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General Theory of Vibrations of Cylindrical Tubes

PART—IV : UNCOUPLED TORSIONAL VIBRATIONS OF OPEN TUBES†

by

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Summary

Based on the ease in satisfying the free edge boundary condition, open tubes are classified into two types, A and B. Type B tubes have more complicated free-edge boundary condition than that of type A tubes. An alternate formulation is evolved for type B tubes; this makes the free-edge boundary condition as simple as in type A tubes.

A simply-supported open tube with the cross-section given by $p = \frac{S}{2\pi} \sin \frac{2\pi s}{S}$, *is analysed exactly as well as by using approximation equations; errors in using the first order approximation equations are discussed. A simply-supported open tube of I-section, representative of type B tubes, is analysed by using first and second approximation equations.*

ADDITIONAL NOTATION**

a, b	—	Typical cross-sectional dimensions of the tube
K_0^4	=	$\frac{\omega^2 \rho L^2 I_p}{E B_{\theta\theta}}$
P	=	a/b
Q	=	L/a

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**In addition to the notation used in Part 1.

R	=	b/t
ε_f	—	Error in natural frequency (Eq. 4.43)
ε_m	—	Error in mode shape (Eq. 4 44)
ζ_3, ζ_4	—	Defined by Eq. (4.81)
$\lambda_1^2, \lambda_2^2, \lambda_3^2$	—	Roots of the polynomial (see Eqs. (4.36) and 4.71)).
μ_θ^2	=	$\dfrac{J_s}{k^2 B_{\theta'\theta'}}$
μ_3^2	—	Defined by Eq. (4.17)
μ_4^2, μ_5^2	—	Defined by Eq. (4.65)
$\upsilon_2^2, \upsilon_3^2$	—	Defined by Eq. (4.65)

4.0 Introduction

For open tubes of doubly-symmetric or doubly anti-symmetric cross-sections, we have uncoupled flexural and torsional vibrations, but each mode, generally involves warping motion also. Owing to the large warping associated with open tubes, beam theories are inadequate. Although the rotation is a function of one variable only, warping displacement depends on two variables, and hence the analysis looses simplicity.

Nevertheless the simplifying assumption of zero centre line shear strains results in a convenient formulation for open tubes Gere's equations[29*] are based on this assumption and they may be good enough for many an application like long open tubes. But short open tubes like cut-outs of aircraft wing structures require the consideration of centre-line shear strains also.

The proposed theory consists in developing the equations governing the natural vibrations of thin-walled cylindrical tubes of arbitrary cross-section, but with the assumption of CSRMD. Kantorovich form of Rayleigh-Ritz method is invoked to yield equations of various orders of approximations. An elegant splitting of warping displacements results in convenient incorporation of centre-line shear strains. Formulation of various equations is presented in Part I.

As different from the analysis of closed tubes, the analysis of open tubes involves the satisfaction of zero shear strain condition at the free-edges. Warping displacement has to satisfy free-edge condition

$$\frac{\partial w}{\partial s} + p\frac{d\theta}{dx} = 0 \qquad \ldots (4.1)$$

*References are given in Part I.

and the end conditions

$$w = 0 \text{ or } \frac{\partial w}{\partial z} = 0 \qquad \dots (4.2)$$

However, if $p = 0$ at free-edges, Eq. (4.1) takes a simple form. The selection of the admissible functions for the method of section 1.6* is also simple. Such tubes are classified as type A (See Fig. 4.1). Nevertheless if $p \neq 0$ at free-edge, it is not easy

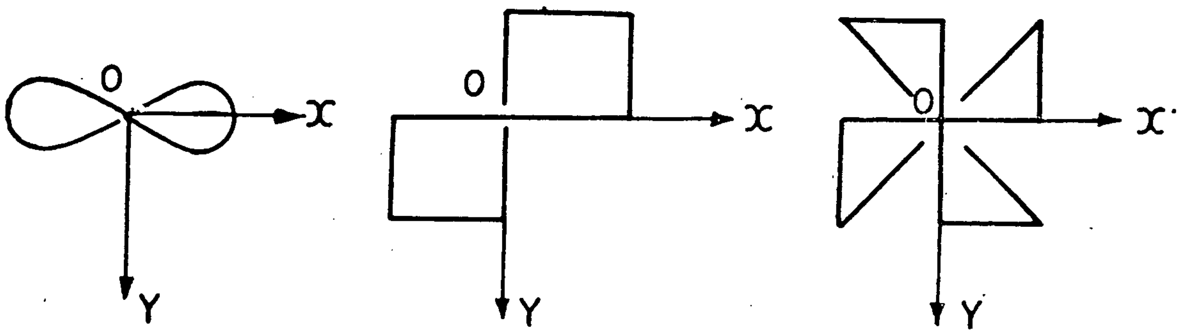

Fig. 4.1 : Open tubes of type A.

to select admissible fuuctions satisfying the free-edge condition. Such tubes are classified as type B (See Fig. 4.2). A modification in the equations, reduces the

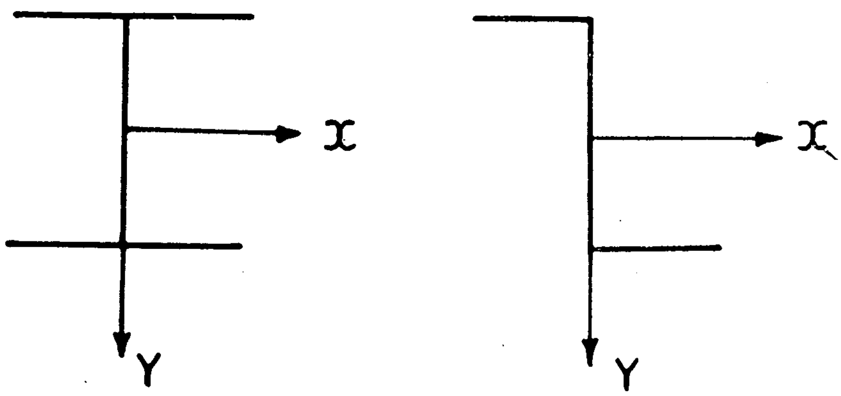

Fig. 4 2 : Open tubes of type B.

application of the method of solution of section 1.6 to type B tubes as easy and straight forward as for type A tubes; these modifications are discussed in section (4.4).

This paper includes an exact solution of simply supported open tube of type A with the cross-section given by $p = \frac{S}{2\pi} \sin \frac{2\pi s}{S}$. The same tube is analysed using first order approximation equations and the error in the natural frequency is discussed. A simply supported tube of I—section, representative of type B tubes, is analysed using first and second order approximation equations.

*Equations and sections referred to as (1.) can be seen in Part I.

4.1 Governing equations for open tubes of type A—rigorous formulation

The governing equations in this case are the same as those for closed tubes (section 2.1)*; one of the boundary conditions here is the zero shear condition at free edges. The equations of equilibrium are

$$\frac{d^2\theta}{dz^2} + k_\theta^2 \theta = -\frac{1}{S_{\theta\theta}} \frac{d}{dz} \oint \frac{\partial w}{\partial s}\, p\ t\ ds$$

$$k^2 \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial s^2} + k_s^2 w = -\frac{dp}{ds}\frac{d\theta}{dz} \qquad \ldots (4.3)$$

and the boundary conditions at each end are

$$\text{either } \theta = 0 \text{ or } \frac{d\theta}{dz} + \frac{1}{S_{\theta\theta}} \oint \frac{\partial w}{\partial s}\, pt\ ds = 0$$

$$\text{either } w = 0 \text{ or } \frac{\partial w}{\partial z} = 0 \qquad \ldots (4.4)$$

and

$$\frac{\partial w}{\partial s} = 0 \text{ at free edges.} \qquad \ldots (4.5)$$

4.2 Simply supported open tube with the boundary of the cross section given by $p = \frac{S}{2\pi} \sin \frac{2\pi s}{S}$ — Exact solution

The boundary conditions in this case are

$$\theta(0) = \theta(1) = 0 \qquad \ldots (4.6)$$

$$\frac{\partial w}{\partial s}(0, s) = \frac{\partial w(1, s)}{\partial s} = 0 \qquad \ldots (4.7)$$

$$\frac{\partial w}{\partial s}(z, 0) = \frac{\partial w(z, S)}{\partial s} = 0 \qquad \ldots (4.8)$$

Eqs. (4.7) and (4.8) suggest expression for w in the form

$$w = \sum_{m=1, 2, 3, \ldots}^{\infty} \sum_{n=1, 2, 3, \ldots}^{\infty} A_{mn} \cos(m\pi z) \cos \frac{2n\pi s}{S} \qquad \ldots (4.9)$$

Substitution of Eq. (4.9), in the first equation of Eqs. (4.3) yields

$$\theta'' + k_\theta^2 \theta = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{2mn\pi^2}{SS_{\theta\theta}} \sin(m\pi z) \oint pt \sin \frac{2n\pi s}{S} ds \qquad \ldots (4.10)$$

*Equations or sections referred to as (2.) appeared in Part II.

From geometry, we have

$$p = \frac{S}{2\pi} \sin \frac{2\pi s}{S} \qquad \dots (4.11)$$

Using Eq. (4.11), Eq. (4.10) can be written as

$$\theta'' + k_\theta^2 \theta = -\sum_{m=1,2,3,\dots}^{\infty} \sum_{n=1,2,3,\dots}^{\infty} A_{mn}\, \delta_{1n} \frac{mn\pi St}{2S_{\theta\theta}} \sin (m\pi z) \qquad \dots (4.12)$$

where δ_{1n} is the Kronecker delta.

The solution of Eq (4.12) is

$$\theta = A_1 \sin k_\theta z + A_2 \cos k_\theta z - \sum_{m=1,2,3,}^{\infty} \sum_{n=1,2,3,}^{\infty} A_{mn} \frac{\delta_{1n}\, mn\pi \sin (m\pi z)}{2S_{\theta\theta}\left(-m^2\pi^2 + k_\theta^2\right)} \qquad \dots (4.13)$$

Satisfaction of the end conditions on θ namely Eqs. (4.6) yields

$$A_1 = A_2 = 0 \qquad \dots (4.14)$$

Hence

$$\theta = -\sum_{m=1,2,3,\dots}^{\infty} \sum_{n=1,2,3,\dots}^{\infty} A_{mn} \frac{\delta_{1n}\, mn\pi St \sin (m\pi z)}{2S_{\theta\theta}\left(-m^2\pi^2 + k_\theta^2\right)} \qquad \dots (4.15)$$

Since

$$S_{\theta\theta} = \oint p^2 t\, ds + \tfrac{1}{3} \oint t^3\, ds = \frac{S^3 t}{8\pi^2}\left(1 + \mu_3^2\right) \qquad \dots (4.16)$$

where $\mu_3^2 = \dfrac{8\pi^2}{3R^2}$, $R = S/t$ $\qquad \dots (4.17)$

we have

$$\theta = -\sum_{m=1,2,3\dots}^{\infty} \sum_{n=1,2,3\dots}^{\infty} A_{mn} \frac{4\,\delta_{1n}\, mn\pi^3}{S^2\left(1+\mu_3^2\right)\left(-m^2\pi^2 + k_\theta^2\right)} \sin (m\pi z)$$

Substituting Eq. (4.18) in the second equation of Eqs. (4.3) and using the orthogonal properties of Fourier series one finds

$$A_{mn}\left(-k^2 m^2 \pi^2 - \frac{4n^2\pi^2}{S^2} + k_s^2\right) - \frac{4m^2 n \pi^4 \delta_{1n}}{S^2\left(1+\mu_3^2\right)\left(-m^2\pi^2 + k_\theta^2\right)} = 0$$

The natural frequencies are obtained for nontrivial values of A'_ms as

$$\left(- k^2 m^2 \pi^2 - \frac{4n^2 \pi^2}{S^2} + k_s^2 \right) - \frac{4m^2 n \pi^4 \delta_{1n}}{S^2\left(1 + \mu_3^2\right)\left(- m^2 \pi^2 + k_\theta^2\right)} = 0$$

$$m, n = 1, 2, 3, \ldots \infty \qquad \ldots (4.20)$$

and the corresponding mode shapes are

$$w = A_{mn} \cos (m\pi z) \cos \frac{2n \pi s}{S}$$

$$\theta = - A_{mn} \frac{4 \delta_{1n} mn \pi^3}{S^2\left(1 + \mu_3^2\right)\left(- m^2 \pi^2 + k_\theta^2\right)} \sin (m\pi z) \qquad \ldots (4.21)$$

Noticing that $\mu_3^2 \ll 1$ in most of the practical tubes, it can be readily seen that the numerical results are going to be nearly same if μ_3 is set equal to zero. Numerical results of a analogous equation (Eq. (3.19))* are discussed in section (3.2). Hence these will not be repeated here.

4.3 Open tubes of type A—first order approximation equations

The governing equations in this case are the same as those for closed tubes (section 2.3) ; however, the open edge condition

$$\frac{d\bar{w}_1}{ds} = 0$$

has to be incorporated while evaluating $\bar{w}_1$ from Eq. (2.33). It can be shown that solution of the simply supported open tube with the cross-section given by $p = \frac{S}{2\pi} \sin \frac{2\pi s}{S}$ by first order approximation equation of open tubes of type A is same as the n = 1 case of exact solutions (Eqs. (4.20) and (4.21)).

4.4 Governing equations for open tubes of type B—rigorous formulation

As mentioned earlier, the governing equations of closed tubes or open tubes of type A govern open tubes of type B also. But there is some difficulty in using the method of section 1.6. In the case of an open tube, the conditions to be satisfied by w, are

$$\frac{\partial w}{\partial s} + p \frac{d\theta}{dz} = 0 \text{ at open edges} \qquad \ldots (4.22)$$

and

$$\text{either } w = 0 \text{ or } \frac{\partial w}{\partial z} = 0 \text{ at ends.} \qquad \ldots (4.23)$$

* Equations or sections referred to as (3.) can be seen in Part III.

Since the above equations involve θ also, it is not possible to select a function for w to satisfy Eqs. (4 22) and (4.23) at the beginning itself, which is a pre-requisite for using method of section 1.6.

However. it may be mentioned here, if one chooses to use the Rayleigh-Ritz method, the formulation of open tubes of type A can also be used here.

The difficulty in selecting a suitable function for w can be avoided by effecting the transformation of the governing equations using the relationship

$$w = -\bar{w}\frac{d\theta}{dz} + w_1(z, s) \qquad \text{...(4 23a)}$$

where $$\bar{w}_1 = \int p\, ds \qquad \text{...(4.24)}$$

because, in this case, θ will be eliminated from Eq. (4.22). The constant of integration in Eq. (4.24) is obtained from the condition of zero net axial force at any cross section due to the first term of Eq. (4.23a), that is

$$\oint \bar{w}_1\, t\, ds = 0\ . \qquad \text{...(4.25)}$$

The expression for w in Eq. (4.23a), but for w_1 (z, s) is same as that used in the well known torsion-bending analysis of open tubes.

Substituting Eq. (4.23a) in the second of Eqs. (4.3), we have

$$-\bar{z} = k^2\frac{\partial^2}{\partial z^2}\left\{-\bar{w}_1\frac{d\theta}{dz} + w_1\right\} + \frac{\partial^2}{\partial s^2}\left\{-\bar{w}_1\frac{d\theta}{dz} + w_1\right\}$$

$$+ k_s^2\left\{-\bar{w}_1\frac{d\theta}{dz} + w_1\right\} + \frac{dp}{ds}\frac{d\theta}{dz} = 0 \qquad \text{...(4.26a)}$$

and noting that $d\bar{w}_1/ds = p$ we get the second of Eqs. (4.27). Substituting Eq. (4.23a) in the first of Eqs. (4.3) and using the condition that

$$\oint \bar{z}\,\bar{w}_1\, ds = 0 \qquad \text{...(4.26b)}$$

one obtains the first of eqs. (4.27).

Thus the equations of equilibrium are

$$\left(k^2 B_{\theta\theta}\frac{d^4\theta}{dz^4} + k_s^2 B_{\theta\theta}\frac{d^2\theta}{dz^2}\right) - \left(J_s\frac{d^2\theta}{dz^2} + k_s^2 I_p\theta\right)$$

$$= k^2\frac{d^2}{dz^2}\oint\frac{\partial w_1}{\partial z}\bar{w}_1\, t\, ds + k_s^2\frac{d}{dz}\oint w_1\,\bar{w}_1\, t\, ds$$

$$k^2\frac{\partial^2 w_1}{\partial z^2} + \frac{\partial^2 w_1}{\partial s^2} + k_s^2 w_1 = k^2\bar{w}_1\left(\frac{d^3\theta}{dz^3} + k_w^2\frac{d\theta}{dz}\right)$$

$$\text{...(4.27)}$$

and the boundary conditions at each end are (these are obtained from Eqs. (4.4) by adopting the same procedure used in obtaining Eqs. (4.27) from Eqs. (4.3)).

$$\text{either } \theta = 0 \text{ or } k^2 B_{\theta\theta} \frac{d^3\theta}{dz^3} + k_s^2 B_{\theta\theta} \frac{d\theta}{dz} - J_s \frac{d\theta}{dz} - k^2 \frac{d}{dz} \oint \frac{\partial w_1}{\partial z} \bar{w}_1 \, t \, ds - k_s^2 \oint w_1 \bar{w}_1 \, t \, ds = 0$$

$$\text{either } \theta' = 0 \text{ or } k^2 B_{\theta\theta} \theta'' - k^2 \oint \frac{\partial w_1}{\partial z} \bar{w}_1 \, t \, ds = 0 \qquad \ldots(4.28)$$

and

$$\frac{\partial w_1}{\partial s} = 0 \text{ at open edges.} \qquad \ldots(4.29)$$

Cross-sectional constants in Eqs. (4.27) and (4.28) are defined by

$$B_{\theta\theta} = \oint \bar{w}_1^2 t \, ds$$

$$I_p = \oint r^2 t \, ds \qquad \ldots(4.30)$$

$$J_s = \frac{1}{3} \oint t^3 \, ds$$

Now if we choose a suitable expression for w_1 to satisfy Eq. (4.29), we are left with four end conditions in all, namely Eqs. (4.28). Substituting this expression for w_1 in first equation of Eqs. (4.27) and solving for θ, the solution involves four additional arbitrary constants and these can be determined using Eq. (4.28). Thus, we will be having expressions for θ and w_1 satisfying all the boundary conditions and the first equation of Eqs. (4.27). As suggested in section 1.6, either the error in second of Eqs. (4.27) can be minimised, or the orthogonality properties can be used to generate simultaneous equations; using these the eigen values and eigen vectors may be computed.

4.5 Open tubes of type B—first order approximation

The governing equation in this case is obtained by putting $w_1 = 0$ in Eqs. (4.27) and using Eq. (4.26b) as

$$k^2 B_{\theta\theta} \frac{d^4\theta}{dz^4} + k_s^2 B_{\theta\theta} \frac{d^2\theta}{dz^2} - \left(J_s \frac{d^2\theta}{dz^2} + k_s^2 I_p \theta \right) = 0 \qquad \ldots(4.31)$$

and the boundary conditions at each end are

$$\text{either } \theta = 0 \text{ or } k^2 B_{\theta\theta} \frac{d^3\theta}{dz^3} + k_s^2 B_{\theta\theta} \frac{d\theta}{dz} - J_s \frac{d\theta}{dz} = 0$$

$$\text{either } \theta' = 0 \text{ or } \theta'' = 0 \qquad \ldots(4\ 32)$$

Eqs. (4.31) and (4.32) may be simplified to yield

$$\theta^{iv} + k_w^2 \theta'' - \mu_\theta^2 \left(\theta'' + k_\theta^2 \theta \right) = 0 \qquad \ldots \quad (4.31a)$$

the boundary conditions at each end are

$$\text{either } \theta = 0 \text{ or } \theta''' + k_w^2 \theta' - \mu_\theta^2 \theta' = 0 \qquad \ldots \quad (4.32a)$$

$$\text{either } \theta' = 0 \text{ or } \theta'' = 0$$

$$\text{where} \quad \mu_\theta^2 = \frac{J_s}{k^2 B_{\theta\theta}} \qquad \ldots \quad (4.33)$$

The equations used by Gere[29] while dealing with torsional vibrations of open tubes can be obtained from Eqs. (431) and (4.32) by putting $k_s^2 = 0$.

4.6 Simply supported open tube with the boundary of the cross section given by $p = \frac{S}{2\pi} \sin \frac{2\pi s}{S}$ — (solutions obtained by using first order approximation equations developed for type B)

Although the tube under analysis is an open tube of type A, the equations of type B are used in order to assess the error, in using these equations; the error is discussed in section (4.7).

The boundary conditions in this case are (Fig. 4.3)

$$\theta(0) = \theta(1) = 0 \qquad \ldots \quad (4.34)$$

$$\theta''(0) = \theta''(1) = 0;$$

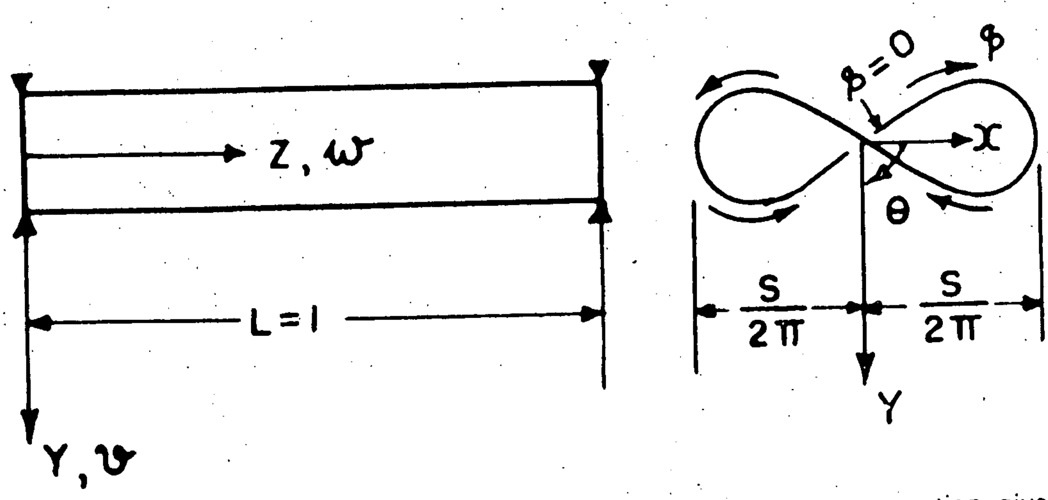

Fig. 4.3 : A simply supported open tube with the boundry of the cross section given by $p = \frac{S}{2\pi} \sin \frac{2\pi s}{S}$

the solution of Eq. (4.31) is

$$\theta = A_1 \sin \lambda_1 z + A_2 \cos \lambda_1 z + A_3 \sinh \lambda_2 z + A_4 \cosh \lambda_2 z \qquad \ldots (4.35)$$

where, $-\lambda_1^2$ and λ_2^2 are the roots of the quadratic equation

$$\xi^2 + \left(k_w^2 - \mu_\theta^2 \right) \xi - \mu_\theta^2 k_\theta^2 = 0 \qquad \ldots (4.36)$$

Using the boundary conditions (4.34), it can be shown that for nontrivial solution

$$\sin \lambda_1 = 0$$

$$\text{or } \lambda_1^2 = m^2 \pi^2 \quad (m = 1, 2, 3, \ldots \infty) \qquad \ldots (4.37)$$

and the mode shapes are given by

$$\theta = A_1 \sin (m\pi z)$$

$$w = A_1 \frac{S^2 m}{4\pi} \cos \frac{2\pi s}{S} \cos (m\pi z) \qquad \ldots (4.38)$$

4.7 Error in using first order approximation equations of open tubes of type B

The first order approximation equations as formulated for open tubes of type B have been widely used in literature for many kinds of open tubes [29, 30]. The object of this section is to estimate the error introduced by using these well known equations already shown to be associated with type B for obtaining solutions for tubes of type A. When this is done besides the error associated with the neglect of w_1 (section 4.5) an additional error is introduced. This is due to the use of equation

$$w = - \bar{w}_1 \frac{d\theta}{dz}$$

(which is used in obtaining first order approximation equation of type B), while a more appropriate expression is

$$w = - \bar{w}_1 \phi_\theta ;$$

ϕ_θ need not necessarily be equal to $d\theta/dz$. ϕ_θ is used in the equation of tubes of type A.

In the case of the example considered in sections (4.2), (4.3) and (4.6) we found that first order approximation equations for tubes of type A gave exact solutions for the case n = 1 (sections (4.2) and (4.3)) while the first order approximation equations derived for tubes of type B introduce an error when applied to tubes of type A. Longitudinal inertia is neglected in the following because its influence is small. The errors are evaluated as follows.

Exact solution

Putting $k_s^2 = 0$ and n = 1 in Eq. (4.20) and ignoring μ_3^2 (because $\mu_3^2 \ll 1$)

one has

$$k_\theta^{*2} = \frac{k_\theta^2}{m^2 \pi^2} = \frac{m^2 \pi^2}{\mu_\theta^2 + m^2 \pi^2} = E_1 \quad \dots (4.39)$$

and the assoociated mode shapes have (Eqs. 4.21)

$$\frac{\theta_m}{w_m} = \frac{4\pi}{mS^2} \quad \frac{1}{\left(1-k_\theta^{*2}\right)} = M_1 \quad \dots (4.40)$$

First order approximation equations of tubes of type B

Putting $\xi = - \lambda_1^2 = - m^2 \pi^2$ in Eq. (4.36) and letting $k_w^2 = 0$, we have

$$k_\theta^{*2} = \frac{k_\theta^2}{m^2 \pi^2} = \frac{m^2 \pi^2}{\mu_\theta^2} = E_2 \quad \dots (4.41)$$

and the mode shapes have the same distribution as in exact solution, but (Eqs. (4.38)).

$$\frac{\theta_m}{w_m} = \frac{4\pi}{mS^2} = M_2 \quad \dots (4.42)$$

Error

The error in natural frequency is

$$\varepsilon_f = E_2 - E_1 = \frac{m^4 \pi^4}{\left(m^2 \pi^2 + \mu_\theta^2\right) \mu_\theta^2} \quad \dots (4.43)$$

and the error in the ratio of maximum amplitudes is

$$\varepsilon_m = M_2 - M_1 = \left(\frac{m^2 \pi^2}{\mu_\theta^2}\right) \left(\frac{4\pi}{mS^2}\right) \quad \dots (4.44)$$

Fig. (4.4) shows the variation of ε_f and ε_m with the length of the tube. From this, it can be seen that the first order approximation equations of tubes of type B, can also be used for type A with small error provided the tube is long, particularly for lower modes.

4.8 Cross sectional constants of an I-section—first order approximation equations of open tubes of type B

It can be seen from Fig. 4.5, that since the tube is doubly symmetric, $\bar{w}_1$ is doubly antisymmetric. Since $\bar{w}_1$ is antisymmetric about BE, it follows that $\bar{w}_1$ in BE

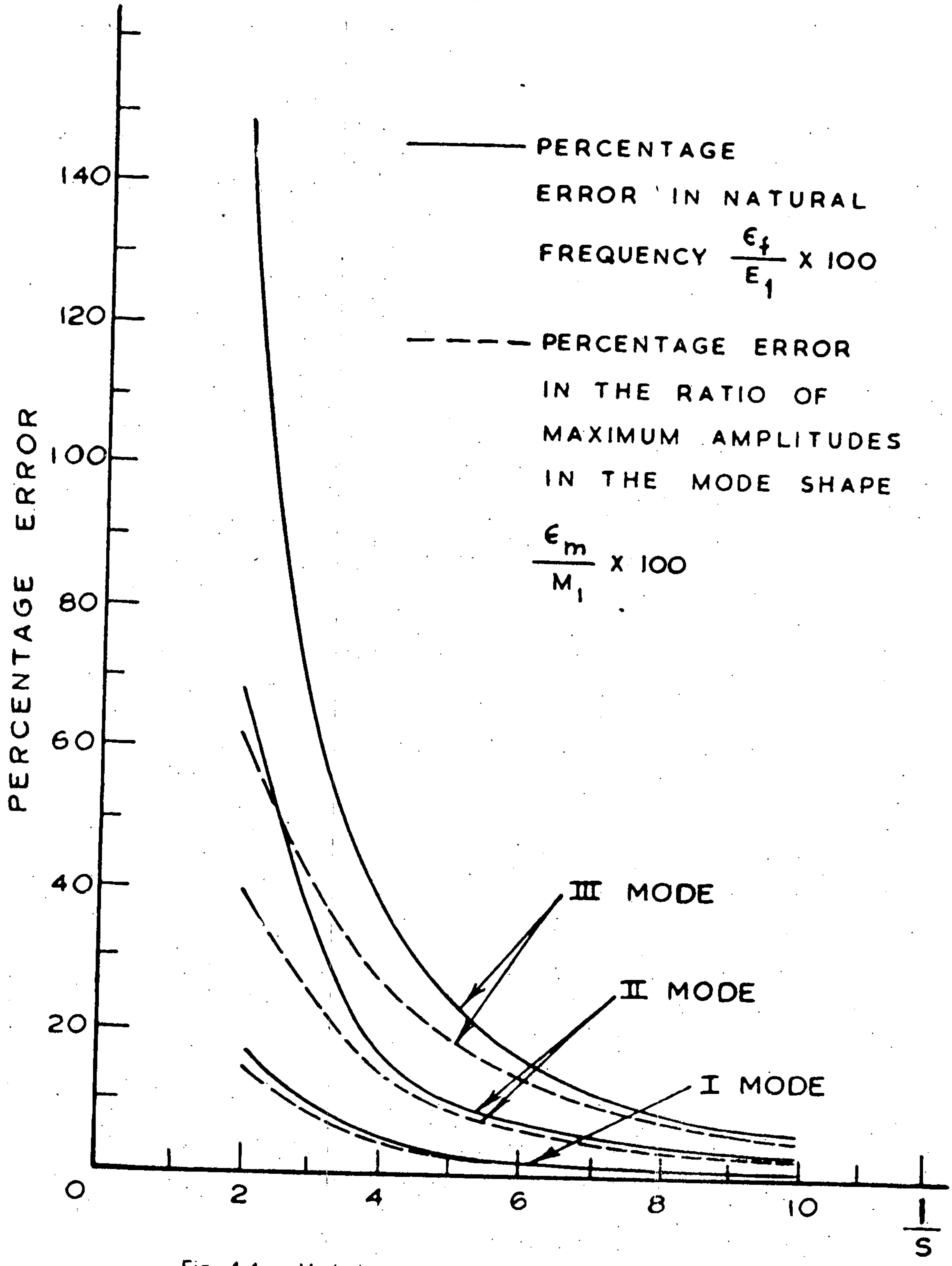

Fig. 4.4 : Variation of percentage errors with length.

is zero. Therefore, it is sufficient if we consider the region AB for obtaining $\bar{w}_1$ since in other regions $\bar{w}_1$ can be obtained by using above features. In obtaining $\bar{w}_1$, it is essential to consider the sign of p; p is positive if the tangential displacement v_t due to a positive rotation θ is in the positive direction of s.

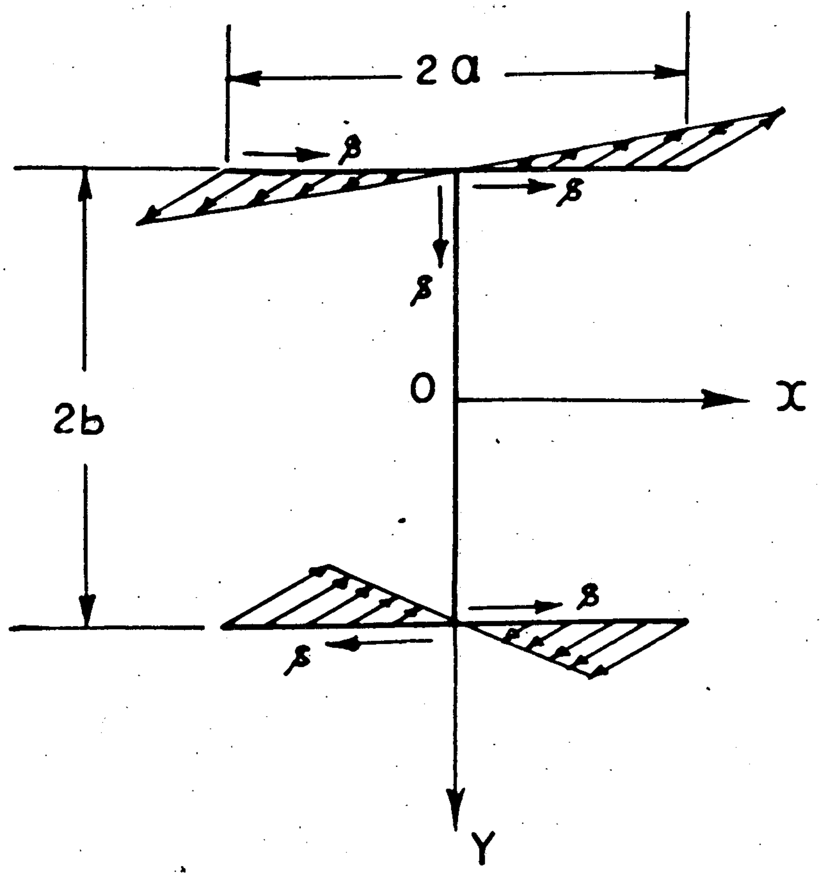

Fig. 4.5 : $\bar{W}_1$ In I Section

p in the region AB is b, $\bar{w}_1$ in this region may be obtained from Eq. (4.24)

$$\bar{w}_1 = \int p \, ds$$

with the condition that $\bar{w}_1$ at B is zero; as

$$\bar{w}_1 = + b \, (s' - a) \text{ in AB} \qquad \ldots \quad (4.45)$$

The distribution of $\bar{w}_1$ on the section is as shown in Fig 4.5. Note that it satisfies

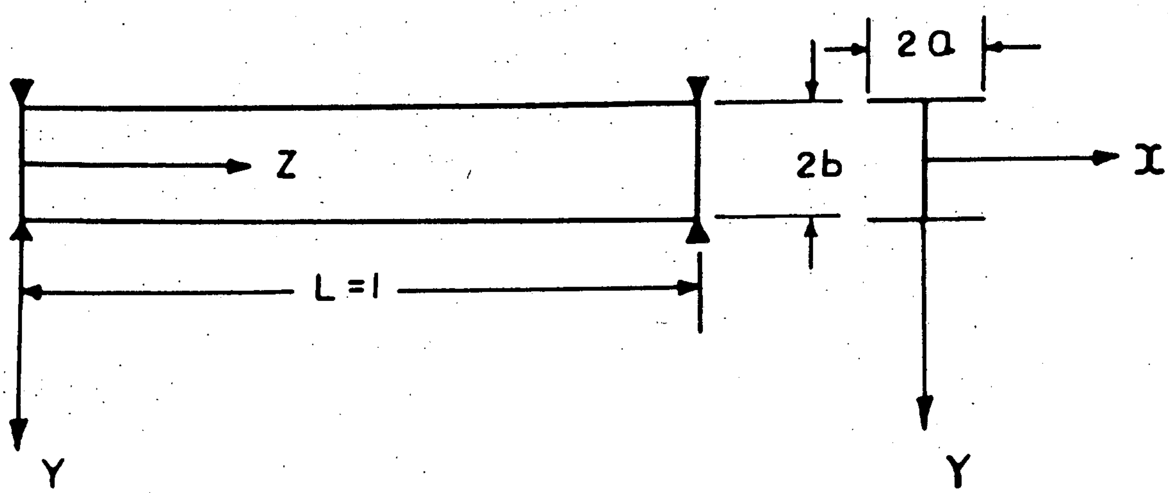

Fig. 4.6 : A simply supported tube of I section

the condition of zero net axial force due to $\bar{w}_1$ system. Using Eq. (4.45), the cross sectional constants may be computed as (Eq. 4.30).

$$B_{\theta\theta} = \frac{4b^2 a^3 t}{3}$$

$$J_s = \frac{2}{3}\left(2a + b\right)t^3 \qquad \ldots (4.46)$$

$$I_p = \frac{2}{3}\, t \left(2a^3 + b^3 + 2b^2 a\right)$$

4.9 Simply supported tube of I-section—first order approximation equations of type B

$$\mu_\theta^2 = \frac{J_s}{k^2 B_{\theta\theta}} = \frac{(P+1)^2 Q^2}{2k^2 R^2 P}$$

and ...(4.47)

$$\frac{k_w^2}{k_\theta^2} = \frac{GJ}{EI_p} = k_{Li}^2 = \frac{(2P+1)}{k^2 R^2 (2P^3+2P+1)}$$

where P = a/b, Q = L/a, R = b/t ...(4.48)

The natural frequencies in this case can be obtained by substituting

$$\xi = -\lambda_1^2 = -m^2\pi^2$$

in Eq. (4.36) and using Eqs. (4.47) and (4.48) as

$$k_\theta^2 = \frac{m^2\pi^4 + \mu_\theta^2 m^2\pi^2}{\mu_0^2 + m^2\pi^2 k_{Li}^2}$$

or

$$k_0^{*2} = \frac{k_\theta^2}{m^2\pi^2} = \frac{m^2\pi^2 + \mu_\theta^2}{\mu_\theta^2 + m^2\pi^2 k_{li}^2} \qquad \ldots (4.49)$$

Neglecting longitudinal inertia one finds

$$k_\theta^{*2} = \frac{m^2\pi^2}{\mu_\theta^2} + 1 \qquad \ldots (4.50)$$

and this is same as the expression given by Gere [29]. Equations (4 49) aud (4.50) can be written in terms of K_θ^4 which is appropriate for short open tubes. K_0^4 is defined as

$$K_\theta^4 = \frac{\rho\omega^2 L^4 I_p}{EB_{\theta\theta}} = k_\theta^2 \mu_\theta^2 \qquad \ldots (4.51)$$

Eq. (4.49) becomes

$$K_\theta^{*4} = \frac{k_\theta^{*2}\,\mu_\theta^2}{m^2\pi^2} = \frac{\mu_\theta^2}{m^2\pi^2}\left(\frac{m^2\pi^2 + \mu_\theta^2}{\mu_\theta^2 + m^2\pi^2 k_{Li}^2}\right) \qquad \text{...(4.52)}$$

Neglecting longitudinal inertia, we have $(k_{Li}^2 = 0)$

$$K_\theta^{*4} = 1 + \frac{\mu_\theta^2}{m^2\pi^2} \qquad \text{... (4.53)}$$

Neglecting St. Venant torsion, $\mu_\theta = 0$ and hence

$$K_\theta^{*4} = 1 \qquad \text{... (4.54)}$$

Figs. (4.8), (4.9) and (4.10) show the variation of K_θ^{*4} with the plan aspect ratio. These reveal that the effect of St. Venant torsion is negligible for short open tubes, and this influence is even smaller at higher frequencies. The influence of longitudinal inertia is small for long tube, but considerable when the tube is short. Thus a separate simplified analysis can be suggested for short and long tubes. The analysis of long tubes must include St. Venant torsion but no longitudinal inertia. The governing equation in this case, as obtained by Gere[33], can be got by putting $k_w^2 = 0$ in Eq. (4.31 a)

$$\theta^{iv} - \mu_\theta^2 (\theta'' - k_\theta^2 \theta) = 0 \qquad \text{... (4.55)}$$

The analysis of short tubes must include longitudinal inertia and no St. Venant torsion. The governing equation in this case is obtained by putting $J_s = 0$ in Eq. (4.31), as

$$\theta^{iv} + k_w^2 \theta'' - \mu_\theta^2 k_\theta^2 = 0 \qquad \text{... (4.56)}$$

From the numerical results presented in Figs. (4.8), (4.9) and (4.10), it can be seen that if the distance between two successive nodal points is less than ten times the flange width, the tube can be treated as short otherwise as long. Further discussion is included in section (4.11).

4.10 Open tubes of type B—second order approximation equations.

The appropriate expression for w in the second order approximation is

$$w = - \bar{w}_1 \frac{d\theta}{dz} - \bar{w}_{2,\theta}\, \psi_\theta \qquad \text{... (4.57)}$$

$\bar{w}_{2,\theta}$ has to be obtained from the relation (section (1.13))

$$\bar{w}_{2\theta} = - \iint \bar{w}_1 \, ds \, ds \qquad \ldots (4.58a)$$

Noticing $\dfrac{d\bar{w}_1}{ds} = p$ the constants of integration are evaluated from the zero shear strain condition,

$$\frac{d\bar{w}_{2,\theta}}{ds} = 0 \text{ at free edges} \qquad \ldots (4.58b)$$

Substituting Eq. (4.57) in the second of Eqs. (4.3) one obtains

$$- \bar{z} = k^2 \frac{\partial^2}{\partial z^2}\left\{ - \bar{w}_1 \frac{d\theta}{dz} - \bar{w}_{2,\theta}\Psi_\theta \right\} + \frac{\partial^2}{\partial s^2}\left\{ - \bar{w}_1 \frac{d\theta}{dz} - \bar{w}_{2,\theta}\Psi_\theta \right\} +$$

$$k_s^2 \left\{ - \bar{w}_1 \frac{d\theta}{dz} - \bar{w}_{2,\theta}\Psi_\theta \right\} + \frac{dp}{ds}\frac{d\theta}{dz} = 0 \qquad \ldots (4.59a)$$

Instead of satisfying Eq. (4.59a), we choose to satisfy

$$\oint \bar{z}\, \bar{w}_1 \, ds = 0$$

$$\oint \bar{z}\, \bar{w}_{2,\theta} \, ds = 0 \qquad \ldots (4.59b)$$

Substituting Eq. (4.57) in the first of Eqs. (4.3), using Eqs. (4.59b) and noticing $\dfrac{d\bar{w}_1}{ds} = p$, we obtain the equations of equilibrium as

$$k^2 B_{\theta\theta} \frac{d^4\theta}{dz^4} + k_w^2 \frac{d^2\theta}{dz^2} + k^2 \bar{L}_{\theta\theta}\left(\frac{d^3\Psi_\theta}{dz^3} + k_w^2 \frac{d\Psi_\theta}{dz} \right)$$

$$- \left(J_s \frac{d^2\theta}{dz^2} + k_s^2 I_p \theta \right) = 0 \qquad \ldots (4.60)$$

$$k^2 \bar{L}_{\theta\theta}\left(\frac{d^3\theta}{dz^3} + k_w^2 \frac{d\theta}{dz} \right) + k^2 \bar{L}_{\theta\theta}\left(\frac{d^2\Psi_\theta}{dz^2} + k_w^2 \Psi_\theta \right) - L_{\theta\theta}\Psi_\theta = 0$$

The boundary conditions at each end are (these are obtained from Eqs. (4.4) by adopting the same procedure used in getting Eqs. (4.60) from Eqs. (4.3)).

$$\text{either } \theta = 0 \text{ or } k^2 B_{\theta\theta}\left(\frac{d^3\theta}{dz^3} + k_w^2 \frac{d\theta}{dz} \right) + k^2 \bar{L}_{\theta\theta}\left(\frac{d^2\Psi_\theta}{dz^2} + k_w^2 \Psi_\theta \right)$$

$$- J_s \frac{d\theta}{dz} = 0$$

$$\text{either } \frac{d\theta}{dz} = 0 \text{ or } k^2 B_{\theta\theta} \frac{d^2\theta}{dz^2} + k^2 \bar{L}_{\theta\theta} \frac{d\Psi_\theta}{dz} = 0$$

$$\text{either } \Psi_\theta = 0 \text{ or } k^2 \bar{L}_{\theta\theta} \frac{d^2\theta}{dz^2} + k^2 \bar{L}_{\theta\theta} \frac{d\Psi_\theta}{dz} = 0 \qquad \ldots (4.61)$$

The following notation is used in Eqs. (4.60) and (4.61)

$$B_{\theta\theta} = \oint \bar{w}_1^2 t \, ds$$

$$\bar{C}_{\theta\theta} = \oint \bar{w}_1 \bar{w}_{2,\theta} t \, ds$$

$$\bar{L}_{\theta\theta} = \oint \bar{w}_{2,\theta}^2 t \, ds$$

$$I_p = \oint r^2 t \, ds$$

$$J_s = \tfrac{1}{3} \oint t^3 \, ds \qquad \ldots \quad (4.62)$$

$$L_{\theta\theta} = \oint \left(\frac{d\bar{w}_{2\cdot\theta}}{ds}\right)^2 t \, ds$$

$$\bar{w}_1 = \int p \, ds \qquad \ldots \quad (4.63)$$

$$\bar{w}_{2\cdot\theta} = - \iint \bar{w}_1 \, ds \, ds \qquad \ldots \quad (4.64)$$

The constant of integration in Eq. (4.63) is obtained from the condition of no net axial force while the integration constants in Eq. (4.64) are obtained from the condition $\frac{d\bar{w}_{2,\theta}}{ds} = 0$ at free edges.

Introducing the notation

$$k_\theta^2 = \frac{k_s^2 I_p}{J_s}; \quad \nu_2^2 = B_{\theta\theta}/\bar{C}_{\theta\theta};$$

$$\mu_4^2 = \frac{J_s}{k^2 B_{\theta\theta}}; \quad \nu_3^2 = \bar{C}_{\theta\theta}/B_{\theta\theta}; \qquad (4.65)$$

$$\mu_5^2 = \frac{L_{\theta\theta}}{k^2 \bar{C}_{\theta\theta}};$$

Eqs. (4.60) become

$$\theta^{iv} + k_w^2 \theta'' + \nu_3^2 \left(\Psi_\theta''' + k_w^2 \Psi_\theta \right) - \mu_4^2 \left(\theta'' + k_\theta^2 \theta \right) = 0$$

$$\theta''' + k_w^2 \theta' + \nu_2^2 \left(\Psi_\theta''' + k_w^2 \Psi_\theta \right) - \mu_5^2 \Psi_\theta = 0 \qquad \ldots \quad (4.66)$$

and the boundary conditions at each end are (from Eqs. (4.61))

either $\theta = 0$ or $\theta''' + k_w^2 \theta' + \nu_3^2 \left(\Psi_\theta'' + k_w^2 \Psi \theta \right) - \mu_4^2 \theta' = 0$

either $\theta = 0''$ or $\theta'' + \nu_3^2 \Psi_\theta' = 0$

either $\Psi_\theta = 0$ or $\theta'' + \nu_2^2 \Psi_\theta' = 0 \qquad \ldots \quad (4.67)$

Combining the two equations of Eqs. (4.66), one finds

$$(D^6 + a_1 D^4 + a_2 D^2 + a_3)(\theta \text{ or } \Psi_\theta) = 0 \qquad \ldots \quad (4.68)$$

where

$$a_1 = 2k_w^2 - \frac{\mu_4^2 \nu_2^2 + \mu_5^2}{\nu_2^2 - \nu_3^2}$$

$$a_2 = k_w^4 - \left(\frac{\mu_4^2 \nu_2^2 + \mu_5^2}{\nu_2^2 - \nu_3^2} \right) k_w^2 + \frac{\mu_4^2 \mu_5^2}{\nu_2^2 - \nu_3^2} - \frac{\mu_4^2 \nu_2^2}{\nu_2^2 - \nu_3^2} k_\theta^2$$

$$a_3 = k_\theta^2 \frac{\mu_4^2 \mu_5^2}{\nu_2^2 - \nu_3^2} - \frac{\mu_4^2 \nu_2^2}{\nu_2^2 - \nu_3^2} k_w^2 \qquad \ldots \quad (4.69)$$

From Eq. (4.68) the form of expressions for θ and Ψ_θ are

$$\theta = A_1 \sin \lambda_1 z + A_2 \cos \lambda_1 z + A_3 \sinh \lambda_2 z + A_4 \cosh \lambda_2 z + A_5 \sinh \lambda_3 z + A_6 \cosh \lambda_3 z$$

$$\Psi\theta = A_1^1 \sin \lambda_1 z + A_2^1 \cos \lambda_1 z + A_3^1 \sinh \lambda_2 z + A_4^1 \cosh \lambda_2 z + A_5^1 \sinh \lambda_3 z + A_6^1 \cosh \lambda_3 z \qquad \ldots \quad (4.70)$$

where $-\lambda_1^2, \lambda_2^2$ and λ_3^2 are the roots of the cubic equation

$$\xi^3 + a_1 \xi^2 + a_2 \xi + a_3 = 0 \qquad \ldots \quad (4.71)$$

In writing Eq. (4.70), it is assumed that λ_1^2, λ_2^2 and λ_3^2 are positive. As the present interest is limited to assessing the influence of shear lag on the natural frequencies associated with primarily rotational motion, additional frequencies arising out of negative λ_2^2 and λ_3^2 are excluded from the present discussion. It is obvious that all arbitrary constants involved in Eqs. (4.70) are not independent since, they have to satisfy any one of Eqs. (4.66). Satisfaction of the second of Eqs. (4.66) requires the following relationships between the constants.

$$A_1 = - \frac{\nu_2^2 \left(-\lambda_1^2 + k_w^2 \right) - \mu_2^2}{\lambda_1 \left(-\lambda_1^2 + k_w^2 \right)} A_2^1$$

$$A_2 = \frac{\nu_2^2 \left(-\lambda_1^2 + k_w^2 \right) - \mu_2^2}{\lambda_1 \left(-\lambda_1^2 + k_w^2 \right)} A_1^1$$

$$A_3 = \frac{\nu_2^2 \left(\lambda_2^2 + k_w^2 \right) - \mu_2^2}{\lambda_2 \left(\lambda_2^2 + k_w^2 \right)} A_4^1 \qquad \ldots \quad (4.72)$$

$$A_4 = \frac{\nu_2^2\left(\lambda_2^2 + k_w^2\right) - \mu_2^2}{\lambda_2\left(\lambda_2^2 + k_w^2\right)} A_3^1$$

$$A_5 = \frac{\nu_2^2\left(\lambda_3^2 + k_w^2\right) - \mu_2^2}{\lambda_3\left(\lambda_3^2 + k_w^2\right)} A_6^1$$

$$A_6 = \frac{\nu_2^2\left(\lambda_3^2 + k_w^2\right) - \mu_2^2}{\lambda_3\left(\lambda_3^2 + k_w^2\right)} A_5^1$$

4.11 Cross sectional constants of I-section—second order approximation equations for open tube of type 'B'

The expression for $\bar{w}_1$ has been discussed in section (4.8). Since the tube is doubly symmetric warp will be antisymmetric about BE and x-axis. It is sufficient if we consider AB for evaluating $\bar{w}_{2,\theta}$ as the expression for $\bar{w}_{2,\theta}$ in other regions can be obtained by using the above features. $\bar{w}_{2,\theta}$ in AB can be obtained from

$$\bar{w}_{2,\theta} = -\iint \bar{w}_1 \, ds \, ds$$

and using the conditions (Eq. (4.58b))

$$\frac{d\bar{w}_{2,\theta}}{ds} = 0 \text{ at A}$$

and

$$\bar{w}_{2,\theta} = 0 \text{ at B}$$

$$\bar{w}_{2,\theta} = -\frac{b}{6}\left(s^3 + a^3 - 3as^2\right) \text{ in AB} \quad \ldots (4.73)$$

$\bar{w}_{2,\theta}$ in I-section is shown in Fig. (4.7). Using (4.73) in Eqs. (4.62) one obtains

$$\bar{L}_{0\theta} = \frac{8}{15}b^2a^3t, \qquad \bar{L}_{\theta\theta} = \frac{68}{315}b^2a^7t \quad \ldots (4.74)$$

$$L_{\theta\theta} = \frac{8}{15}b^2a^5t\ ;$$

other cross-sectional constants are given in Eq. (4.46).

4.12 A simply supported tube of I-section—second order approximation equations of open tube of type 'B'

The boundary conditions in this case (Fig. 4.6) are

$$\theta(0) = \theta''(0) = \Psi_\theta'(0) = 0 \quad \ldots (4.75)$$

$$\theta(l) = \theta''(l) = \Psi_\theta'(l) = 0 \quad \ldots (4.76)$$

The appropriate expressions for θ and Ψ_θ are given in Eqs. (4.70). Satisfying the boundary conditions (4.73) and using Eqs. (4.72), one finds

$$A_2 = A_4 = A_6 = A_1^1 = A_3^1 = A_5^1 = 0 \quad \ldots (4.77)$$

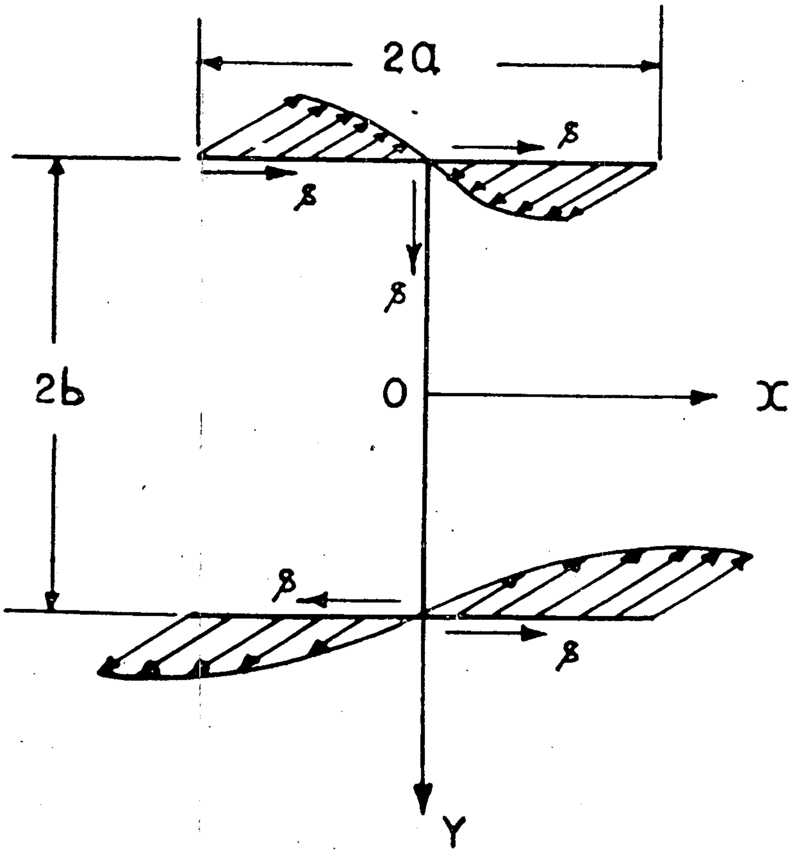

Fig. 4.7 : $\overline{W}_{2,\theta}$ In I section

Satisfying Eqs. (4.76) and using Eqs. (4.77) one obtains

$$A_1 \sin \lambda_1 + A_3 \sinh \lambda_2 + A_5 \sinh \lambda_3 = 0$$

$$-\lambda_1^2 A_1 \sin \lambda_1 + \lambda_2^2 A_3 \sinh \lambda_2 + \lambda_3^2 A_5 \sinh \lambda_3 = 0$$

$$-\lambda_1 A_2^1 \sin \lambda_1 + \lambda_2 A_4^1 \sinh \lambda_2 + \lambda_3 A_6^1 \sinh \lambda_3 = 0 \qquad (4.78)$$

Using Eq. (4.72) one finds for the nontrivial values of constants in Eqs. (4.78),

$$\sin \lambda_1 \sinh \lambda_2 \sinh \lambda_3 = 3 \qquad ...(4.79)$$

or

$$\lambda_1 = m\pi \; ; \quad m = 1, 2, 3, \; ... \; \infty \qquad ...(4.80)$$

Neglecting longitudincl inertia a simple expression fer natural frequency can be obtained. Substituting $\xi = -\lambda_1^2 = -m^2 \pi^2$ in Eq. (4.71), putting $k_w^2 = 0$ and using the notation

$$\zeta_3^2 = \frac{\nu_2^2}{\nu_2^2 + \nu_3^2}$$

$$\zeta_4^2 = \frac{\mu_5^2}{\nu_2^2 + \nu_3^2} \qquad ...(4.81)$$

$$\mu_4^2 = \mu_\theta^2$$

one obtains

$$k_\theta^{*2} = \frac{k_\theta^2}{m^2\pi^2} = 1 + \frac{m^4\pi^4 + m^2\pi^2\zeta_4^2}{\mu_0^2(\zeta_3^2 m^2\pi^2 + \zeta_4^2)}$$

or

$$K_\theta^{*4} = \frac{k_\theta^{*2}}{m^2\pi^2} = \frac{\mu_\theta^2}{m^2\pi^2}\left\{1 + \frac{m^4\pi^4 + m^2\pi^2\zeta_4^2}{\mu_0^2(\zeta_3^2 m^2\pi^2 + \zeta_4^2)}\right\} \quad \ldots(4.82)$$

Using Eqs. (4.74), (4.65), (4.81), (4.47) and (4.48) and considering the case of a/b=1, it can be shown that

$$\mu_\theta^2 = \frac{(P+1)Q^2}{2k^2 \cdot R^2}$$

$$\zeta_3^2 = 85 \quad \ldots(4.83)$$

$$\zeta_4^2 = \frac{210Q^2}{k^2}$$

the variations of k_θ^{*4}, given in Eq. (4.82), with plan aspect ratio is presented in Figs. (4.8), (4.9) and (4.10). These show that the influence of shear lag is small in long tubes ; as such Eq. (4.55) can be used. But the influence of shear lag in short tubes is considerable. Hence in short tubes, Eqs. (4.66) have to be used ; neglect of J_R involves only small error.

4.13 Conclusions

In this part, problems of torsional vibrations of open tubes are discussed. The governing equations of open tubes are same as those for closed tubes, but for minor modifications. Unless the tube is such that p = O at open edges (open tubes of type A) the method of solution of section 1.6, cannot be used. To facilitate the use of this method of section 1.6 modified governing equations are derived (for tubes of type B). The exact solution of a simply-supported tube with the boundary of cross section given by $p = \frac{S}{2\pi} \sin \frac{2\pi S}{S}$ brings out a doubly infinite set of frequencies, besides an infinite set of frequencies associated with primarily rotational motion. One infinite set of these additional frequencies involves small rotations also while the others are pure warping modes.

The same example is worked out using first order approximation equations of open tubes of type B, and the results are compared with exact solutions. This comparison shows that the first approximation equations of tubes of type B give adequate accuracy even for tubes of type A, when the length of the tube is large.

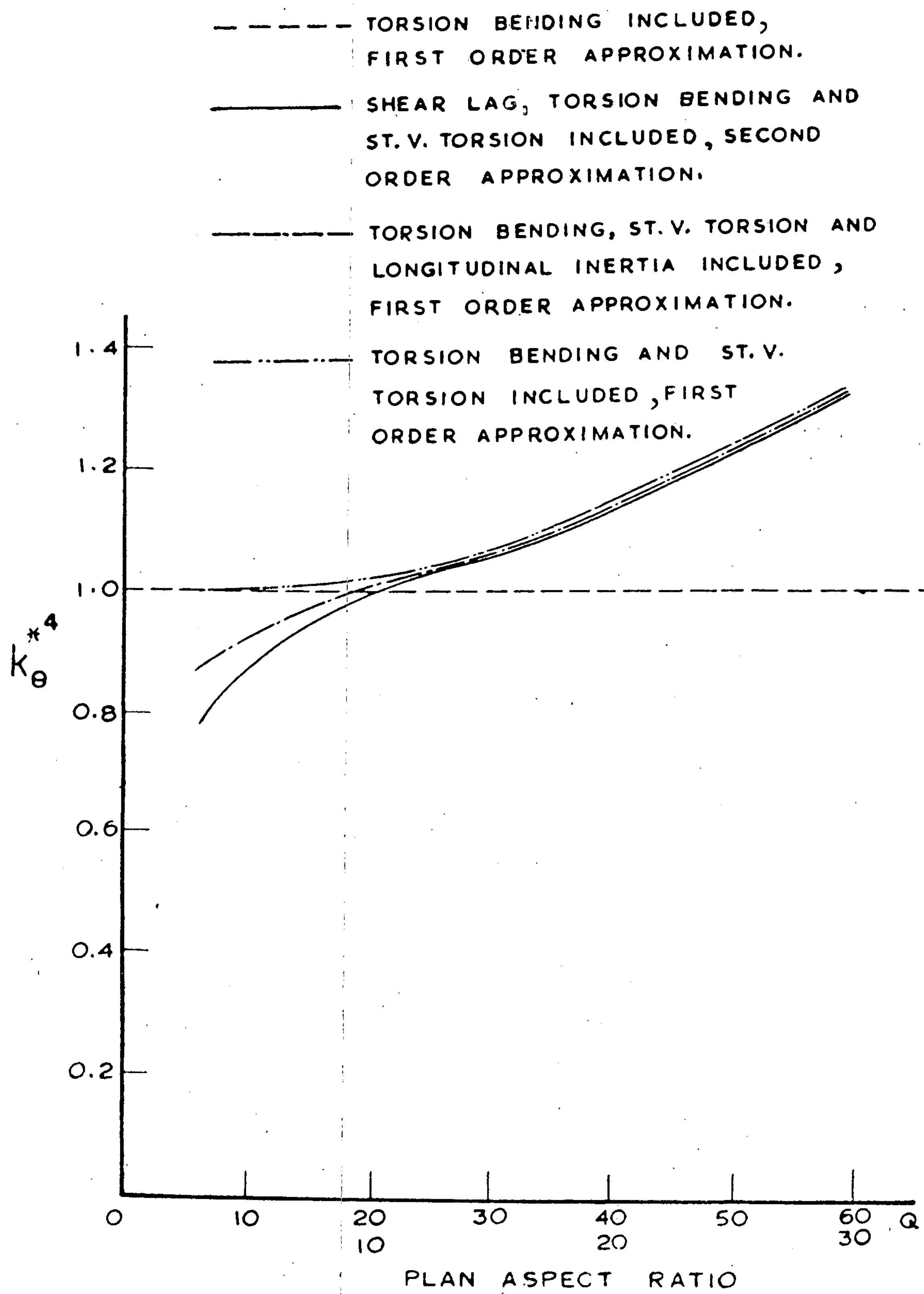

Fig. 4.8 : Influence of plan aspect ratio on the frequency parameter for a simply supported tube of I section with a/b = 1, fundamental.

A simply supported tube of I—section is analysed by using first and second approximation equations of tubes of type B. The results of this case show that in

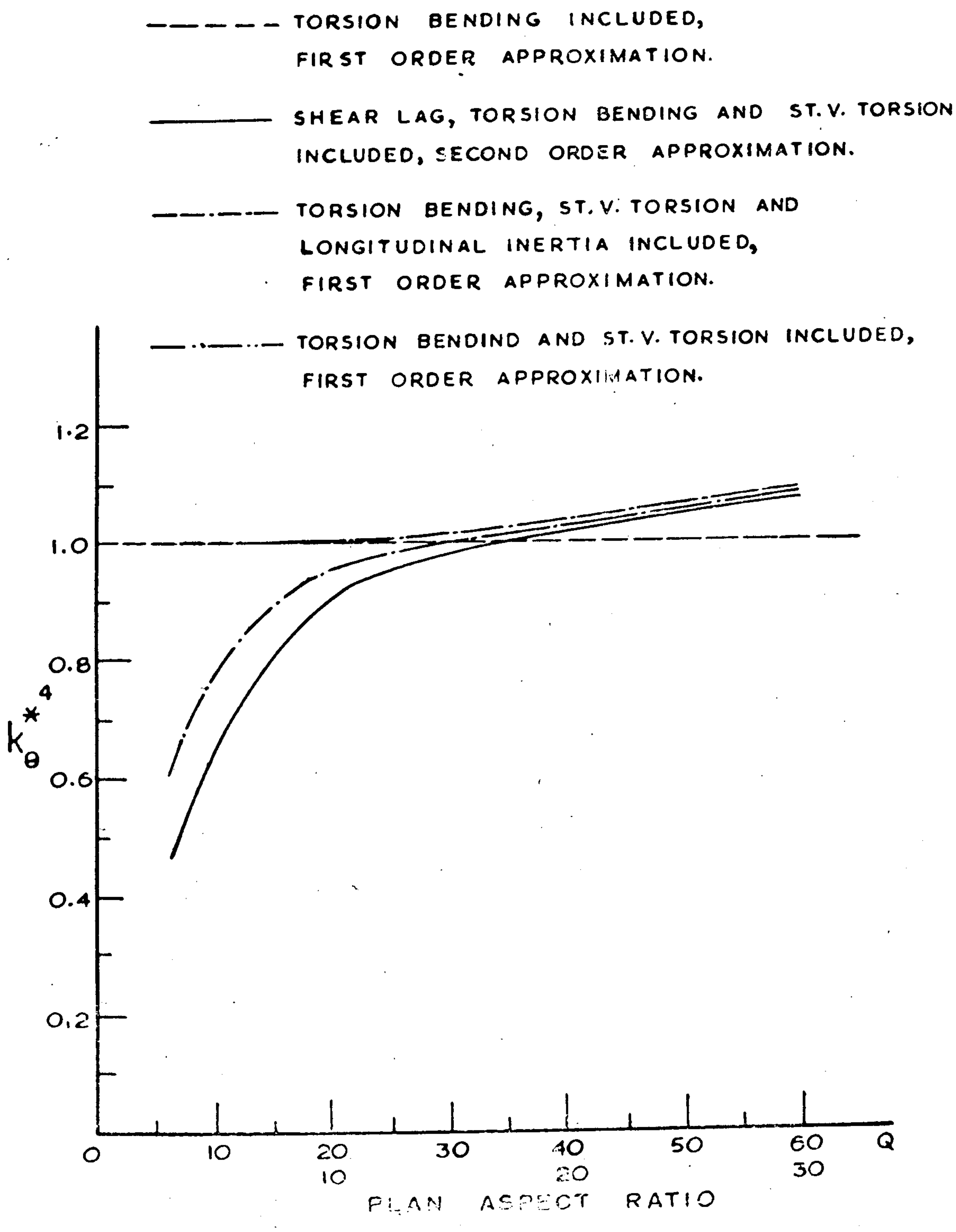

Fig. 4.9 : Influence of plan aspect ratio on the frequency parameter for a simply supported tube of I section with a/b=1, second mode

long tubes, shear lag and longitudinal inertia are negligible while St. Venant torsion plays an important role ; but in short tubes St. Venant torsion effect is small while shear lag and longitudinal inertia have considerable effect. Depending on the numerical results presented, one can say, that if the distance between two successive

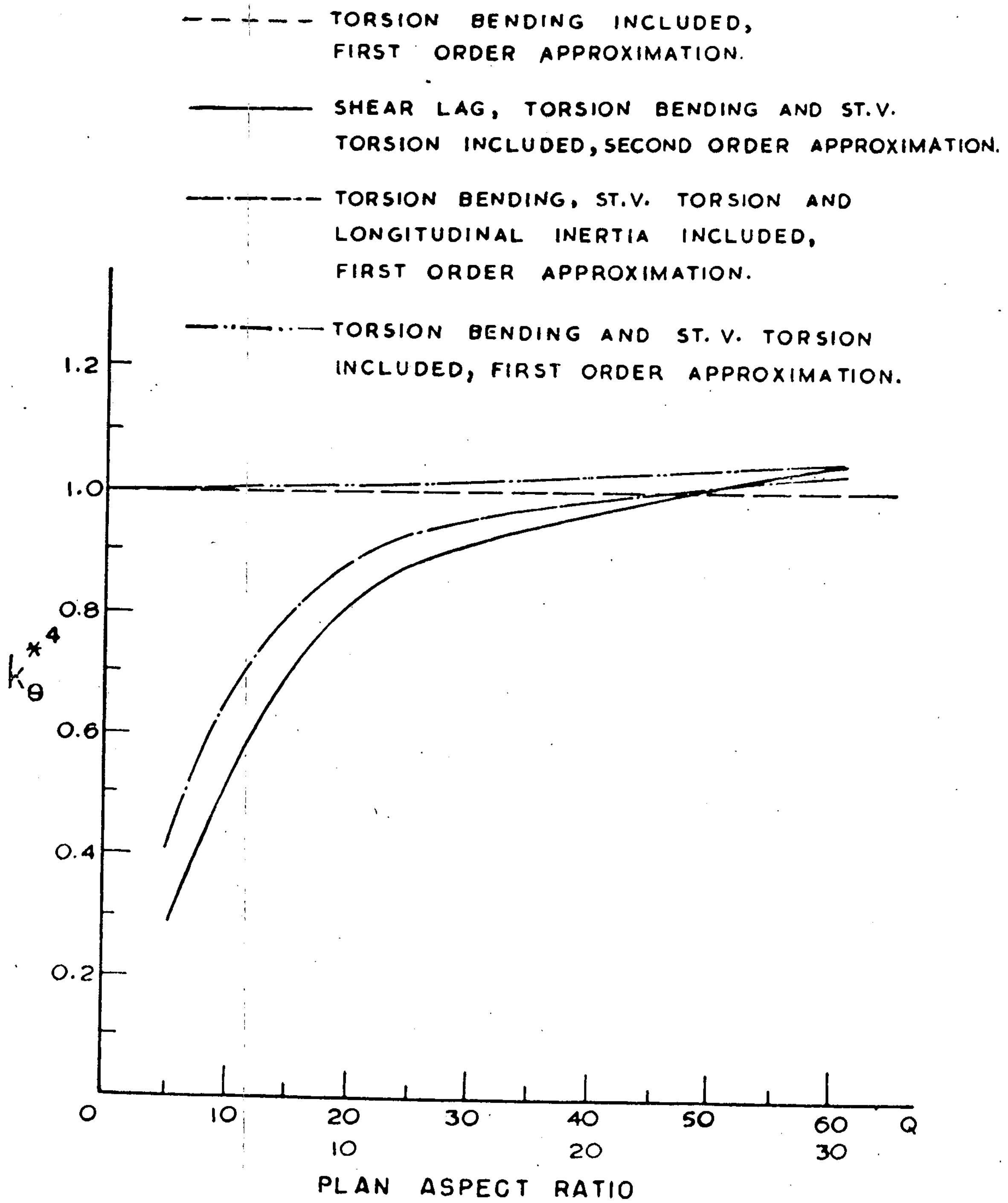

Fig. 4.10 : Influence of plan aspect ratio on the frequency parameter for a simply supported tube of I section with a/b=1, third mode.

nodal points is less than ten times the flange width it may be classified as a short tube ; otherwise it is to be treated as a long tube.

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