Borel's contributions to arithmetic groups and their cohomology

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Armand Borel made profound contributions to the study of arithmetic groups and their cohomology. The general theory of reductive groups over arbitrary fields, which he developed with Jacques Tits, played a fundamental role in this and several other areas.

The purpose of this article is to give a glimpse of Borel's work on arithmetic groups. We will only consider reductive algebraic groups defined over $\mathbb{Q}$, since using the Levi decomposition one can easily reduce to the case where the group is reductive, and given a reductive group over a number field, by restriction of scalars we obtain a reductive group over $\mathbb{Q}$. So let us assume that G is a connected reductive algebraic group defined over $\mathbb{Q}$. We realize G as a subgroup of GL_n, for some n, in terms of a fixed $\mathbb{Q}$-embedding of G in GL_n. Let $G_{\mathbb{Z}} := G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$. A subgroup Γ of $G(\mathbb{Q})$ is said to be an *arithmetic subgroup* of G if it is commensurable with $G_{\mathbb{Z}}$, i. e., if $\Gamma \cap G_{\mathbb{Z}}$ is of finite index in both Γ and $G_{\mathbb{Z}}$. It is easy to see that the notion of arithmetic subgroups is independent of the specific embedding of G in GL_n. It was observed by Minkowski that the *principal congruence subgroup* of $\mathrm{GL}_n(\mathbb{Z})$ of level m, i. e., the subgroup of matrices congruent to the identity matrix modulo an integer m, is torsion-free provided $m \geqslant 3$. Thus, any arithmetic subgroup contains a torsion-free subgroup of finite index.

An arithmetic subgroup is obviously a discrete subgroup of $G(\mathbb{R})$, and it is usually a rather "large" subgroup ; in fact, as Borel showed in (70), if G is semi-simple and does not contain a nontrivial connected normal $\mathbb{Q}$-subgroup N such that $N(\mathbb{R})$ is compact, then Γ is dense in G in the Zariski-topology. These subgroups arise in several different contexts, for example, in the special orthogonal, symplectic or unitary group of a rational quadratic, alternating or hermitian form, or as the fundamental group of locally symmetric spaces of finite volume. Their study is an integral part of the general theory of automorphic forms, Shimura varieties, and the Langlands program.

The origins of the theory of arithmetic groups can be traced back to the work of Lagrange, and later Gauss, on the reduction theory of binary quadratic forms (this work provided the reduction theory for the arithmetic subgroup $\mathrm{GL}_2(\mathbb{Z})$ of $\mathrm{GL}_2(\mathbb{R})$). Generalizing Lagrange's approach, Hermite studied positive-definite, as well as indefinite, quadratic forms in several variables. Minkowski then developed a reduction theory for positive definite quadratic forms giving a fundamental domain, for the action of $\mathrm{GL}_n(\mathbb{Z})$ on the space of $n \times n$-positive symmetric matrices, which is a convex polyhedral cone. (For a nice account of all this, see [SO].) After this, Siegel developed the reduction theory for the arithmetic subgroups Γ of the automorphism group G of a semi-simple $\mathbb{Q}$-algebra with involution. He showed that the

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"fundamental set" he constructed has the following three properties, and moreover it is of finite volume with respect to any Haar measure on $G(\mathbb{R})$ (which implies that $G(\mathbb{R})/\Gamma$ is of finite volume) if the central torus of G is anisotropic (over $\mathbb{Q}$).

(i) For some maximal compact subgroup K of $G(\mathbb{R})$, $K \cdot \Omega = \Omega$,

(ii) $\Omega \cdot \Gamma = G(\mathbb{R})$;

and furthermore the following property, known as the *Siegel property*,

(iii) for any $g \in G(\mathbb{Q})$, the set $\{\, \gamma \in \Gamma \mid \Omega g \cap \Omega\gamma \neq \varnothing \,\}$ is finite.

A subset Ω of $G(\mathbb{R})$ is said to be a *fundamental set* for Γ if it has the above three properties. The goal of the reduction theory is to construct a "nice" fundamental set for any arithmetic subgroup.

In the 1950's, Weil gave a classification of classical semi-simple groups. He showed that up to isogeny these groups are the automorphism groups of semi-simple algebras with involutions. Thus Siegel's reduction theory covered most of the classical groups. Weil taught a course on Siegel's reduction theory at the University of Chicago (mimeographed notes of his course were prepared and distributed in 1958). In this course, he also proved the following compactness criterion for several of the classical groups : $G(\mathbb{R})/\Gamma$ is compact if and only if G is of $\mathbb{Q}$-rank zero, or, equivalently, either Γ, or the Lie algebra $\mathfrak{g}(\mathbb{Q})$ of G, consists entirely of semi-simple elements. (Based on the evidence provided by this result, and the classical results on the orthogonal groups, it was later conjectured by Godement that this compactness criterion holds for all semi-simple groups.) Subsequently Weil gave some lectures on this topic at Paris.

Harish-Chandra attended these lectures in Paris and found the situation – where results were known only for certain reductive groups, and where an explicit description of these groups was required for the study of their arithmetic subgroups – to be quite unsatisfactory, so he determined to find a general theory. This turned out to be very fortuitous, both for the subject and for his own later work on the general theory of automorphic forms where reduction theory of arithmetic subgroups plays an absolutely crucial role. He was able to quickly prove important results on arithmetic groups in a general set-up. The famous paper (58) of Borel and Harish-Chandra was an outgrowth of these results.

In the Borel and Harish-Chandra paper, a nice fundamental set for any arithmetic subgroup is constructed using the Minkowski-Siegel fundamental set for $\mathrm{GL}_n(\mathbb{Z})$ in $\mathrm{GL}_n(\mathbb{R})$, and the Godement compactness criterion is proved in full generality (an independent and more popular proof of this criterion was given by Mostow and Tamagawa). I will now describe the fundamental set constructed by Borel and Harish-Chandra, and a more usable variant given by Borel (for a different approach to reduction theory due to Godement and Weil which leads to a similar fundamental set, see [G]).

Standard Siegel sets in $\mathrm{GL}_n(\mathbb{R})$: Let K be the orthogonal group $\mathrm{O}_n(\mathbb{R})$, which is known to be a maximal compact subgroup of $\mathrm{GL}_n(\mathbb{R})$. Let A be the subgroup of diagonal matrices with strictly positive entries, and N be the subgroup of upper triangular unipotent matrices. For positive real numbers c and t, let

$$A_t = \{\, a \in A \mid \ a_{i,i} \leqslant t a_{i+1,i+1} \,\},$$

and

$$N_c = \{\, x \in N \mid \ |x_{i,j}| \leqslant c, \ \ 1 \leqslant i < j \leqslant n \,\}.$$

The subset $\mathfrak{S}_{t,c} = KA_tN_c$ is called a standard Siegel set (in $\mathrm{GL}_n(\mathbb{R})$). It is known that if $t \geqslant 2/\sqrt{3}$ and $c \geqslant 1/2$, then $\mathfrak{S}_{t,c}$ is a fundamental set for $\mathrm{GL}_n(\mathbb{Z})$ in $\mathrm{GL}_n(\mathbb{R})$; that it has the Siegel property was shown by Siegel.

The fundamental set constructed by Borel and Harish-Chandra is given in terms of an embedding of the underlying reductive group in GL_n. So now let G be a connected reductive $\mathbb{Q}$-subgroup of GL_n. It is known that we can find an element $a \in \mathrm{GL}_n(\mathbb{R})$ such that $aG(\mathbb{R})a^{-1}$ is self-adjoint. We fix such an a. Then given an arithmetic subgroup Γ of $G(\mathbb{Q})$, and a standard Siegel set $\mathfrak{S}$ in $\mathrm{GL}_n(\mathbb{R})$ which is a fundamental set for $\mathrm{GL}_n(\mathbb{Z})$, there exist finitely many elements $x_i \in \mathrm{GL}_n(\mathbb{Z})$ such that $\Omega = G(\mathbb{R}) \cap \bigcup a^{-1}\mathfrak{S}x_i$ is a fundamental set for Γ in $G(\mathbb{R})$. The main ingredient in the proof of this assertion is the important "finiteness lemma" which appears in §6 of Borel's book [B]. This book gives an excellent exposition of the general theory of arithmetic groups and it contains complete proofs of the main results. In this book, Borel used the above fundamental set to construct an intrinsically described fundamental set as follows. Let K be a maximal compact subgroup of $G(\mathbb{R})$. Let P be a minimal parabolic $\mathbb{Q}$-subgroup of G and let U be the unipotent radical of P. Let S be a maximal $\mathbb{Q}$-split torus of G contained in P. The centralizer M of S in G is actually contained in P and is a Levi-subgroup of P, i. e., it is a connected reductive subgroup and $P = M \cdot U$ (a semi-direct product). Let Δ be the basis of the root system of G, with respect to S, determined by P. Given a positive real number t and a relatively compact subset ω of $P(\mathbb{R})$, the subset $\mathfrak{S} = \mathfrak{S}_{t,\omega} = K \cdot A_t \cdot \omega$, where

$$A_t = \{\, x \in A := S(\mathbb{R})^\circ \mid \ \alpha(x) \leqslant t \ \text{ for all } \alpha \in \Delta \,\},$$

is called a *Siegel set* of $G(\mathbb{R})$ (with respect to K, P and S). In §13 of [B], Borel showed that there exists a Siegel set $\mathfrak{S}$ ($= \mathfrak{S}_{t,\omega}$, for some t and ω), and a finite subset C of $G(\mathbb{Q})$ such that $\Omega \subset \mathfrak{S} \cdot C$; where Ω is the fundamental set for the arithmetic subgroup Γ given above. Then $G(\mathbb{R}) = \mathfrak{S} \cdot C \cdot \Gamma$. Making a clever use of the Bruhat and Iwasawa decompositions, Harish-Chandra showed that any Siegel set $\mathfrak{S}$ has the Siegel property, see [B : §15]. Consequently, the set $\mathfrak{S} \cdot C$ also has the Siegel property and so it is a fundamental set for Γ .

The existence of a fundamental set (recall that such a set has the Siegel property by definition) implies (see §5 of (61)) that Γ has a finite presentation, and its finite subgroups form finitely many conjugacy classes. It is not difficult to see that if G is semi-simple, or, more generally, if G does not admit a nontrivial character defined over $\mathbb{Q}$, then the volume of any Siegel set $\mathfrak{S}$, with respect to a Haar measure on $G(\mathbb{R})$, is finite, and hence, $G(\mathbb{R})/\Gamma$ carries a finite $G(\mathbb{R})$-invariant Borel measure, i. e., Γ is a lattice in $G(\mathbb{R})$.

Borel (and independently, Godement and Weil, see [G]) showed that $P(\mathbb{Q})\backslash G(\mathbb{Q})/\Gamma$ is finite (or, equivalently, there are only finitely many Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G), and for a finite subset C of $G(\mathbb{Q})$, $P(\mathbb{Q}) \cdot C \cdot \Gamma = G(\mathbb{Q})$ if and only if there is a Siegel set $\mathfrak{S}$, with respect to K, P, and S, such that $G(\mathbb{R}) = \mathfrak{S} \cdot C \cdot \Gamma$. The finiteness of the set of Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups is a very useful result.

In (60), Borel employed the ideas and results of his joint paper (58) with Harish-Chandra to construct a fundamental set for the discrete subgroup $G(k)$ of the adele group $G(A)$, where G is a connected reductive algebraic group defined over a number field k, and A is the k-algebra of adeles of k. As a consequence, he deduced that $G(k)$ is a lattice in $G(A)$ if G does not admit a nontrivial character defined over k. He also proved the finiteness of the "class number", and the finiteness of the subset of the Galois cohomology set $H^1(k, G)$ consisting of cohomology-classes which are trivial at every place of k, using an adelic analogue of the "finiteness lemma" mentioned above.

Existence of cocompact discrete subgroups. Borel used the Godement compactness criterion to show that any connected semi-simple real Lie group $\mathcal{G}$ contains a cocompact discrete subgroup (62). It would clearly suffice to prove the existence of such a discrete subgroup assuming that $\mathcal{G}$ is an adjoint group. In this case Borel proved, by showing that the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ has a form defined over a totally real extension of $\mathbb{Q}$ of degree > 1, all but one of whose conjugates are compact, that there exists a connected semi-simple algebraic group G, which is defined and anisotropic over $\mathbb{Q}$, such that $G(\mathbb{R})^\circ$ is isomorphic to the direct product of $\mathcal{G}$ and a compact group. Now, in view of the Godement compactness criterion, the projection into $\mathcal{G}$ of any arithmetic subgroup of $G(\mathbb{Q})$ would be a discrete cocompact subgroup of $\mathcal{G}$.

In a latter joint work (109) of Borel with Harder, it is shown that given a number field k and a finite set S of places of k, and an absolutely simple algebraic k-group G, the natural map

$$H^1(k, \mathrm{Aut}\, G) \to \prod_{v \in S} H^1(k_v, \mathrm{Aut}\, G)$$

is surjective. This theorem provides, in particular, S-arithmetic cocompact subgroups in a finite product of simple real and p-adic groups of the same type.

Some finiteness results. In the joint work (139) of Borel with the present author, a natural Haar measure on any semi-simple group over a local (i. e., locally compact nondiscrete) field is given and it is shown that, up to natural equivalence, there are only finitely many S-arithmetic subgroups whose covolume with respect to this Haar measure is less than a given positive number c. Here the underlying global field k, the semi-simple k-group, and the finite set S of places of k, are all allowed to vary arbitrarily. This paper also provides new, more geometric proofs of several other finiteness assertions, including the finiteness of class numbers. It uses a formula for the covolume of *principal* S-arithmetic subgroups given in [P].

Compactifications of locally symmetric spaces and cohomology of arithmetic groups. I will next describe Borel's two important contributions to compactifications of locally symmetric spaces $V = X/\Gamma$, where G is a connected reductive group defined over $\mathbb{Q}$, Γ is a torsion-free arithmetic subgroup of $G(\mathbb{Q})$, and X is the symmetric space $K\backslash G(\mathbb{R})$ of $G(\mathbb{R})$, K being a maximal compact subgroup of the latter. The first one (69) is a joint paper with Baily which was inspired, in part, by an earlier work on compactifications by Satake [S]. In this, X is assumed to be hermitian (i. e., it carries a $G(\mathbb{R})$-invariant complex structure). Baily and Borel used Harish-Chandra's realization of X as a bounded symmetric domain to give a compactification V^* of X/Γ. On V^* they introduced a natural structure of an irreducible normal analytic variety and provided an embedding of V^* into a complex projective space, both by means of the Poincaré-Eisenstein series. The main results of (95) imply that there is, in fact, a unique structure of algebraic variety on X/Γ compatible with the complex analytic structure on it. It turns out that, if Γ is a congruence subgroup, the projective variety V^* is definable over an explicit number field (Shimura) and it has much arithmetic significance.

Zucker conjectured that the L^2-cohomolgoy of $V\ (=X/\Gamma)$ (see below) is isomorphic to the middle intersection cohomology of V^*. This conjecture was proved by Borel (122) for groups G of $\mathbb{Q}$-rank 1, and Borel and Casselman (131) for groups of $\mathbb{Q}$-rank 2. Looijenga, and Saper and Stern, proved it in general independently and by entirely different arguments.

Borel's second important contribution to compactification was his joint paper (98) with Serre. To describe it, let us assume that G is a connected semi-simple $\mathbb{Q}$-group, Γ is a torsion-free arithmetic subgroup, and X is the symmetric space of $G(\mathbb{R})$. Using the reduction theory, Raghunathan [R] constructed a proper Morse function on X/Γ, with finitely many critical values. Existence of such a Morse function implies that X/Γ is diffeomorphic to the interior of a compact manifold with boundary, which in turn implies that the trivial $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$ admits a finite free resolution in which each term is of finite rank. This gives several finiteness results on the Eilenberg-MacLane cohomology of Γ. To obtain more precise results about the cohomology of Γ, we need to know the boundary of a nice compactification of X/Γ. This information is not available for the compactification of X/Γ given by Raghunathan's Morse function. A different, and more canonical, construction of a compactification of X/Γ was given by Borel and Serre in (98). They used the "geodesic action" of the identity component of the center of the group of $\mathbb{R}$-points of a Levi subgroup of any parabolic $\mathbb{Q}$-subgroup P of G on X to construct a boundary face $e(P)$ associated with P ; $e(P)$ is diffeomorphic to a Euclidean space and $e(G)=X$. The corner $X(P)$ associated with P is, by definition, the disjoint union of boundary faces $e(Q)$, $Q\supset P$; $\bar{X}$ is the union of all the $X(P)$'s, so it is the disjoint union of all the $e(P)$'s. Borel and Serre introduced a Hausdorff topology on $\bar{X}$ under which it becomes a manifold with corners (topologically a manifold with boundary) whose interior is X. It is obvious from the construction of $\bar{X}$ that $G(\mathbb{Q})$ operates on it ; the Siegel property of Siegel sets is used to show that Γ operates properly on $\bar{X}$. The fact that there exists a Siegel set $\mathfrak{S}$, and a finite subset C of $G(\mathbb{Q})$, such that $G(\mathbb{R})=\mathfrak{S}\cdot C\cdot\Gamma$ quickly implies that $\bar{X}/\Gamma$ is compact. Thus $\bar{X}/\Gamma$ is a compact manifold with corners whose interior is X/Γ.

The boundary $\partial\bar{X}$ of $\bar{X}$ has the homotopy type of the Tits building of $G(\mathbb{Q})$; so, according to a theorem of Solomon and Tits, it has the homotopy type of an (infinite) bouquet of spheres of dimension $\ell-1$, where $\ell=\mathbb{Q}$-rank G. Thus the reduced homology $\widetilde{H}_i(\partial\bar{X})$ of $\partial\bar{X}$, with coefficients in $\mathbb{Z}$, vanishes except in dimension $\ell-1$, and $\widetilde{H}_{\ell-1}(\partial\bar{X})=:I$ is the Steinberg module of $G(\mathbb{Q})$. From Poincaré-duality for manifolds with boundary we get an isomorphism $H^i_c(\bar{X})\simeq H_{d-i}(\bar{X},\partial\bar{X})$, where d is the dimension of X. Hence, the cohomology group $H^i_c(\bar{X})$ is trivial unless $i=d-\ell$ in which case it is I. The fact that $\bar{X}/\Gamma$ is compact implies that $H^i(\Gamma,\mathbb{Z}[\Gamma])\simeq H^i_c(\bar{X})$, and we conclude that $H^i(\Gamma,\mathbb{Z}[\Gamma])$ vanishes for all i except $i=d-\ell$ and $H^{d-\ell}(\Gamma,\mathbb{Z}[\Gamma])=I$. Thus Γ is a generalized Poincaré-duality group, in the sense of Bieri and Eckmann, with dualizing module I, and its cohomological dimension is $d-\ell$.

A nice and comprehensive treatment of various compactifications of symmetric and locally symmetric spaces is given in a forthcoming book by Borel and Lizhen Ji.

L^2-cohomology. Let M be a Riemannian manifold. The Riemannian metric defines a positive definite scalar product $(\ ,\ )_x$ on the exterior algebra of the cotangent space at each point x of M, and hence a scalar product, which may possibly be infinite, on $\Omega^p(M)$, the space of real-valued smooth p-forms on M, given by

$$(\omega, \omega') = \int_M (\omega_x, \omega'_x)_x \, dv,$$

where dv is the Riemannian volume-form on M. An exterior p-form ω is said to be *square-integrable* if (ω, ω) is finite. Let $\Omega^*_{(2)}(M)$ be the subcomplex of the de Rham complex $\Omega^*(M)$ of M consisting of square-integrable forms ω such that $d\omega$ is also square-integrable. The p-th cohomology, denoted $H^p_{(2)}(M, \mathbb{R})$, of this subcomplex is called the p-th L^2-cohomology group of M, with coefficients in $\mathbb{R}$. There is a boundary operator $\partial : \Omega^p(M) \to \Omega^{p-1}(M)$ which is formally adjoint to d. A square-integrable form ω is said to be *harmonic* if $d\omega = 0 = \partial\omega$. Let $\mathcal{H}^*_{(2)}(M, \mathbb{R})$ be the graded group of harmonic forms. Then there is a homomorphism $\mathcal{H}^*_{(2)}(M, \mathbb{R}) \to H^*_{(2)}(M, \mathbb{R})$, which is injective if M is complete, and the cokernel is known to be either trivial or infinite-dimensional. According to a result of Kodaira, if a de Rham cohomology class is represented by a square-integrable form then it is also represented by a L^2-harmonic form.

Now let G be a connected semi-simple group defined over $\mathbb{Q}$ and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. As before, let X be the symmetric space of $G(\mathbb{R})$. Borel and Casselman showed in (126) that the L^2-cohomology of X/Γ is finite dimensional if the absolute rank of G equals the rank of a maximal compact subgroup of $G(\mathbb{R})$.

Stable cohomology of arithmetic groups. I shall now describe Borel's results on "stable" cohomology of arithmetic groups (100). Let G and Γ be as above and X be the symmetric space of $G(\mathbb{R})$ Let I_G be the space of smooth differential forms on the symmetric space X which are invariant under $G(\mathbb{R})^\circ$. Such forms are known to be harmonic and, as was shown by E. Cartan, I_G is the cohomology, with real coefficients, of the "compact dual" X_u of X. The inclusion of I^Γ_G in the de Rham complex of X/Γ induces a natural homomorphism $j^*_\Gamma : I^\Gamma_G \to H^*(X/\Gamma, \mathbb{R})$. As X is contractible, $H^*(X/\Gamma, \mathbb{R})$ is, in fact, the Eilenberg-MacLane cohomology $H^*(\Gamma, \mathbb{R})$, of Γ, with coefficients in $\mathbb{R}$. The main result of (100) gives a range in which j^p_Γ is an isomorphism. (For $\mathbb{Q}$-isotropic groups, this range is roughly $\frac{1}{4}(\mathbb{Q}\text{-rank}\, G)$.) If $G(\mathbb{R})/\Gamma$ is compact, then by Hodge theory j^*_Γ is injective, and Matsushima showed that there exists a positive constant $m(G(\mathbb{R}))$ such that j^p_Γ is surjective for all $p \leqslant m(G(\mathbb{R}))$. Garland observed that even when X/Γ is noncompact, Matsushima's argument can be used to show that j^p_Γ is surjective for a positive integer $p \leqslant m(G(\mathbb{R}))$, provided every cohomology class of X/Γ of dimension p is represented by a square-integrable exterior p-form. Together with Hsiang, Garland also gave a "square-integrabilty criterion" which gave a range up to which this condition is satisfied. (The work of Garland and Hsiang was inspired by a paper of Raghunathan on the first cohomology of an arithmetic subgroup of a connected semi-simple $\mathbb{Q}$-group G of $\mathbb{Q}$-rank 1, with coefficients in a rational G-module, in which he showed that any cohomology class is represented by a square-integrable 1-form.)

Now to find a range in which j_Γ^p is an isomorphism, Borel worked with the subcomplex C^*, of the de Rham complex $\Omega^*(X/\Gamma)$, consisting of forms which together with their exterior derivative have "logarithmic growth" near the boundary $\partial(\bar{X}/\Gamma)$ of X/Γ. He showed that (i) the inclusion $C \to \Omega^*(X/\Gamma)$ induces an isomorphism in the cohomology, (ii) there is a positive constant $c(G)$ such that for all $p \leqslant c(G)$, C^p consists of square-integrable forms, and (iii) $I_G^\Gamma \subset C^*$. From this it follows that j_Γ^p is injective for $p \leqslant c(G)$, and an isomorphism for $p \leqslant \min(c(G), m(G(\mathbb{R})))$. By explicitly computing $c(G)$ and $m(G(\mathbb{R}))$, Borel showed that j_Γ^p is an isomorphism for $p < \frac{1}{4}(\mathbb{Q}\text{-rank}\, G)$. Therefore, for p in this range, $H^p(\Gamma, \mathbb{R})$ coincides with the p-th cohomology group $H^p(X_u, \mathbb{R})$ of the compact dual X_u.

If we consider a sequence (H_n, f_n) of classical simple algebraic $\mathbb{Q}$-groups, where $f_n : H_n \to H_{n+1}$ is the natural inclusion, for example, $H_n = \mathrm{SL}_n$, then $\varprojlim I_{H_n}$ is known. In a given dimension p, the sequence $I_{H_n}^p$ is stationary, i. e., there is an integer $n(p)$ such that $I_{H_m}^p = \varprojlim I_{H_n}^p$ for $m \geqslant n(p)$. One can use this to compute, for example, the cohomology of $\mathrm{SL}(\mathfrak{o}) = \bigcup_n \mathrm{SL}_n(\mathfrak{o})$ and $\mathrm{Sp}(\mathfrak{o}) = \bigcup_n \mathrm{Sp}_{2n}(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers of a number field k.

Applications to K-theory of number fields. Quillen has shown that the groups $K_i(\mathfrak{o})$ are finitely generated abelian groups, and the rank of $K_i(\mathfrak{o})$ is equal to the dimension of the space of indecomposable elements (in the sense of Hopf algebras) in $H^*(\mathrm{SL}(\mathfrak{o}), \mathbb{R})$ of degree i. Using his results on cohomology of $\mathrm{SL}(\mathfrak{o})$, described above, in (100) Borel was able to determine the rank of $K_i(\mathfrak{o})$. He was able to similarly determine the ranks of Karoubi's L-groups of $\mathfrak{o}$. Borel's computation shows that the rank s_i of $K_i(\mathfrak{o})$ is zero if i is even and it is $r_1 + r_2$ or r_2, according as i is congruent to 1 or 3 modulo 4, where, as usual, r_1 is the number of real places of k and r_2 is the number of complex places. In an interesting work (108), Borel showed further that for odd i, say $i = 2m - 1$, there is a natural map (called these days the "Borel regulator map") from $K_i(\mathfrak{o})$ into the space of indecomposable elements of $H^*(\mathrm{SL}(\mathfrak{o}), \mathbb{R})$ of degree i (note that this vector space has a natural $\mathbb{Q}$-structure given by $H^*(\mathrm{SL}(\mathfrak{o}), \mathbb{Q})$) whose image is a lattice of covolume a rational multiple of ${D_k}^{1/2}\zeta_k(m)\pi^{-d(m+1)}$, where ζ_k is the Dedekind ζ-function of k, $d = [k : \mathbb{Q}]$, and D_k is the absolute value of the discriminant of k.

A vanishing theorem. Using Langlands' classification of irreducible admissible representations of real reductive groups, Borel and Wallach [BW], and independently, Zuckerman [Z], proved the following important vanishing theorem. Let G be a connected semi-simple algebraic group defined over $\mathbb{R}$, $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$ and K be a maximal compact subgroup. Let H be an infinite dimensional irreducible unitary $(\mathfrak{g}, K)$-module and F be a finite dimensional $(\mathfrak{g}, K)$-module, then $H^p(\mathfrak{g}, K, H \otimes F)$ *vanishes for all* $p < \mathbb{R}$*-rank* G. For a cocompact discrete subgroup Γ of $G(\mathbb{R})$, this vanishing theorem is known to imply that $H^p(\Gamma, \mathbb{R}) \simeq H^p(\mathfrak{g}, K, \mathbb{R})$ for $p < \mathbb{R}\text{-rank}\, G$. As $H^p(\mathfrak{g}, K, \mathbb{R})$ is isomorphic to the p-th cohomology of the compact dual X_u of the symmetric space associated with $G(\mathbb{R})$, we conclude that $j_\Gamma^p : I_G^\Gamma \to H^p(\Gamma, \mathbb{R})$ is an isomorphism for $p < \mathbb{R}\text{-rank}\, G$. This strengthens a result of Matsushima mentioned above. Kumaresan [K], using an idea of Parthasarathy, gave an elegant proof of a vanishing theorem which subsumes the vanishing theorems of Borel, Wallach and Zuckerman.

Most of the results mentioned above have analogues for S-arithmetic groups, and there are also results on cohomology of arithmetic and S-arithmetic groups with nontrivial coefficients, but describing them would have made this article too long.

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