to assume importance in the natural vibration analysis of modern aircraft wings.

6. Conclusions

In this paper we have extended the theory for the determination of natural vibrations of single cell tubes proposed in Ref. 1 to the case of multi-cell tubes. The difference is primarily in the determination of cross-sectional constants. Comparison of theoretical result with experimentally observed frequency indicates that the present proposal yields results to a satisfactory degree of accuracy. This theory can easily be extended to the cases of flexural and coupled oscillations of multi-cell tubes.

7. References

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APPENDIX A

Bredt-Batho Torsion Constant

From the general theory of wing stressing[3] it will be possible to deduce the Bredt-Batho torsion constant for a multi-cell tube. However for the sake of completeness we intend presenting here the derivation of the Bredt-Batho torsion constant for a four-cell tube. Fig. 6 shows the cross-section of a multi-cell tube subjected to a uniform torque loading T. The expression for the shear strain may be written as

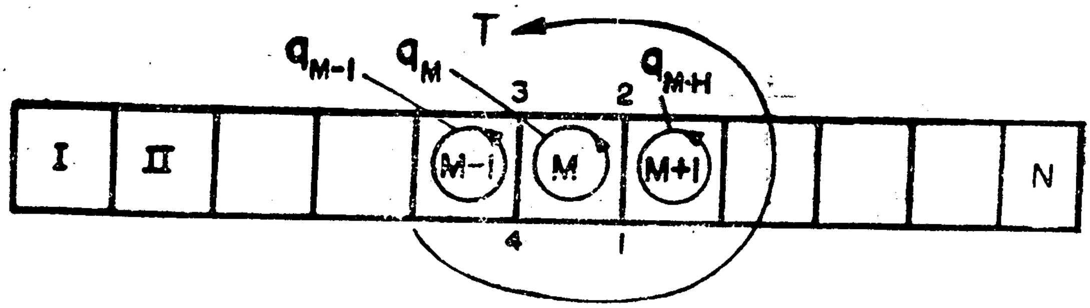

Fig. 6 A Multi—Cell Tube

$$\frac{\partial w}{\partial s} = \frac{q}{Gt} - p\,\frac{d\theta}{dz} \tag{A1}$$

continuity of w in each cell demands

$$2A_M\,G\,\frac{d\theta}{dz} = \oint q\,\frac{ds}{t} \tag{A2}$$

where A_M is the area of the M^{th} cell. Assuming the circulating shear flow to be constant in each cell, equation (A2) may be written as

$$2A_M\,G\frac{d\theta}{dz} = -\,q_{M-1}\,\Delta_{M-1,M} + q_M\,\Delta_M - q_{M+1}\,\Delta_{M,M+1}$$

$$M = 1, 2, 3 \ldots N \tag{A3}$$

where

$$\Delta_{M-1,M} = \int_3^4 \frac{ds}{t}$$

$$\Delta_M = \oint_M \frac{ds}{t}$$

$$\Delta_{M,M+1} = \int_1^2 \frac{ds}{t} \tag{A4}$$

The total torque carried by the tube is the sum of the torques carried by each cell. Hence

$$T = \sum_{M=1,2,\ldots}^{N} 2A_M q_M \tag{A5}$$

equations (A3) and (A5) are sufficient for the determination of shear flows and the torsion constant.

In the case of a four-cell tube subjected to torque, the shear flows are given by (see Fig. 7)

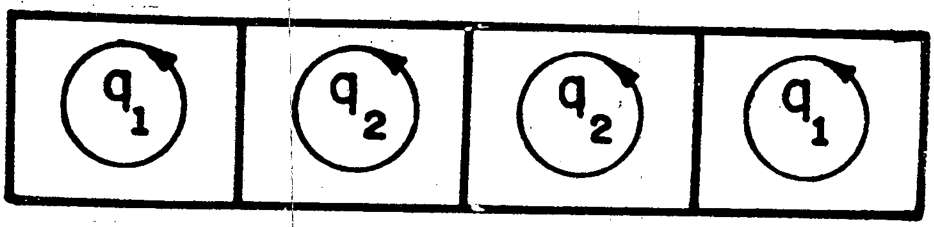

Fig. 7 Shear flows in Doubly—Symmetric Four—Cell Tube

$$q_1 = 2AG \frac{d\theta}{dz} \frac{\Delta_1}{\Delta_1^2 - \Delta_{12}^2 - \Delta_1 \Delta_{12}}$$

$$q_2 = 2AG \frac{d\theta}{dz} \frac{\Delta_1 + \Delta_{12}}{\Delta_1^2 - \Delta_{12}^2 - \Delta_1 \Delta_{12}} \tag{A6}$$

Using equation (A6) in equations (A5) and rewriting, one obtains the expression for Bredt-Batho torsion constant for a four-cell tube as

$$J_B = \frac{T}{G \frac{d\theta}{dz}} = \frac{8A^2 (2\Delta_1 + \Delta_{12})}{\Delta_1^2 - \Delta_{12}^2 - \Delta_1 \Delta_{12}} \tag{A7}$$

where

$$A = 2ab$$

$$\Delta_1 = \frac{2a + 4b}{t}$$

$$\Delta_{12} = \frac{2b}{t} \tag{A8}$$