# NONEXISTENCE OF ARITHMETIC FAKE COMPACT HERMITIAN SYMMETRIC SPACES OF TYPE OTHER THAN $A_n$ ($n \leqslant 4$)

Gopal Prasad and Sai-Kee Yeung

**Abstract.** *The quotient of a hermitian symmetric space of non-compact type by a torsion-free cocompact arithmetic subgroup of the identity component of the group of isometries of the symmetric space is called an arithmetic fake compact hermitian symmetric space if it has the same Betti numbers as the compact dual of the hermitian symmetric space. This is a natural generalization of the notion of "fake projective planes " to higher dimensions. Study of arithmetic fake compact hermitian symmetric spaces of type $A_n$ with even n has been completed in* [PY1], [PY2]. *The results of this paper, combined with those of* [PY2], *imply that there does not exist any arithmetic fake compact hermitian symmetric space of type other than $A_n$, $n \leqslant 4$ (see Theorems 1 and 2 in the Introduction below and Theorem 2 of* [PY2]*). The proof involves the volume formula given in* [P], *the Bruhat-Tits theory of reductive p-adic groups, and delicate estimates of various number theoretic invariants.*

*Keywords:* arithmetic lattices, Bruhat-Tits theory, volume formula, cohomology.

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## 1. Introduction

**1.1.** Let $\overline{\mathscr{G}}$ be a connected real semi-simple Lie group with trivial center and with no non-trivial compact normal subgroups, and $\mathfrak{g}$ be its Lie algebra. The group $\mathrm{Aut}(\overline{\mathscr{G}})$ (=$\mathrm{Aut}(\mathfrak{g})$) of automorphisms of $\overline{\mathscr{G}}$ is a Lie group with finitely many connected components, and its identity component is $\overline{\mathscr{G}}$. We will denote the identity component of $\mathrm{Aut}(\overline{\mathscr{G}})$ in the *Zariski-topology* by $\mathrm{Int}(\overline{\mathscr{G}})$. Let $X$ be the symmetric space of $\overline{\mathscr{G}}$ ($X$ is the space of maximal compact subgroups of $\overline{\mathscr{G}}$), and $X_u$ be the compact dual of $X$. There is a natural identification of the group of isometries of $X$ with $\mathrm{Aut}(\overline{\mathscr{G}})$. We assume in this paper that $X$ (and hence $X_u$) is hermitian. Then every holomorphic automorphism of $X$ is an isometry. The group $\mathrm{Hol}(X)$ of holomorphic automorphisms of $X$ is a subgroup of finite index of the group $\mathrm{Aut}(\overline{\mathscr{G}})$ of isometries, and it is known (see [Ta], the remark in §5) that $\mathrm{Hol}(X) \cap \mathrm{Int}(\overline{\mathscr{G}}) = \overline{\mathscr{G}}$.

**1.2.** We will say that the quotient $X/\Pi$ of $X$ by a *torsion-free* cocompact discrete subgroup $\Pi$ of $\overline{\mathscr{G}}$ is a *fake compact hermitian symmetric space*, or a *fake* $X_u$, if its Betti numbers are same as that of $X_u$; $X/\Pi$ is an *arithmetic fake compact hermitian symmetric space*, or an

G. Prasad: University of Michigan, Ann Arbor, MI 48109
e-mail: gprasad@umich.edu
S.-K. Yeung: Purdue University, West Lafayette, IN 47907
(corresponding author) email: yeung@math.purdue.edu.

*arithmetic fake* $X_u$, if, moreover, $\Pi$ is irreducible (i.e., no subgroup of $\Pi$ of finite index is a direct product of two infinite normal subgroups) and it is an arithmetic subgroup of $\overline{\mathscr{G}}$. Any such space can be endowed with the structure of a smooth complex projective variety.

We note that if $\overline{\mathscr{G}}$ contains an irreducible arithmetic subgroup, then the simple factors of its complexification are isomorphic to each other, see [Marg], Corollary 4.5 in Ch. IX. Also, if $\mathbb{R}$-rank of $\overline{\mathscr{G}}$ is at least 2, which is the case for all $\overline{\mathscr{G}}$ to be considered in §§4–7 of this paper, then by Margulis' arithmeticity theorem ([Marg], Ch. IX]), any irreducible discrete cocompact subgroup of $\overline{\mathscr{G}}$ (in fact, any irreducible lattice) is arithmetic.

If $\Pi$ is a torsion-free cocompact discrete subgroup of $\overline{\mathscr{G}}$, then there is a natural embedding of $H^*(X_u, \mathbb{C})$ in $H^*(X/\Pi, \mathbb{C})$, see [B], 3.1 and 10.2, and hence $X/\Pi$ is a fake $X_u$ if and only if this embedding is an isomorphism.

**1.3.** Let $\overline{\mathscr{G}}$, $X$ and $X_u$ be as above, and let $\Pi$ be a torsion-free cocompact discrete subgroup of $\overline{\mathscr{G}}$. Let $Z = X/\Pi$. If $Z$ is a fake $X_u$, then the Euler-Poincaré characteristic $\chi(Z)$ of $Z$, and so the Euler-Poincaré characteristic $\chi(\Pi)$ of $\Pi$, equals $\chi(X_u)$. As $X$ has been assumed to be hermitian, the Euler-Poincaré characteristic of $X_u$ is positive. On the other hand, it follows from Hirzebruch proportionality principle, see [Ser], Proposition 23, that the Euler-Poincaré characteristic of $X/\Pi$ is positive if and only if the complex dimension of $X$ is even. Using the results of [BP], we can easily conclude that there are only finitely many irreducible arithmetic fake compact hermitian symmetric spaces of types other than $A_1$. It is of interest to determine them all.

**1.4.** Hermitian symmetric spaces have been classified by Élie Cartan; see [H], Ch. IX. We recall that the noncompact irreducible hermitian symmetric spaces are the symmetric spaces of Lie groups $\mathrm{SU}(n+1-m, m)$, $\mathrm{SO}(2, 2n-1)$, $\mathrm{Sp}(2n)$, $\mathrm{SO}(2, 2n-2)$, $\mathrm{SO}^*(2n)$, an absolutely simple real Lie group of type $E_6$ with Tits index ${}^2E_{6,2}^{16'}$, and an absolutely simple real Lie group of type $E_7$ with Tits index $E_{7,3}^{28}$ (for Tits indices see Table II in [Ti1]). The complex dimensions of these spaces are $(n+1-m)m$, $2n-1$, $n(n+1)/2$, $2n-2$, $n(n-1)/2$, 16 and 27 respectively. The Lie groups listed above are of type $A_n$, $B_n$, $C_n$, $D_n$, $D_n$, $E_6$ and $E_7$ respectively. We will say that a symmetric space is one of these types if it is a product of symmetric spaces of noncompact simple Lie groups of that type, and say that a hermitian locally symmetric space is of one of these types if its simply connected cover is a hermitian symmetric space of that type.

The purpose of this paper is to prove the following two theorems.

**Theorem 1.** *There does not exist an irreducible arithmetic fake compact hermitian symmetric space of type other than* $A_n$.

Regarding spaces of type $A_n$, we have the following result.

**Theorem 2.** *There does not exist an irreducible arithmetic fake compact hermitian symmetric space of type* $A_n$ *with* $n > 4$.

The proof of Theorem 1 is carried out in §§4–7. Arithmetic fake compact hermitian symmetric spaces of type $A_n$, with $n$ even, have been studied in detail in [PY1] and [PY2]. In [PY1] we have given a classification of "fake projective planes", the first of which was constructed by David Mumford in [Mu] using $p$-adic uniformization. Note that fake projective planes are arithmetic fake compact hermitian symmetric spaces of type $A_2$. Using ingenious computer-assisted group theoretic computations, Cartwright and Steger ([CS]) have shown that the twenty eight classes of fake projective planes of [PY1] altogether contain *fifty* distinct fake projective planes up to isometry with respect to the Poincaré metric [CS]. Since each of them supports two distinct complex structures [KK,§5], there are exactly *one hundred* fake projective planes counted up to biholomorphism. In [PY2] we have shown that arithmetic fake compact hermitian symmetric spaces of type $A_n$, with $n$ even, can exist only for $n = 2$, 4, and have constructed four arithmetic fake $\mathbf{P}^4_{\mathbb{C}}$, four arithmetic fake Grassmannians $\mathbf{Gr}_{2,5}$, and five (irreducible) arithmetic fake $\mathbf{P}^2_{\mathbb{C}}\times\mathbf{P}^2_{\mathbb{C}}$. (Fake $\mathbf{P}^4_{\mathbb{C}}$ and fake $\mathbf{Gr}_{2,5}$ are of type $A_4$ and every fake $\mathbf{P}^2_{\mathbb{C}}\times\mathbf{P}^2_{\mathbb{C}}$ is of type $A_2$.) To prove Theorem 2 we therefore assume that $n$ is odd and $> 3$. The proof occupies §§8–9. We also prove some results for $n = 3$, see Proposition 3 at the end of §8, and 9.3.

In the following subsection we will explain the strategy of the proof, and fix notation which will be used throughout the paper.

**1.5.** Let $\overline{\mathscr{G}}$, $X$, $X_u$ be as in 1.1; $X$ will be assumed to be a hermitian symmetric space of one of the following types: $A_n$ with $n > 3$ odd, $B_n$, $C_n$, $D_n$, $E_6$ and $E_7$. Assume, if possible, that $\overline{\mathscr{G}}$ contains a cocompact irreducible arithmetic subgroup $\Pi$ whose orbifold Euler-Poincaré characteristic $\chi(\Pi)$ equals $\chi(X_u)$. Then there exist a number field $k$, a connected adjoint absolutely simple algebraic $k$-group $\overline{G}$ of same type as $X$, real places $v_1,\dots,v_r$ of $k$ such that $\overline{G}(k_v)$ is compact for every real place $v$ different from $v_1,\dots,v_r$, $\overline{\mathscr{G}}$ is isomorphic to $\prod_{j=1}^r\overline{G}(k_{v_j})^\circ$ (and will be identified with it), and $\Pi$ is an arithmetic subgroup contained in $\overline{G}(k)$. It is obvious from this that $k$ is totally real. If $\overline{G}$ is either of type $A_n$ ($n > 1$ arbitrary) or $D_n$ with $n$ odd, or of type $E_6$, then for every real place $v$ of $k$, $\overline{G}$ is an outer form over $k_v$, and hence the unique quadratic extension $\ell$ of $k$ over which $\overline{G}$ is an inner form is totally complex. If $\overline{G}$ is of type $D_n$ with $n$ even, and it is *not* a triality form of type $D_4$, then at every real place $v$ of $k$, $\overline{G}$ is an inner form, and hence either $\overline{G}$ is an inner $k$-form, or the unique quadratic extension $\ell$ over which $\overline{G}$ is an inner form is a totally real field. If $\overline{G}$ is a triality form of type $D_4$, let $\ell$ be a fixed cubic extension of $k$ contained in the smallest Galois extension of $k$ over which $\overline{G}$ is an inner form. For triality forms occuring in this paper, $\ell$ is totally real. As $\Pi$ is cocompact, by Godement compactness criterion $\overline{G}$ is anisotropic over $k$ (i.e., its $k$-rank is 0).

Let $\pi : G \to \overline{G}$ be the simply connected cover of $\overline{G}$ defined over $k$. The kernel of the isogeny $\pi$ is the center $C$ of the simply connected $k$-group $G$.

*Description of C:* For a positive integer $a$, let $\mu_a$ be the kernel of the endomorphism $x \mapsto x^a$ of $\mathrm{GL}_1$. Then if $G$ is of type ${}^2A_n$, its center is $k$-isomorphic to the kernel of the norm map $N_{\ell/k} : R_{\ell/k}(\mu_{n+1}) \to \mu_{n+1}$, and if $G$ is of type ${}^2E_6$, its center is $k$-isomorphic to the kernel of the norm map $N_{\ell/k} : R_{\ell/k}(\mu_3) \to \mu_3$. If $G$ is of type $B_n$, $C_n$ or $E_7$, then $C$ is $k$-isomorphic to $\mu_2$. If $G$ is an inner $k$-form of type $D_n$ with $n$ even, then $C$ is $k$-isomorphic to $\mu_2\times\mu_2$, and if $G$ is a non-triality outer form of type $D_n$, $C$ is $k$-isomorphic to $R_{\ell/k}(\mu_2)$ or to the kernel of the norm map $N_{\ell/k} : R_{\ell/k}(\mu_4) \to \mu_4$ according as $n$ is even or odd. If $G$ is a triality form of type $D_4$, let the cubic extension $\ell$ of $k$ be as above. Then $C$ is $k$-isomorphic to the kernel of the norm map $N_{\ell/k} : R_{\ell/k}(\mu_2) \to \mu_2$.

It is known, and easy to see using the above description of $C$, that for any real place $v$ of $k$, the order of the kernel of the induced homomorphism $G(k_v) \to \overline{G}(k_v)$ is $n+1$ if $G$ is of type ${}^2A_n$, is of order 2 if $G$ is of type $B_n$, $C_n$ or $E_7$, is of order 4 if $G$ is of type $D_n$, and of order 3 if it is of type ${}^2E_6$. Moreover, as $G(k_v)$ is connected, $\pi(G(k_v)) = \overline{G}(k_v)^\circ$. Let $\mathscr{G} = \prod_{j=1}^r G(k_{v_j})$, and let $\widetilde{\Pi}$ be the inverse image of $\Pi$ in $\mathscr{G}$. Then the kernel of the homomorphism $\pi : \mathscr{G} \to \overline{\mathscr{G}}$ is of order $s^r$, and hence the orbifold Euler-Poincaré characteristic $\chi(\widetilde{\Pi})$ of $\widetilde{\Pi}$ equals $\chi(\Pi)/s^r = \chi(X_u)/s^r$, where here, and in the sequel, $s = n+1$ if $G$ is of type $A_n$, $s = 2$ if $G$ is of type $B_n$, $C_n$ or $E_7$, $s = 4$ if $G$ is of type $D_n$, and $s = 3$ if $G$ is of type $E_6$. Now let $\Gamma$ be a maximal discrete subgroup of $\mathscr{G}$ containing $\widetilde{\Pi}$. Then the orbifold Euler-Poincaré characteristic $\chi(\Gamma)$ of $\Gamma$ is a submultiple[1] of $\chi(\widetilde{\Pi}) = \chi(X_u)/s^r$. Using the volume formula of [P], some nontrivial number theoretic estimates, the Bruhat-Tits theory, and the Hasse principle for semi-simple groups (Proposition 7.1 of [PR]), we will show that $\mathscr{G}$ does not contain such a subgroup $\Gamma$. This will prove Theorems 1 and 2.

## 2. Preliminaries

**2.1.** We will use the notations introduced in 1.5. Thus $k$ will be a totally real number field, $G$ an absolutely simple simply connected algebraic $k$-group (of one of the following nine types: ${}^2A_n$ with $n\,(>3)$ odd, $B_n$, $C_n$, ${}^1D_n$, ${}^2D_n$, ${}^3D_4$, ${}^6D_4$, ${}^2E_6$, and $E_7$), $C$ its center, $\mathscr{G} = \prod_{j=1}^r G(k_{v_j})$. We will think of $G(k)$ as a subgroup of $\mathscr{G}$ in terms of its diagonal embedding.

$V_f$ (resp. $V_\infty$) will denote the set of nonarchimedean (resp., archimedean) places of $k$. As $k$ admits at least $r$ distinct real places, see 1.5, $d := [k : \mathbb{Q}] \geqslant r$. For $v \in V_f$, $q_v$ will denote the cardinality of the residue field $\mathfrak{f}_v$ of $k_v$. If $G$ is an outer form, $\ell$ will denote the quadratic or cubic extension of $k$ as in 1.5. If $G$ is an inner form, let $\ell = k$.

As explained in 1.5, to prove Theorems 1 and 2 it will suffice to show that $\mathscr{G}$ does not contain a maximal arithmetic subgroup $\Gamma$ ($\Gamma$ arithmetic with respect to the $k$-structure on $G$) whose orbifold Euler-Poincaré characteristic is a submultiple of $\chi(X_u)/s^r$. Assume, if possible, that such a $\Gamma$ exists. Then $\Lambda := \Gamma \cap G(k)$ is a "principal" arithmetic subgroup, i.e., for every nonarchimedean place $v$ of $k$, the closure $P_v$ of $\Lambda$ in $G(k_v)$ is a parahoric subgroup and $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$, moreover, $\Gamma$ is the normalizer of $\Lambda$ in $\mathscr{G}$; see Proposition 1.4(iv) of [BP]. Let the "type" $\Theta_v$ of $P_v$ be as in [BP], 2.2, and $\Xi_{\Theta_v}$ be as in 2.8 there. If $P_v$ is hyperspecial, then $\Xi_{\Theta_v}$ is trivial. The order of $\Xi_{\Theta_v}$ is always a divisor of $s$ ($s$ as in 1.5). We note that for all but finitely many $v \in V_f$, $P_v$ is hyperspecial.

In terms of the normalized Haar-measure $\mu$ on $\mathscr{G} = \prod_{j=1}^r G(k_{v_j})$ used in [P] and [BP], and to be used in this paper, $|\chi(\Gamma)| = \chi(X_u)\mu(\mathscr{G}/\Gamma)$ (see [BP], 4.2). Thus the condition that $\chi(\Gamma)$ is a submultiple of $\chi(X_u)/s^r$ is equivalent to the condition that $\mu(\mathscr{G}/\Gamma)$ is a submultiple of $1/s^r$. We will show below that $\mathscr{G}$ does not contain a maximal arithmetic subgroup $\Gamma$ such that $\mu(\mathscr{G}/\Gamma)$ is a submultiple of $1/s^r$.

For a comprehensive survey of the basic notions and the main results of the Bruhat-Tits theory of reductive groups over nonarchimedean local fields, used in this paper, see [Ti2].

[1]given two nonzero real numbers $x$ and $y$, we say that $y$ is a *submultiple* of $x$ if $x/y$ is an integer

**2.2.** All unexplained notations are as in [BP] and [P]. Thus for a number field $K$, $D_K$ will denote the absolute value of its discriminant, $h_K$ its class number, i.e., the order of its class group $Cl(K)$. We will denote by $h_{K,s}$ the order of the subgroup of $Cl(K)$ consisting of the elements of order dividing $s$, where, as in 1.5, $s = n+1$ if $G$ is of type $A_n$, $s = 2$ if $G$ is of type $B_n$ or $C_n$, $s = 4$ if $G$ is of type $D_n$, and $s = 3$ if $G$ is of type $E_6$. Then $h_{K,s}|h_K$. We will denote by $U_K$ the multiplicative-group of units of $K$, and by $K_s$ the subgroup of $K^\times$ consisting of the elements $x$ such that for every normalized valuation $v$ of $K$, $v(x) \in s\mathbb{Z}$.

**2.3.** For a parahoric subgroup $P_v$ of $G(k_v)$, we define $e(P_v)$ and $e'(P_v)$ by the following formulae (cf. Theorem 3.7 of [P]):

$$e(P_v) = \frac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)}. \tag{1}$$

$$e'(P_v) = e(P_v)\cdot\frac{\#\overline{\mathscr{M}}_v(\mathfrak{f}_v)}{q_v^{\dim \overline{\mathscr{M}}_v}} = q_v^{(\dim \overline{M}_v - \dim \overline{\mathscr{M}}_v)/2}\cdot\frac{\#\overline{\mathscr{M}}_v(\mathfrak{f}_v)}{\#\overline{M}_v(\mathfrak{f}_v)}. \tag{2}$$

**2.4.** Let $m_1, \ldots, m_n$ ($m_1 \leqslant \cdots \leqslant m_n$), where $n$ is the absolute rank of $G$, be the exponents of the Weyl group of $G$. For type $A_n$, $m_j = j$; for types $B_n$ and $C_n$, $m_j = 2j-1$; for type $D_n$ the exponents are 1, 3, 5, ..., $2n-5$, $2n-3$ and $n-1$ (the multiplicity of $n-1$ is two when $n$ is even); for type $E_6$, the exponents are 1, 4, 5, 7, 8 and 11; and for type $E_7$, the exponents are 1, 5, 7, 9, 11, 13 and 17. Then

• if either $G$ is of inner type, or $v$ completely splits in $\ell$,

$$e'(P_v) = e(P_v)\prod_{j=1}^{n}\Big(1 - \frac{1}{q_v^{m_j+1}}\Big);$$

• if $v$ does not split in $\ell$ and $G$ is of type ${}^2A_n$ with $n$ odd, then

$$e'(P_v) = e(P_v)\Big(1 - \frac{1}{q_v^{n+1}}\Big)\prod_{j=1}^{(n-1)/2}\Big(1 - \frac{1}{q_v^{2j}}\Big)\Big(1 + \frac{1}{q_v^{2j+1}}\Big),$$

or

$$e'(P_v) = e(P_v)\prod_{j=1}^{(n+1)/2}\Big(1 - \frac{1}{q_v^{2j}}\Big)$$

according as $v$ *does not* or *does* ramify in $\ell$.

• if $G$ is of type ${}^2D_n$ and $v$ does not split in $\ell$,

$$e'(P_v) = e(P_v)\Big(1 + \frac{1}{q_v^{n}}\Big)\prod_{j=1}^{n-1}\Big(1 - \frac{1}{q_v^{2j}}\Big),$$

or

$$e'(P_v) = e(P_v)\prod_{j=1}^{n-1}\Big(1 - \frac{1}{q_v^{2j}}\Big)$$

according as $v$ *does not* or *does* ramify in $\ell$.

• if $G$ is a triality form (i.e., of type ${}^3D_4$ or ${}^6D_4$) and $v$ does not completely split in $\ell$, let $\omega$ be a nontrivial cube root of unity, then

(i) if $\ell_v := \ell \otimes_k k_v$ is a (cubic) field extension of $k_v$,

$$e'(P_v) = e(P_v)\Big(1 - \frac{1}{q_v^2}\Big)\Big(1 - \frac{\omega}{q_v^4}\Big)\Big(1 - \frac{\omega^2}{q_v^4}\Big)\Big(1 - \frac{1}{q_v^6}\Big),$$

or

$$e'(P_v) = e(P_v)\Big(1 - \frac{1}{q_v^2}\Big)\Big(1 - \frac{1}{q_v^6}\Big)$$

according as $\ell_v$ is a *unramified* or a *ramified* extension of $k_v$,

(ii) if $\ell \otimes_k k_v$ is a direct product of $k_v$ and a quadratic field extension of $k_v$, then

$$e'(P_v) = e(P_v)\Big(1 + \frac{1}{q_v^4}\Big)\Big(1 - \frac{1}{q_v^2}\Big)\Big(1 - \frac{1}{q_v^4}\Big)\Big(1 - \frac{1}{q_v^6}\Big),$$

or

$$e'(P_v) = e(P_v)\Big(1 - \frac{1}{q_v^2}\Big)\Big(1 - \frac{1}{q_v^4}\Big)\Big(1 - \frac{1}{q_v^6}\Big)$$

according as the quadratic extension is *unramified* or *ramified*

• if $G$ is of type ${}^2E_6$ and $v$ does not split in $\ell$,

$$e'(P_v) = e(P_v)\Big(1 - \frac{1}{q_v^2}\Big)\Big(1 + \frac{1}{q_v^5}\Big)\Big(1 - \frac{1}{q_v^6}\Big)\Big(1 - \frac{1}{q_v^8}\Big)\Big(1 + \frac{1}{q_v^9}\Big)\Big(1 - \frac{1}{q_v^{12}}\Big),$$

or

$$e'(P_v) = e(P_v)\Big(1 - \frac{1}{q_v^2}\Big)\Big(1 - \frac{1}{q_v^6}\Big)\Big(1 - \frac{1}{q_v^8}\Big)\Big(1 - \frac{1}{q_v^{12}}\Big)$$

according as $v$ *does not* or *does* ramify in $\ell$.

**2.5.** Since $q_v^{\dim \overline{\mathscr{M}}_v} > \#\overline{\mathscr{M}}_v(\mathfrak{f}_v)$ (cf. 2.6 of [P]), $e'(P_v) < e(P_v)$. It is not difficult to check by case-by-case computations, using (2) and the Bruhat-Tits theory, that *for all $v \in V_f$, and an arbitrary parahoric subgroup $P_v$ of $G(k_v)$, $e'(P_v)$ is an integer.* If, for example, either $G$ is quasi-split over $k_v$ and splits over the maximal unramified extension of $k_v$ (equivalently, $G(k_v)$ contains a hyperspecial parahoric subgroup), or it does not split over the maximal unramified extension of $k_v$, then explicit computations can be avoided using the fact that the order of a subgroup of a finite group divides the order of the latter, an analogue (see [Gi]) for reductive groups over finite fields of a result of Borel and de Siebenthal on subgroups of maximal rank of a compact Lie group, and the fact that over a finite field $\mathfrak{f}$, the groups of $\mathfrak{f}$-rational points of connected absolutely simple $\mathfrak{f}$-groups of types $B_m$ and $C_m$, for an arbitrary $m$, have equal order. A detailed proof of the integrality of $e'(P_v)$ for groups of type $A_n$ is given in [GM].

**2.6.** Now we will use the volume formula of [P] to write down the precise value of $\mu(\mathscr{G}/\Lambda)$. As the Tamagawa number $\tau_k(G)$ of $G$ equals 1, Theorem 3.7 of [P] (recalled in 3.7 of [BP]), for $S = V_\infty$, provides us the following:

$$\mu(\mathscr{G}/\Lambda) = D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}\mathfrak{s}}\Big(\prod_{j=1}^{n}\frac{m_j!}{(2\pi)^{m_j+1}}\Big)^d\mathscr{E}, \tag{3}$$

where $n$ is the absolute rank of $G$, $\mathfrak{s}$ is $(n-1)(n+2)/2$ if $G$ is of type ${}^2A_n$ with $n$ odd, $2n-1$ if $G$ is of type ${}^2D_n$, 7 if $G$ is a triality form (i.e., of type ${}^3D_4$ or ${}^6D_4$), 26 if $G$ is of type ${}^2E_6$, and 0 for all other groups under consideration in this paper, and

$$\mathscr{E} = \prod_{v\in V_f} e(P_v),$$

with $e(P_v)$ as in 2.3.

**2.7.** Let $\zeta_k$, $\zeta_\ell$ be the Dedekind zeta-functions of $k$ and $\ell$ respectively. We will let $\zeta_{\ell|k}$ denote the function $\zeta_\ell/\zeta_k$. If $\ell$ is a quadratic extension of $k$, which will often be the case in this paper, $\zeta_{\ell|k}$ is the Hecke $L$-function associated to the nontrivial Dirichlet character of $\ell/k$. Recall that

$$\zeta_k(a) = \prod_{v\in V_f} \Big(1-\frac{1}{q_v^a}\Big)^{-1},$$

and if $\ell$ is a quadratic extension of $k$,

$$\zeta_{\ell|k}(a) = \prod{}' \Big(1-\frac{1}{q_v^a}\Big)^{-1} \prod{}'' \Big(1+\frac{1}{q_v^a}\Big)^{-1},$$

where $\prod'$ is the product over the nonarchimedean places $v$ of $k$ which split in $\ell$, and $\prod''$ is the product over the nonarchimedean places $v$ which do not split and also do not ramify in $\ell$. We will let the reader write down a similar product expression for $\zeta_{\ell|k}(a) = \zeta_\ell(a)/\zeta_k(a)$ when $\ell$ is a cubic extension of $k$.

Using the values of $e'(P_v)$ given in 2.4 we will rewrite the Euler product $\mathscr{E}$ appearing in (3). For this purpose we define

$$\mathscr{Z} = \prod_{j=1}^{n} \zeta_k(m_j+1)$$

if $G$ is of inner type;

$$\mathscr{Z} = \zeta_k(n+1) \prod_{j=1}^{(n-1)/2} \zeta_k(2j)\zeta_{\ell|k}(2j+1)$$

if $G$ is of type ${}^2A_n$ with $n$ odd;

$$\mathscr{Z} = \zeta_{\ell|k}(n) \prod_{j=1}^{n-1} \zeta_k(2j)$$

if $G$ is of type ${}^2D_n$;

$$\mathscr{Z} = \zeta_k(2)\zeta_{\ell|k}(4)\zeta_k(6)$$

if $G$ is a triality form;

$$\mathscr{Z} = \zeta_k(2)\zeta_{\ell|k}(5)\zeta_k(6)\zeta_k(8)\zeta_{\ell|k}(9)\zeta_k(12)$$

if $G$ is of type ${}^2E_6$. Then for all $G$,

$$\mathscr{E} = \mathscr{Z} \prod_{v\in V_f} e'(P_v). \tag{4}$$

**2.8.** If $G$ is of inner type, let

$$\mathscr{R} = 2^{-dn}\Big|\prod_{j=1}^{n}\zeta_k(-m_j)\Big|. \tag{5}$$

If $G$ is of type ${}^2A_n$ with $n$ odd, let

$$\mathscr{R} = 2^{-dn}\Big|\zeta_k(-n)\prod_{j=1}^{(n-1)/2}\zeta_k(1-2j)\zeta_{\ell|k}(-2j)\Big|. \tag{6}$$

If $G$ is of type ${}^2D_n$ (recall that $\ell$ is totally real or totally complex according as $n$ is even or odd), let

$$\mathscr{R} = 2^{-dn}\Big|\zeta_{\ell|k}(1-n)\prod_{j=1}^{n-1}\zeta_k(1-2j)\Big|. \tag{7}$$

If $G$ is a triality form (then $\ell$ is a totally real cubic extension of $k$), let

$$\mathscr{R} = 2^{-4d}|\zeta_k(-1)\zeta_{\ell|k}(-3)\zeta_k(-5)|. \tag{8}$$

If $G$ is of type ${}^2E_6$ (then $\ell$ is a totally complex quadratic extension of $k$), let

$$\mathscr{R} = 2^{-6d}|\zeta_k(-1)\zeta_{\ell|k}(-4)\zeta_k(-5)\zeta_k(-7)\zeta_{\ell|k}(-8)\zeta_k(-11)|. \tag{9}$$

Using the following functional equations for any totally real $k$ and respectively a totally real extension of arbitrary degree and a totally complex quadratic extension $\ell$ of $k$,

$$\zeta_k(2a) = D_k^{\frac{1}{2}-2a}\Big(\frac{(-1)^a 2^{2a-1}\pi^{2a}}{(2a-1)!}\Big)^d \zeta_k(1-2a),$$

$$\zeta_{\ell|k}(2a) = \Big(\frac{D_k}{D_\ell}\Big)^{2a-\frac{1}{2}}\Big(\frac{(-1)^a 2^{2a-1}\pi^{2a}}{(2a-1)!}\Big)^{d([\ell:k]-1)}\zeta_{\ell|k}(1-2a),$$

and

$$\zeta_{\ell|k}(2a+1) = \Big(\frac{D_k}{D_\ell}\Big)^{2a+\frac{1}{2}}\Big(\frac{(-1)^a 2^{2a}\pi^{2a+1}}{(2a)!}\Big)^d \zeta_{\ell|k}(-2a),$$

for every positive integer $a$, and the fact that $\dim G = n + 2\sum m_j$, the volume formula (3) and the explicit value of $\mathscr{E}$ given for each case in 2.7, we find that

$$\mu(\mathscr{G}/\Lambda) = \mathscr{R}\prod_{v\in V_f} e'(P_v), \tag{10}$$

where $\mathscr{R}$ is as above.

**2.9.** We have the following

$$\mu(\mathscr{G}/\Gamma) = \frac{\mu(\mathscr{G}/\Lambda)}{[\Gamma:\Lambda]} = \frac{\mathscr{R}\prod_{v\in V_f} e'(P_v)}{[\Gamma:\Lambda]}. \tag{11}$$

Let $s$ be as in 1.5. Proposition 2.9 of [BP] applied to $G' = G$ and $\Gamma' = \Gamma$ implies that any prime divisor of $[\Gamma:\Lambda]$ divides $s$. Now since $e'(P_v)$ is an integer for all $v \in V_f$, we conclude from (11) that if $\mu(\mathscr{G}/\Gamma)$ is a submultiple of 1, then any prime which divides the numerator of the rational number $\mathscr{R}$ is a divisor of $s$. We record this observation as the following proposition.

**Proposition 1.** *If $\mu(\mathscr{G}/\Gamma)$ is a submultiple of 1 (or, equivalently, the orbifold Euler-Poincaré characteristic $\chi(\Gamma)$ of $\Gamma$ is a submultiple of $\chi(X_u)$), then every prime divisor of the numerator of the rational number $\mathscr{R}$ divides s.*

**2.10.** Let $\mathscr{T}$ be the set of all nonarchimedean places $v$ of $k$ such that *either* (i) $v$ does not ramify in $\ell$ (equivalently, $G$ splits over the maximal unramified extension of $k_v$) and $P_v$ is not a hyperspecial parahoric subgroup of $G(k_v)$, *or* (ii) $v$ ramifies in $\ell$, $G$ is quasi-split over $k_v$ and $P_v$ is not special. It can be easily seen, using the relative local Dynkin diagram of $G/k_v$ given in 4.3 of [Ti2], that if $v \notin \mathscr{T}$, then $\Xi_{\Theta_v}$ is trivial; if $v \in \mathscr{T}$ ramifies in $\ell$, then $\#\Xi_{\Theta_v} \leqslant 2$.

If for a $v \in V_f$, $P_v$ is hyperspecial, then obviously $e'(P_v) = 1$. On the other hand, it is not difficult to see, by direct computation, that $e'(P_v) > s$ for all $v \in \mathscr{T}$. Therefore, $\mathscr{E} = \prod_{v\in V_f} e(P_v) > \prod_{v\in V_f} e'(P_v) > s^{\#\mathscr{T}}$ (cf. 2.5), and hence, we see from (3) that

$$\mu(\mathscr{G}/\Lambda) > D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s}\Big(\prod_{j=1}^{n}\frac{m_j!}{(2\pi)^{m_j+1}}\Big)^d s^{\#\mathscr{T}}. \tag{12}$$

Since $\mu(\mathscr{G}/\Gamma) = \mu(\mathscr{G}/\Lambda)/[\Gamma : \Lambda]$ is a submultiple of $1/s^r$ (see 2.1), we conclude that $\mu(\mathscr{G}/\Lambda) \leqslant [\Gamma : \Lambda]/s^r$. From bound (12) we now obtain:

$$D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s}\Big(\prod_{j=1}^{n}\frac{m_j!}{(2\pi)^{m_j+1}}\Big)^d s^{\#\mathscr{T}} < [\Gamma : \Lambda]/s^r. \tag{13}$$

## 3. Discriminant bounds

We will recall discriminant bounds required in later discussions. We define $M_r(d) = \min_K D_K^{1/d}$, where the minimum is taken over all totally real number fields $K$ of degree $d$. Similarly, we define $M_c(d) = \min_K D_K^{1/d}$, by taking the minimum over all totally complex number fields $K$ of degree $d$.

The precise values of $M_r(d), M_c(d)$ for low values of $d$ are given in the following table (cf. [N]).

| $d$ : | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $M_r(d)^d$ : | 5 | 49 | 725 | 14641 | 300125 | 20134393 | 282300416 |
| $M_c(d)^d$ : | 3 | | 117 | | 9747 | | 1257728. |

The following proposition can be proved in the same way as Proposition 2 in [PY2] has been proved.

**Proposition 2.** *Let $k$ and $\ell$ be a totally real number field and a totally complex number field of degree $d$ respectively.*

| $\forall d \geqslant$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $D_k^{1/d} >$ | 2.23 | 3.65 | 5.18 | 6.8 | 8.18 | 11.05 | 11.38 |
| $D_\ell^{1/d} >$ | 1.73 | | 3.28 | | 4.62 | | 5.78. |

## 4. $G$ of type $B_n$ or $C_n$

**4.1.** In this section we assume that $G$ is of type $B_n$ or $C_n$ with $n > 1$. Then its dimension is $n(2n+1)$. The $j$-th exponent $m_j = 2j-1$, $\mathfrak{s} = 0$, and the complex dimension of the symmetric space $X$ of $\mathscr{G} = \prod_{j=1}^{r} G(k_{v_j})$ is $r(2n-1)$ if $G$ is of type $B_n$, and is $rn(n+1)/2$ if $G$ is of type $C_n$. The center $C$ of $G$ is $k$-isomorphic to $\mu_2$ and $s = 2$. The Galois cohomology group $H^1(k, C)$ is isomorphic to $k^\times/k^{\times 2}$. The order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G' = G$ and $S = V_\infty$, is $2^{r-1}$. From the proof of Proposition 0.12 of [BP], we easily conclude that $\#k_2/k^{\times 2} \leqslant h_{k,2}2^d$. Let $\mathscr{T}$ be as in 2.10. We can adapt the argument used to prove Proposition 5.1 in [BP], and the argument in 5.5 of [BP], for $S = V_\infty$ and $G' = G$, to derive the following bound from Proposition 2.9 of [BP]:

(14) $$[\Gamma : \Lambda] \leqslant h_{k,2} 2^{d+r-1+\#\mathscr{T}}.$$

Hence, from (13) we obtain

(15) $$D_k^{1/d} < f_1(n, d, h_{k,2}) := \Big[\{2\prod_{j=1}^{n} \frac{(2\pi)^{2j}}{(2j-1)!}\}^d \cdot \frac{h_{k,2}}{2}\Big]^{\frac{2}{dn(2n+1)}}.$$

According to the Brauer-Siegel Theorem, for a totally real number field $k$ of degree $d$, and all real $\delta > 0$,

(16) $$h_k R_k \leqslant 2^{1-d}\delta(1+\delta)\Gamma((1+\delta)/2)^d (\pi^{-d} D_k)^{\frac{(1+\delta)}{2}} \zeta_k(1+\delta),$$

where $R_k$ is the regulator of $k$. Now from (15) we get the following bound:

(17) $$D_k^{1/d} < f_2(n, d, R_k, \delta)$$

$$:= \Big[\{\frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2}} \prod_{j=1}^{n} \frac{(2\pi)^{2j}}{(2j-1)!}\} \cdot \{\frac{\delta(1+\delta)}{R_k}\}^{\frac{1}{d}}\Big]^{\frac{2}{(2n^2+n-1-\delta)}},$$

since $\zeta_k(1+\delta) \leqslant \zeta(1+\delta)^d$, where $\zeta = \zeta_{\mathbb{Q}}$. Using the lower bound $R_k \geqslant 0.04\, e^{0.46d}$, for a totally real number field $k$, due to R. Zimmert [Z], we obtain the following bound from (17):

(18) $$D_k^{1/d} < f_3(n, d, \delta)$$

$$:= \Big[\{\frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2} e^{0.46}} \prod_{j=1}^{n} \frac{(2\pi)^{2j}}{(2j-1)!}\} \cdot \{25\delta(1+\delta)\}^{\frac{1}{d}}\Big]^{\frac{2}{(2n^2+n-1-\delta)}}.$$

**4.2.** It is obvious that for fixed $n \geqslant 2$ and $\delta \in [0.04,\ 9]$, $f_3(n, d, \delta)$ decreases as $d$ increases. Now we observe that for $n \geqslant 9$, $(2n-1)! > (2\pi)^{2n}$. From this it is easy to see that if for a given $d$, $\delta \in [0.04, 9]$, and $n \geqslant 8$, $f_3(n, d, \delta) \geqslant 1$, then $f_3(n+1, d, \delta) < f_3(n, d, \delta)$, and if $f_3(n, d, \delta) < 1$, then $f_3(n+1, d, \delta) < 1$. In particular, if for given $d$, and $\delta \in [0.04,\ 9]$, $f_3(8, d, \delta) < c$, with $c \geqslant 1$, then $f_3(n, d', \delta) < c$ for all $n \geqslant 8$ and $d' \geqslant d$.

We obtain by a direct computation the following upper bound for the value of $f_3(n, 2, 3)$ for $6 \leqslant n \leqslant 14$.

| $n$ : | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 |
|---|---|---|---|---|---|---|---|---|---|
| $f_3(n, 2, 3) <$ | 1 | 1.1 | 1.2 | 1.3 | 1.4 | 1.6 | 1.8 | 2.1 | 2.4. |

From the bounds provided by this table and the properties of $f_3$ mentioned in the preceding paragraph we conclude that $f_3(n,d,3) < 2.1$ for all $n \geqslant 7$, and $d \geqslant 2$. As $D_k^{1/d} < f_3(n,d,3)$, Proposition 2 implies that unless $k = \mathbb{Q}$ (i.e., $d = 1$), $n \leqslant 6$.

We assert now that $n \leqslant 13$. To prove this, we can assume, in view of the result established in the preceding paragraph, that $k = \mathbb{Q}$. By a direct computation we see that $f_1(14,1,1) < 1$. Hence, $f_1(n,1,1) < 1$ for all $n \geqslant 14$. As $D_{\mathbb{Q}} = 1$, from bound (15) we conclude that $n \leqslant 13$.

We will now assume that $d \geqslant 2$ and consider each of the possible cases $2 \leqslant n \leqslant 6$ separately.

• $n = 6$: For $d \geqslant 2$, $D_k^{1/d} < f_3(6,d,1) \leqslant f_3(6,2,1) < 2.4$. Therefore, by Proposition 2, $d = 2$ and $D_k < 6$, which implies that $k = \mathbb{Q}(\sqrt{5})$ is the only possibility.

• $n = 5$: For $d \geqslant 2$, $D_k^{1/d} < f_3(5,d,1) \leqslant f_3(5,2,1) < 2.9$. Therefore, we infer from Proposition 2 that $d = 2$ and $D_k < 9$. So there are two possible real quadratic fields $k$, their discriminants are 5 and 8. Both the fields have class number 1, and we use the bound (15) to obtain $D_k^{1/2} < f_1(5,2,1) < 2.8$. So only $D_k = 5$ can occur.

• $n = 4$: For $d \geqslant 3$, $D_k^{1/d} < f_3(4,d,1) \leqslant f_3(4,3,1) < 3.62$, and from Proposition 2 we conclude that if $n = 4$, then $d < 3$. Let us assume that $d = 2$. Then since $D_k^{1/2} < f_3(4,2,1.1) < 3.76$, $D_k < 15$ and so the possible values of $D_k$ are $5, 8, 12$ or $13$. The quadratic fields with these $D_k$ have class number 1. Now from bound (15) we obtain $D_k^{1/2} < f_1(4,2,1) < 3.4$. Hence, $D_k < 12$, and only $D_k = 5, 8$ can occur.

• $n = 3$: For $d \geqslant 4$, as $D_k^{1/d} < f_3(3,d,1) \leqslant f_3(3,4,1) < 5.1$, from Proposition 2 we infer that if $n = 3$, then $d < 4$. If $d = 3 = n$, $D_k < 133$ from which we find that $D_k = 49$ or 81. Now we consider the case where $d = 2$ (and $n = 3$). Since $D_k^{1/2} < f_3(3,2,1) < 5.6$, $D_k < 32$, and in this case the possible values of $D_k$ are $5, 8, 12, 13, 17, 21, 24, 28$ or $29$. The quadratic fields with these discriminants have class number 1, and we use bound (15) to obtain $D_k^{1/2} < f_1(3,2,1) < 4.52$. Hence, $D_k < 21$ and only $D_k = 5, 8, 12, 13, 17$ can occur.

• $n = 2$: As $D_k^{1/d} < f_3(2,7,1) < 9$, Proposition 2 implies that $d \leqslant 6$.

♦ $n = 2$ and $d = 6$: As $D_k^{1/6} < f_3(2,6,1) < 9$, $D_k < 531441$. One can check from the table in [1] that $h_k = 1$ for all the five number fields satisfying this bound. We now use bound (15) to obtain $D_k^{1/6} < f_1(2,6,1) < 7.2$. But according to Proposition 2 there is no totally real number field $k$ for which this bound holds.

♦ $n = 2$ and $d = 5$: As $D_k^{1/5} < f_3(2,5,1) < 9.3$, $D_k < 69569$. Again, one can check from the table in [1] that there are five such number fields and the class number of each of them is 1. Now we use bound (15) to obtain $D_k^{1/5} < f_1(2,5,1) < 7.1$. Hence, $D_k < 18043$. From [1] we find that $D_k = 14641$ is the only possibility.

♦ $n = 2$ and $d = 4$: As $D_k^{1/4} < f_3(2,4,0.9) < 9.74$, $D_k < 9000$. According to [1], there are 45 totally real quartic number fields with discriminant $< 9000$, all of them have class number 1. We use bound (15) to obtain $D_k^{1/4} < f_1(2,4,1) < 7.04$. Hence, $D_k < 2457$. We find from [1] that there are eight totally real quartic number fields $k$ with $D_k < 2457$. Their discriminants are

$$725, 1125, 1600, 1957, 2000, 2048, 2225, 2304.$$

♦ $n = 2$ and $d = 3$: As $D_k^{1/3} < f_3(2, 3, 0.8) < 10.5$, $D_k < 1158$. From table B.4 of [C] we find that there are altogether 31 totally real cubics satisfying this discriminant bound. Each of these fields have class number 1. We use bound (15) to obtain $D_k^{1/3} < f_1(2, 3, 1) < 7$, which implies that $D_k < 343$. There are eight real cubic number fields satisfying this bound. The values of $D_k$ are

$$49, 81, 148, 169, 229, 257, 316, 321.$$

♦ $n = 2$ and $d = 2$: As $D_k^{1/2} < f_3(2, 2, 0.5) < 12$, $D_k < 144$. From table B.2 of totally real quadratic number fields given in [C], we check that the class number of all these fields are bounded from above by 2. Hence, $D_k^{1/2} < f_1(2, 2, 2) < 7.3$. So $D_k \leqslant 53$. Among the real quadratic fields with $D_k \leqslant 53$, there is only one field whose class number is 2, it is the field with $D_k = 40$. All the rest have class number 1, and from bound (15) we conclude that $D_k^{1/2} < f_1(2, 2, 1) < 6.8$, i.e., $D_k < 47$. Therefore, the following is the list of the possible values of $D_k$:

$$5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44.$$

To summarize, for $G$ of type $B_n$ or $C_n$, the possible $n$, $d$ and $D_k$ are given in the following table.

| $n$ | $d$ | $D_k$ |
|---|---|---|
| $2, \ldots, 13$ | 1 | 1 |
| 6 | 2 | 5 |
| 5 | 2 | 5 |
| 4 | 2 | 5, 8 |
| 3 | 3 | 49, 81 |
| 3 | 2 | 5, 8, 12, 13, 17 |
| 2 | 5 | 14641 |
| 2 | 4 | 725, 1125, 1600, 1957, 2000, 2048, 2225, 2304 |
| 2 | 3 | 49, 81, 148, 169, 229, 257, 316, 321 |
| 2 | 2 | 5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44. |

**4.3.** We will show that none of the possibilities listed in the above table actually give rise to an arithmetic fake compact hermitian symmetric space of type $B_n$ or $C_n$. For this we recall first of all that $\overline{G}$, and so also $G$, is anisotropic over $k$ (1.5). Now we observe that if $G$ is a group of type $B_n$ ($n \geqslant 2$), then it is $k$-isotropic if and only if it is isotropic at all the real places of $k$ (this follows from the classical Hasse principle for quadratic forms which says that a quadratic form over $k$ is isotropic if and only if it is isotropic at every place of $k$, and the well-known fact that a quadratic form of dimension $> 4$ is isotropic at every nonarchimedean place). Also, a $k$-group of type $C_n$ ($n \geqslant 2$) is $k$-isotropic if it is isotropic at all the real places of $k$ (this is known, and follows, for example, from Proposition 7.1 of [PR]). These results imply that if $d = 1$, i.e., if $k = \mathbb{Q}$, then $G$ is isotropic, and so $d = 1$ is not possible.

Now let us take up the case where $d = 2$, i.e., $k$ is a real quadratic field, and $n = 2$, 5 or 6. Then for any real place $v$ of $k$ where $G$ is isotropic, the complex dimension of the symmetric space of $G(k_v)$ is odd (recall from 1.4 that the complex dimension of the symmetric space of

$G(k_v)$ is $2n-1$ if $G$ is of type $B_n$, and it is $n(n+1)/2$ if $G$ is of type $C_n$). But as the complex dimension of the hermitian symmetric space $X$ is even (since the orbifold Euler-Poincaré characteristic of $\Gamma$ is positive, see 1.3), we conclude that $G$ must be isotropic at both the real places of $k$ (note that $G$ is anisotropic at a place $v$ of $k$ if and only if $G(k_v)$ is compact). From this observation we conclude that $G$ is $k$-isotropic also in case $d = 2$, and $n = 2$, 5 or 6. Therefore these cases do not occur.

**4.4.** To rule out the remaining cases listed in the table in 4.2, we compute the value of $\mathscr{R}$ in each case ($\mathscr{R}$ as in (5)). The following table provides the minimal monic polynomial defining $k$ and the values of $\zeta_k$ needed for the computation of $\mathscr{R}$. It turns out that in none of the remaining cases the numerator of $\mathscr{R}$ is a power of 2 and Proposition 1 then eliminates these cases.

| $n$ | $d$ | $k$ | $D_k$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ | $\zeta_k(-5)$ | $\zeta_k(-7)$ |
|---|---|---|---|---|---|---|---|
| 4 | 2 | $x^2-5$ | 5 | 1/30 | 1/60 | 67/630 | 361/120 |
| 4 | 2 | $x^2-2$ | 8 | 1/12 | 11/120 | 361/252 | 24611/240. |

| $n$ | $d$ | $k$ | $D_k$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ | $\zeta_k(-5)$ |
|---|---|---|---|---|---|---|
| 3 | 3 | $x^3-x^2-2x+1$ | 49 | $-1/21$ | 79/210 | $-7393/63$ |
| 3 | 3 | $x^3-3x-1$ | 81 | $-1/9$ | 199/90 | $-50353/27$ |
| 3 | 2 | $x^2-17$ | 17 | 1/3 | 41/30 | 5791/63 |
| 3 | 2 | $x^2-13$ | 13 | 1/6 | 29/60 | 33463/1638 |
| 3 | 2 | $x^2-3$ | 12 | 1/6 | 23/60 | 1681/126 |
| 3 | 2 | $x^2-2$ | 8 | 1/12 | 11/120 | 361/252 |
| 3 | 2 | $x^2-5$ | 5 | 1/30 | 1/60 | 67/630. |

| $n$ | $d$ | $k$ | $D_k$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ |
|---|---|---|---|---|---|
| 2 | 5 | $x^5-x^4-4x^3+3x^2+3x-1$ | 14641 | $-20/33$ | 1695622/165 |
| 2 | 4 | $x^4-4x^2+1$ | 2304 | 1 | 22011/10 |
| 2 | 4 | $x^4-x^3-5x^2+2x+4$ | 2225 | 4/5 | 9202/5 |
| 2 | 4 | $x^4-4x^2+2$ | 2048 | 5/6 | 87439/60 |
| 2 | 4 | $x^4-5x^2+5$ | 2000 | 2/3 | 3793/3 |
| 2 | 4 | $x^4-4x^2-x+1$ | 1957 | 2/3 | 3541/3 |
| 2 | 4 | $x^4-6x^2+4$ | 1600 | 7/15 | 17347/30 |
| 2 | 4 | $x^4-x^3-4x^2+4x+1$ | 1125 | 4/15 | 2522/15 |
| 2 | 4 | $x^4-x^3-3x^2+x+1$ | 725 | 2/15 | 541/15 |
| 2 | 3 | $x^3-x^2-4x+1$ | 321 | $-1$ | 555/2 |
| 2 | 3 | $x^3-x^2-4x+2$ | 316 | $-4/3$ | 874/3 |
| 2 | 3 | $x^3-x^2-4x+3$ | 257 | $-2/3$ | 1891/15 |
| 2 | 3 | $x^3-4x-1$ | 229 | $-2/3$ | 1333/15 |
| 2 | 3 | $x^3-x^2-4x-1$ | 169 | $-1/3$ | 11227/390 |
| 2 | 3 | $x^3-x^2-3x+1$ | 148 | $-1/3$ | 577/30 |
| 2 | 3 | $x^3-3x-1$ | 81 | $-1/9$ | 199/90 |
| 2 | 3 | $x^3-x^2-2x+1$ | 49 | 1/21 | 79/210. |

## 5. $G$ of type $D_n$

We will consider hermitian symmetric spaces associated to Lie groups of type $D_n$, with $n \geqslant 4$. The noncompact irreducible hermitian symmetric spaces of these types are $\mathrm{SO}^*(2n)/\mathrm{U}(n)$ and $\mathrm{SO}(2, 2n-2)/\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(2n-2))$. In the terminology of Élie Cartan, these are hermitian symmetric spaces of types DIII and BDI respectively.

We note that any absolutely simple algebraic group $G$ over $\mathbb{Q}$ of type ${}^1D_n$ or ${}^2D_n$, with $n \geqslant 4$, or a triality form of type $D_4$, whose $\mathbb{R}$-rank is at least 2, is $\mathbb{Q}$-isotropic (note that if $G$ is a triality form, then as at the unique real place of $\mathbb{Q}$ the relative rank of $G$ is 2, we see that in the Tits index of $G$ over $k_v$ the central vertex is distinguished for every place $v$ of $\mathbb{Q}$, and then it follows from Proposition 7.1 of [PR] that $G$ is isotropic over $\mathbb{Q}$), and hence, by Godement compactness criterion, its arithmetic subgroups are non-cocompact in $G(\mathbb{R})$. Since we are only interested in compact hermitian locally symmetric spaces (and $\mathrm{SO}(2, 2n-2)$ is of $\mathbb{R}$-rank 2, and for $n \geqslant 4$, $\mathbb{R}$-rank of $\mathrm{SO}^*(2n)$ is at least 2) *in this section the number field $k$ will be a nontrivial extension of* $\mathbb{Q}$.

**5.1.** The exponents of the Weyl group of $G$ of type $D_n$ are $1, 3, 5, \ldots, 2n-5, 2n-3$, together with $n-1$ which has multiplicity two if $n$ is even and multiplicity 1 if $n$ is odd. The center of $G$ is of order 4 and $\dim G = n(2n-1)$. Let $\mathscr{T}$ be as in 2.10.

The following bounds for $[\Gamma : \Lambda]$ can be obtained from Propositions 0.12, 2.9, 5.1 and the considerations in 5.5 of [BP].

**Case (a):** $n$ is even, and $G$ is of type ${}^1D_n$, i.e., it is of inner type. Then

$$[\Gamma : \Lambda] \leqslant h_{k,2}^2 2^{2(d+r-1+\#\mathscr{T})}. \tag{19}$$

**Case (b):** $n$ is even and $G$ is of type ${}^2D_n$. Then $\ell$ is a totally real quadratic extension of $k$ (see 1.5), and

$$[\Gamma : \Lambda] \leqslant h_{\ell,2} 2^{2(d+r+\#\mathscr{T})-1} D_\ell / D_k^2. \tag{20}$$

**Case (c):** $n$ is odd. Then $G$ is of type ${}^2D_n$, $\ell$ is a totally complex quadratic extension of $k$ (see 1.5), and

$$[\Gamma : \Lambda] \leqslant h_{\ell,4} 2^{2(d+r+\#\mathscr{T})}. \tag{21}$$

**Case (d):** $n = 4$, $G$ is a triality form of type $D_4$, $\ell$ is a totally real cubic extension of $k$ such that over the normal closure of $\ell/k$, $G$ is an inner form of a split group.

$$[\Gamma : \Lambda] \leqslant h_{\ell,2} 2^{2(d+r+\#\mathscr{T})} D_\ell / D_k^3. \tag{22}$$

### Case (a)

**5.2.** In this case, $n\,(\geqslant 4)$ is even, $G$ is of inner type, and (5) provides the following value of $\mathscr{R}$:

$$\mathscr{R} := 2^{-dn} |\zeta_k(1-n) \prod_{j=1}^{n-1} \zeta_k(1-2j)|$$

Letting

$$A(n) = \frac{(2\pi)^n}{(n-1)!} \cdot \prod_{j=1}^{n-1} \frac{(2\pi)^{2j}}{(2j-1)!}$$

and using the bounds (13) and (19) we obtain the following:

$$D_k^{1/d} < a_1(n,d,h_{k,2}) := \Big[\{4A(n)\}^d \cdot \frac{h_{k,2}^2}{4}\Big]^{\frac{2}{dn(2n-1)}}. \tag{23}$$

Using the Brauer-Siegel bound (16) and lower bound for the regulator for totally real field $k$ of degree $d$ recalled in 4.1 we obtain the following bound.

$$D_k^{1/d} < a_2(n,d,\delta) := \Big[\{\frac{[\Gamma((1+\delta)/2)\zeta(1+\delta)]^2}{(\pi)^{1+\delta}e^{0.92}}A(n)\} \cdot \{25\delta(1+\delta)\}^{\frac{2}{d}}\Big]^{\frac{2}{(2n^2-n-2-2\delta)}}. \tag{24}$$

The argument for the proof of the following Lemma, which will be used in later sections as well, is the same as in the first paragraph of 4.2.

**Lemma 1.** *Let* $\delta \in [0.04, 9]$. *For fixed values of $n$ and $\delta$, $a_2(n,d,\delta)$ decreases as $d$ increases. Furthermore, for fixed values of $d$ and $\delta$, if $n \geqslant 8$, then* $a_2(n+1,d,\delta) < \max(1, a_2(n,d,\delta))$.

We obtain by a direct computation the following upper bound for the value of $a_2(n,2,4)$ for small $n$.

| $n$ : | 4 | 6 | 8 |
|---|---|---|---|
| $a_2(n,2,4) <$ | 10.7 | 3.33 | 2.13 |

From Proposition 2 we now infer that $k = \mathbb{Q}$ for all even $n \geqslant 8$. But as $k \neq \mathbb{Q}$, $n = 4$ or 6.

Consider first the case $n = 6$. For $d \geqslant 2$, $D_k^{1/d} < a_2(6,2,1) < 3.2$. Now using Proposition 2 we conclude that $d = 2$ and $D_k < 11$, hence, $D_k = 5$ or 8. Since the class number of the corresponding fields is 1, $D_k^{1/2} < a_1(6,2,1) < 2.82$. As $2.82^2 < 8$, we conclude that $D_k = 5$. Then $k = \mathbb{Q}(\sqrt{5})$ and for this field $\zeta_k(-1) = 1/30$, $\zeta_k(-3) = 1/60$, $\zeta_k(-5) = 67/630$, $\zeta_k(-7) = 361/120$ and $\zeta_k(-9) = 412751/1650$. Using these values, we compute $\mathscr{R}$ and find that its numerator is not a power of 2, now Proposition 1 rules out the case $n = 6$.

Consider now $n = 4$. For $d \geqslant 4$, $D_k^{1/d} < a_2(4,4,1) < 5.7$. Therefore, $d \leqslant 4$, and for $d = 4$, $D_k < 1056$. From the list of number fields in [1] we find that the only possible value is $D_k = 725$ and the class number of the corresponding number field is 1. Hence $D_k^{1/4} < a_1(4,4,1) < 4.9$. According to Proposition 2 no such number field exists.

For $d = 3$, $D_k^{1/3} < a_2(4,3,1) < 6$. From the table of totally real cubics in [1] we find that the class number of each of the four number fields satisfying the above bound is 1. Hence, $D_k^{1/3} < a_1(4,3,1) < 5$. So $D_k$ can only take one of the following two values,

$$49,\ 81.$$

For $d = 2$, $D_k^{1/2} < a_2(4,2,1) < 6.7$; hence, $D_k < 45$. From the list of real quadratics in [C] we find for real quadratic $k$ with $D_k < 45$, $h_k \leqslant 2$. But then $D_k^{1/2} < a_1(4,2,2) < 5$. We conclude that $D_k < 25$ and then $h_k = 1$. It follows that $D_k^{1/2} < a_1(4,2,1) < 4.73$. We conclude that $D_k$ can only take one of the following values,

$$5, 8, 12, 13, 17, 21.$$

Following is thus the list of possible totally real number fields $k$.

| $n$ | $d$ | $D_k$ |
|---|---|---|
| 4 | 3 | 49, 81 |
| 4 | 2 | 5, 8, 12, 13, 17, 21. |

In the following table, for each of these fields, we give the values of $\zeta_k$ required for the computation of $\mathscr{R}$ for $n = 4$

| $n$ | $d$ | $D_k$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ | $\zeta_k(-5)$ |
|---|---|---|---|---|---|
| 4 | 3 | 49 | −1/21 | 79/210 | −7393/63 |
| 4 | 3 | 81 | −1/9 | 199/90 | −50353/27 |
| 4 | 2 | 5 | 1/30 | 1/60 | 67/630 |
| 4 | 2 | 8 | 1/12 | 11/120 | 361/252 |
| 4 | 2 | 12 | 1/6 | 23/60 | 1681/126 |
| 4 | 2 | 13 | 1/6 | 29/60 | 33463/1638 |
| 4 | 2 | 17 | 1/3 | 41/30 | 5791/63 |
| 4 | 2 | 21 | 1/3 | 77/30 | 17971/63. |

Now computing $\mathscr{R}$ we find that its numerator is not a power of 2 and hence according to Proposition 1 none of the $k$ as above can give rise to an arithmetic fake compact hermitian space of type $D_n$ with $G$ of inner type.

**Case (b)**

**5.3.** In this case, $n\,(\geqslant 4)$ is an even integer, $G$ is of type ${}^2D_n$, $\mathfrak{s} = 2n-1$, $s = 4$, and $\ell$ is a totally real quadratic extension of $k$. The following value of $\mathscr{R}$ is provided by (7):

$$\mathscr{R} = 2^{-dn}|\zeta_{\ell|k}(1-n)\prod_{j=1}^{n-1}\zeta_k(1-2j)|.$$

Letting

$$A(n) = \frac{(2\pi)^n}{(n-1)!}\cdot\prod_{j=1}^{n-1}\frac{(2\pi)^{2j}}{(2j-1)!}$$

and using the bounds (13) and (20), we obtain the following bounds:

$$D_k^{1/d} < b_1(n,d,h_{\ell,2}) := \Big[\{4A(n)\}^d\cdot\frac{h_{\ell,2}}{2}\Big]^{\frac{2}{dn(2n-1)}}, \tag{25}$$

$$D_k^{1/d} < b_2(n,d,\delta) := \Big[\{\frac{[\Gamma((1+\delta)/2)\zeta(1+\delta)]^2}{(\pi)^{1+\delta}e^{0.92}}A(n)\}\cdot\{25\delta(1+\delta)\}^{\frac{1}{d}}\Big]^{\frac{2}{(2n^2-n-2-2\delta)}}, \tag{26}$$

$$D_\ell^{1/2d} < \mathfrak{t}_1(n,d,D_k,h_{\ell,2}) := \Big[2^{2d-1}A(n)^d h_{\ell,2}D_k^{\frac{n(5-2n)-6}{2}}\Big]^{\frac{1}{d(2n-3)}}, \tag{27}$$

$$\begin{aligned} D_\ell^{1/2d} &< \mathfrak{t}_2(n,d,D_k,R_\ell/w_\ell,\delta) \\ &:= \Big[\frac{\delta(1+\delta)}{2R_\ell/w_\ell}D_k^{\frac{n(5-2n)-6}{2}}\{A(n)\frac{[\Gamma((1+\delta)/2)\zeta(1+\delta)]^2}{\pi^{1+\delta}}\}^d\Big]^{\frac{1}{d(2n-4-\delta)}}, \end{aligned} \tag{28}$$

$$(29)\quad D_\ell^{1/2d} < \mathrm{t}_3(n,d,D_k,\delta) := \Big[25\delta(1+\delta)D_k^{\frac{n(5-2n)-6}{2}}\{A(n)\frac{[\Gamma((1+\delta)/2)\zeta(1+\delta)]^2}{\pi^{1+\delta}e^{0.92}}\}^d\Big]^{\frac{1}{d(2n-4-\delta)}}.$$

To obtain the above bounds we have used $D_\ell \geqslant D_k^2$, the Brauer-Siegel bound (16) for totally real fields, and the bound for the regulator due to Zimmert given in 4.1.

**5.4.** We obtain by a direct computation the following upper bound for the value of $b_2(n,2,4)$ for small $n$.

| $n$: | 4 | 6 | 8 |
|---|---|---|---|
| $b_2(n,2,4) <$ | 7.6 | 3 | 2.1. |

From Proposition 2, and Lemma 1, where in the latter the function $a_2(n,d,\delta)$ is replaced by $b_2(n,d,\delta)$, we conclude that $k=\mathbb{Q}$ for all even $n\geqslant 8$. But $k\neq\mathbb{Q}$, and hence $n\leqslant 6$.

Consider now $n=6$. For $d\geqslant 3$, $D_k^{1/d} < b_2(6,2,2) < 3$. Therefore, $d=2$ and $D_k = 5, 8$ are the only possibilities. Let us take up first the case where $D_k = 8$. As $D_\ell^{1/4}\leqslant \mathrm{t}_3(6,2,8,2) < 3.2$, Proposition 2 rules out this case. Consider now the case where $D_k=5$, i.e., $k=\mathbb{Q}(\sqrt{5})$. The following argument involving Hilbert class fields will be used repeatedly. As $D_\ell^{1/4}\leqslant \mathrm{t}_3(6,2,5,1) < 7.3$. The Hilbert class field of $\ell$ is a totally real number field (since $\ell$ is totally real) of degree $h_\ell$ over $\ell$ (hence of degree $4h_\ell$ over $\mathbb{Q}$), and its root discriminant equals $D_\ell^{1/4}$ which is $< 7.3$. On the other hand, according to Proposition 2 $M_r(6) > 8.18$. So we conclude that $4h_\ell < 6$ and, $h_\ell \leqslant \lfloor 5/4\rfloor = 1$. Where here, and in the sequel, we use $\lfloor x\rfloor$ to denote the integral part of $x$. It follows that $D_\ell^{1/4}\leqslant \mathrm{t}_1(6,2,5,1) < 5.5$. From [1] we see that there is only one number field $\ell$ containing $k=\mathbb{Q}(\sqrt{5})$ with this root discriminant bound. For this $\ell$, $D_\ell = 725$, and $\zeta_{\ell|k}(-5) = 2164$. Now using this value of $\zeta_{\ell|k}(-5)$ and the values of $\zeta_k$ (for $k=\mathbb{Q}(\sqrt{5})$) given in 5.2 we compute the value of $\mathscr{R}$ and find that its numerator is not a power of 2. So Proposition 1 rules out $n=6$ with $D_k=5$.

Consider now $n=4$. As $D_k^{1/d} < b_2(4,4,2) < 5.17$, Proposition 2 implies that $d\leqslant 3$.

For $d=3$, we know from Proposition 2 that $D_k\geqslant 49$. Hence, $D_\ell^{1/6}\leqslant \mathrm{t}_3(4,3,49,1) < 17$. From Table IV of [Mart], $M_r(14) > 17$. So by considering the Hilbert class field of $\ell$, we obtain $h_\ell \leqslant \lfloor 13/6\rfloor = 2$. It follows that $D_\ell^{1/6}\leqslant \mathrm{t}_1(4,3,49,2) < 8.6$. But according to Proposition 2, $M_r(7) > 11.05$. Hence, $h_\ell\leqslant\lfloor 8/6\rfloor = 1$. This in turn implies that $D_\ell^{1/6} < \mathrm{t}_1(4,3,49,1) < 8.2$, and therefore, $D_\ell < 304007$. It is seen from table t66.001 in [1] that there is only one totally real number field $\ell$ of degree 6 for which this bound holds. For this $\ell$, $D_\ell = 300125$. Hence the only possibility for $d=3$ is $(D_k, D_\ell) = (49, 300125)$.

Let us assume now that $d=2$. For a real quadratic field $k$, either $D_k = 5$ or 8 or $D_k\geqslant 12$. Consider first the quadratic fields $k$ with $D_k\geqslant 8$. Since $D_\ell^{1/4} < \mathrm{t}_3(4,2,8,0.5) < 39.2$. From Table IV of [Mart] we find that $M_r(80) > 39.4$. Hence, by considering the Hilbert class field of $\ell$, we infer that $h_\ell\leqslant\lfloor 79/4\rfloor = 19$. Hence $h_{\ell,2}\leqslant 16$. It follows that $D_\ell^{1/4}\leqslant \mathrm{t}_1(4,2,8,16) < 16.79$. As $M_r(14) > 17$, by considering the Hilbert class field of $\ell$, we conclude that $h_\ell\leqslant\lfloor 13/4\rfloor = 3$. So $h_{\ell,2}\leqslant 2$. But then $D_\ell^{1/4}\leqslant \mathrm{t}_1(4,2,8,2) < 13.637$ and hence, $D_\ell\leqslant 34584$.

Let us now consider real quadratic fields $k$ with $D_k\geqslant 12$. The discussion in the preceding paragraph implies that $h_{\ell,2}\leqslant 2$. As $D_\ell^{1/4}\leqslant \mathrm{t}_1(4,2,12,2) < 9.47$. Proposition 2 gives that

$M_r(8) > 11.38$. Hence, by considering the Hilbert class field of $\ell$, we conclude that $h_\ell \leqslant \lfloor 7/4 \rfloor = 1$. But then as $D_\ell^{1/4} \leqslant \mathfrak{t}_1(4,2,12,1) < 8.834$, so $D_\ell \leqslant 6090$.

From t44.001 again, we check that there are only 24 such totally real quartics, with $D_\ell$ given below:

$$725, 1125, 1600, 1957, 2000, 2048, 2225, 2304, 2525, 2624, 2777, 3600,$$

$$3981, 4205, 4225, 4352, 4400, 4525, 4752, 4913, 5125, 5225, 5725, 5744.$$

Furthermore, with our assumption that $D_k \geqslant 12$, we know that $h_\ell = 1$, and as $D_k^{1/2} \leqslant b_1(4,2,1) < 4.84$, $D_k$ can only be one of $12, 13, 17, 21$. Since $D_\ell$ is an integral multiple of $D_k^2$, we check easily that for $D_k \geqslant 12$, the only possible values for $(D_k, D_\ell)$ are $(17, 4913), (13, 4225), (12, 2304)$, $(12, 3600)$ and $(12, 4752)$.

Consider now the case $D_k = 5$, i.e., $k = \mathbb{Q}(\sqrt{5})$, and $\ell$ is a totally real number field of degree 4 containing $\mathbb{Q}(\sqrt{5})$. We will show that $D_\ell^{1/4} \leqslant 55$. Assume to the contrary that $D_\ell^{1/4} > 55$. We will first prove that $R_\ell \geqslant 1.64$. For this we shall use some results of [F], §3. In the following paragraph all unexplained notation are from [F], §3, in which $k$ has been replaced by $\ell$.

Recall that the image of the group of units of $\ell$ under the logarithmic embedding $\ell - \{0\} \to \mathbb{R}^4$ forms a lattice $\Lambda_\ell$ of rank 3. Let $0 < m_\ell(\varepsilon_1) \leqslant m_\ell(\varepsilon_2) \leqslant m_\ell(\varepsilon_3)$ be the successive minima of the Euclidean abasolute value on $\Lambda_\ell$. Consider first the case where $\mathbb{Q}(\varepsilon_1) = \ell$. In this case, using Remak's estimate as stated in (3.15) of [F], we see that the following lower bound for the regulator of $\ell$ holds:

$$R_\ell \geqslant \Big(\frac{\log D_\ell - 4\log 4}{40^{1/2}}\Big)^3 > 4.5.$$

Let us assume now that $\mathbb{Q}(\varepsilon_1)$ is a proper subfield of $\ell$. Then $\mathbb{Q}(\varepsilon_1)$ is a real quadratic field. Among such fields, $\mathbb{Q}(\sqrt{5})$ has the smallest regulator (see the corollary in §3 of [Z]). Hence, the smallest fundamental unit $\varepsilon_1$ can be taken to be $\frac{1+\sqrt{5}}{2}$. Then $m_\ell(\varepsilon_1) = 2\log(\frac{1+\sqrt{5}}{2})$. So, $m_\ell(\varepsilon_2) \geqslant 2\log(\frac{1+\sqrt{5}}{2})$. From a result of Remak and Friedman, cf. (3.2) of [F], we know that

$$m_\ell(\varepsilon_3) \geqslant 2(\frac{1}{4}\log|D_\ell| - \frac{1}{2}\log 5 - \log 2) > 2(\log(55) - \log(2\sqrt{5})).$$

(Note that $A(\ell/k) = (\frac{2}{3}(8-2))^{1/2} = 2$ in the notation of [F], page 611.) Hence from the bound (3.12) of [F] we obtain the following:

$$\begin{aligned} R_\ell &\geqslant \frac{1}{2\sqrt{2}}\prod_{i=1}^{3} m_\ell(\varepsilon_i) \geqslant \sqrt{2}\Big(\log(\frac{1+\sqrt{5}}{2})\Big)^2 m_\ell(\varepsilon_3) \\ &> 2\sqrt{2}\Big(\log(\frac{1+\sqrt{5}}{2})\Big)^2\Big(\log(55) - \log(2\sqrt{5})\Big) > 1.64. \end{aligned}$$

This proves our assertion about $R_\ell$. Now since $w_\ell = 2$, we conclude that

$$D_\ell^{1/4} < \mathfrak{t}_2(4, 2, 5, 1.64/2, 0.5) < 55,$$

contradicting the assumption that $D_\ell^{1/4} > 55$. Thus we have proved that $D_\ell^{1/4} \leqslant 55$.

We find, using Table IV of [Mart], that $M_r(800) > 55$. As $D_\ell^{1/4} \leqslant 55$, by considering Hilbert class field of $\ell$, we conclude that $h_\ell \leqslant \lfloor 799/4 \rfloor = 199$. Hence, $h_{\ell,2} \leqslant 128$. Then $D_\ell^{1/4} < \mathrm{t}_1(4, 2, 5, 128) < 31.6$. From Table IV of [Mart] we see that $M_r(41) > 31.7$. Hence, again by considering the Hilbert class field of $\ell$ we conclude that $h_\ell \leqslant \lfloor 40/4 \rfloor = 10$, so $h_{\ell,2} \leqslant 8$. But then $D_\ell^{1/2d} \leqslant \mathrm{t}_1(4, 2, 5, 8) < 24$. Again, from Table IV of [Mart] we see that $M_r(24) > 24$. By considering the Hilbert class field of $\ell$, we infer that $h_\ell \leqslant \lfloor 23/4 \rfloor = 5$. Hence, $h_{\ell,2} \leqslant 4$. But then $D_\ell^{1/4} < \mathrm{t}_1(4, 2, 5, 4) < 22.32$, and so $D_\ell < 248186$. From the tables t44001-t44003 of [1] we find that for $D_\ell \leqslant 248186$, $h_\ell \leqslant 3$, and so $h_{\ell,2} \leqslant 2$. It then follows that $D_\ell \leqslant \lfloor \mathrm{t}_1(4, 2, 5, 2)^4 \rfloor \leqslant 187789$.

Here is the list of all the possibilities:

| $n$ | $d$ | $D_k$ | $D_\ell$ |
|---|---|---|---|
| 4 | 3 | 49 | 300125 |
| 4 | 2 | 17 | 4913 |
| 4 | 2 | 13 | 4225 |
| 4 | 2 | 12 | 2304, 3600, 4752 |
| 4 | 2 | 8 | $\leqslant$ 34584 |
| 4 | 2 | 5 | $\leqslant$ 187789. |

**5.5.** Malle has provided us the list of pairs $(k, \ell)$ satisfying the above constraints. The values of $\zeta_k$ and $\zeta_{\ell|k}$ required to compute $\mathscr{R}$ for each of the possible pairs $(k, \ell)$, with $D_k \geqslant 12$, have been tabulated below.

| $n$ | $d$ | $D_k$ | $D_\ell$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ | $\zeta_k(-5)$ | $\zeta_{\ell\|k}(-3)$ |
|---|---|---|---|---|---|---|---|
| 4 | 3 | 49 | 300125 | $-1/21$ | $79/210$ | $-7393/63$ | 8202104 |
| 4 | 2 | 17 | 4913 | $1/3$ | $41/30$ | $5791/63$ | $366280/17$ |
| 4 | 2 | 13 | 4225 | $1/6$ | $29/60$ | $33463/1638$ | 35936 |
| 4 | 2 | 12 | 2304 | $1/6$ | $23/60$ | $1681/126$ | 5742 |
| 4 | 2 | 12 | 3600 | $1/6$ | $23/60$ | $1681/126$ | 25776 |
| 4 | 2 | 12 | 4752 | $1/6$ | $23/60$ | $1681/126$ | 68944. |

Using the values of $\zeta_k$ and $\zeta_{\ell|k}$ given above, we can compute $\mathscr{R}$. We see that its numerator is not a power of 2 for any of the above $(k, \ell)$, and Proposition 1 rules out all these pairs.

For $k = \mathbb{Q}(\sqrt{2})$, for which $D_k = 8$, there are 32 number fields $\ell$ containing $k$ and with $D_\ell \leqslant 34584$. For $k = \mathbb{Q}(\sqrt{5})$, for which $D_k = 5$, there are 363 number fields $\ell$ containing $k$ and with $D_\ell \leqslant 187789$. In each of these 32 + 363 cases, we have computed $\mathscr{R}$ (interested readers my write to either of the authors to obtain the values). The numerator of $\mathscr{R}$ in none of the cases is a power of 2. Proposition 1 thus eliminates Case (b).

## Case (c)

**5.6.** In Case (c), $n$ is odd and $G$ is of type ${}^2D_n$, $\mathfrak{s} = 2n - 1$, $s = 4$, $\ell$ is a totally complex quadratic extension of totally real $k$ ($k \neq \mathbb{Q}$). Equation (7) provides the following value of

$\mathscr{R}$:

$$\mathscr{R} = 2^{-dn}|\zeta_{\ell|k}(1-n)\prod_{j=1}^{n-1}\zeta_k(1-2j)|.$$

Letting

$$A(n) = \frac{(2\pi)^n}{(n-1)!}\cdot\prod_{j=1}^{n-1}\frac{(2\pi)^{2j}}{(2j-1)!},$$

from bounds (13) and (21), using the following bound provided by the Brauer-Siegel Theorem for a totally complex number field $\ell$ of degree $2d$,

$$h_\ell R_\ell \leqslant w_\ell \delta(1+\delta)\Gamma(1+\delta)^d((2\pi)^{-2d}D_\ell)^{(1+\delta)/2}\zeta_\ell(1+\delta), \tag{30}$$

where $\delta > 0$, $h_\ell$ is the class number and $R_\ell$ is the regulator of $\ell$, and $w_\ell$ is the order of the finite group of roots of unity contained in $\ell$, and the bound $R_\ell \geqslant 0.02 w_\ell\, e^{0.1d}$ due to R. Zimmert [Z], we obtain the following bounds:

$$D_k^{1/d} < c_1(n,d,h_{\ell,4}) := \left[\{4A(n)\}^d h_{\ell,4}\right]^{\frac{2}{dn(2n-1)}}, \tag{31}$$

$$D_k^{1/d} < c_2(n,d,\delta) := \left[\{4A(n)\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\}\cdot\{50\delta(1+\delta)\}^{\frac{1}{d}}\right]^{\frac{2}{2n^2-n-2-2\delta}}, \tag{32}$$

$$D_\ell/D_k^2 < \mathfrak{t}(n,d,D_k,h_{\ell,4}) := \left(4^d A(n)^d h_{\ell,4}\right)^{\frac{2}{2n-1}} D_k^{-n}, \tag{33}$$

$$D_\ell^{1/2d} < \mathfrak{u}_1(n,d,D_k,h_{\ell,4}) := \left[\left(4^d A(n)^d h_{\ell,4}\right)^{\frac{2}{2n-1}} D_k^{2-n}\right]^{\frac{1}{2d}}, \tag{34}$$

$$\begin{aligned} D_\ell^{1/2d} &< \mathfrak{u}_2(n,d,D_k,R_\ell/w_\ell,\delta) \\ &:= \left[\frac{\delta(1+\delta)}{R_\ell/w_\ell} D_k^{\frac{n(5-2n)-2}{2}}\{4A(n)\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}}\}^d\right]^{\frac{1}{d(2n-2-\delta)}}, \end{aligned} \tag{35}$$

$$D_\ell^{1/2d} < \mathfrak{u}_3(n,d,D_k,\delta) := \left[50\delta(1+\delta)D_k^{\frac{n(5-2n)-2}{2}}\{4A(n)\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\}^d\right]^{\frac{1}{d(2n-2-\delta)}}. \tag{36}$$

**5.7.** We obtain by a direct computation the following upper bound for $c_2(n,2,3)$ for small $n$.

| $n:$ | 5 | 7 | 9 |
|---|---|---|---|
| $c_2(n,2,2.6) <$ | 4.2 | 2.5 | 1.78. |

It is obvious that the conclusion of Lemma 1 holds with the function $a_2(n,d,\delta)$ replaced by $c_2(n,d,\delta)$. Also, for fixed $d$ and $\delta$, $c_2((n,d,\delta)$ clearly decreases as $n$ increases. As $k \neq \mathbb{Q}$, using Proposition 2 we conclude that $n \leqslant 7$.

Consider now $n = 7$. For $d \geqslant 2$, since $D_k^{1/d} \leqslant c_2(7,2,2) < 2.5$, Proposition 1 implies that $d = 2$ and $D_k = 5$ is the only possibilty. But if $D_k = 5$, $D_\ell^{1/4} \leqslant \mathfrak{u}_3(7,2,5,1.5) \leqslant 3.2$, which according to Proposition 2 is not possible.

**5.8.** Consider now the case $n = 5$. As $c_2(5,3,1) < 4$, Proposition 2 implies that $d \leqslant 3$. If $d = 2$, then $D_k < 4^2 = 16$, and hence, $D_k = 5, 8, 12$ or 13. On the other hand, if $d = 3$, then $D_k < 4^3 = 64$, and $D_k = 49$ is the only possibility. But then $D_\ell^{1/6} \leqslant \mathfrak{u}_3(5,3,49,1) < 4.4$. According to Proposition 2 there does not exist a totally complex $\ell$ of degree 6 satisfying this bound for $D_\ell$. We conclude therefore that $d = 2$.

Let now $d = 2$. If $D_k \geqslant 8$, as $D_\ell^{1/4} < \mathfrak{u}_1(5,2,8,2) < 5.52$, and according to Proposition 2, $M_c(8) > 5.78$, we conclude using the Hilbert class field of $\ell$ that $h_\ell \leqslant \lfloor 7/4 \rfloor = 1$.

Let us consider the case $D_k = 13$. As $\mathfrak{t}(5,2,13,1) < 1.1$, $D_\ell = 169$. However, there is no totally complex quartic field with discriminant 169. Hence $D_k$ cannot be 13.

Let us now assume that $D_k = 12$. As $\mathfrak{t}(5,2,12,1) < 1.7$, the only possibility for $\ell$ is $D_\ell = 144$.

Suppose $D_k = 8$. Then $k = \mathbb{Q}(\sqrt{2})$. As $\mathfrak{t}(5,2,8,1) < 12.36$, $D_\ell \leqslant 12 \cdot 8^2 \leqslant 768$. From the list of totally complex quartics in table t40.001 of [1] we see that those which contain $\mathbb{Q}(\sqrt{2})$, and have discriminant in the above range, have discriminant in $\{256, 320, 512, 576\}$. (Note that there are two totally complex quartics with discriminant 576, but only one of them contains $\mathbb{Q}(\sqrt{2})$. Only the one containing $\mathbb{Q}(\sqrt{2})$ is of interest to us.)

Now let us assume that $D_k = 5$. Then $k = \mathbb{Q}(\sqrt{5})$. As $D_\ell^{1/4} \leqslant \mathfrak{u}_3(5,2,5,0.7) < 12.4$, $M_c(36) > 12.5$ (Table IV of [Mart]), by considering the Hilbert class field of $\ell$, we infer that $h_\ell \leqslant \lfloor 35/4 \rfloor = 8$. But then as $D_\ell^{1/4} \leqslant \mathfrak{u}_1(5,2,5,8) < 8.5$, and $M_c(16) > 8.7$ (Table IV of [Mart]), by again considering the Hilbert class field of $\ell$, we conclude that $h_\ell \leqslant \lfloor 15/4 \rfloor = 3$. It follows that $h_{\ell,4} \leqslant 2$ and hence, $D_\ell^{1/4} < \mathfrak{u}_1(5,2,5,2) < 7.85$. Therefore, $D_\ell \leqslant 3797$. Moreover, $D_\ell$ is a multiple of $D_k^2 = 25$, so $D_\ell \leqslant 3775$. According to the table in [1], The discriminant of the totally complex quartic fields $\ell$ with $D_\ell \leqslant 3775$, and which contain $\mathbb{Q}(\sqrt{5})$, is one of the following:

$$125, 225, 400, 1025, 1225, 1525, 1600, 2725, 3025, 3625, 3725.$$

The class number of $\ell$ with $D_\ell = 3725$ is 1 according to [Mart]. But $\mathfrak{t}(5,2,5,1) < 130$ and hence $D_\ell \leqslant 5^2 \cdot 129 < 3225$. Hence $\ell$ with $D_\ell = 3725$ can be excluded.

**5.9.** Among the pairs of $(D_k, D_\ell)$ obtained above, only some of them can be discriminants of number fields $k$ and $\ell$ such that $\ell$ is a totally complex quadratic extension of $k$. We eliminate the rest. In conclusion, here are all the possibilities in Case (c): $n = 5$, $d = 2$, $\ell$ is a totally complex quadratic extension of a real quadratic number field $k$, and

| $n$ | $d$ | $D_k$ | $D_\ell$ |
|---|---|---|---|
| 5 | 2 | 12 | 144 |
| 5 | 2 | 8 | 256, 576 |
| 5 | 2 | 5 | 125, 225, 400, 1025, 1225, 1525, 1600, 2725, 3025, 3625. |

Using the values of $\zeta_k$ and $\zeta_{\ell|k}$ given in the following table for the pairs $(k, \ell)$ listed above, we computed $\mathscr{R}$. Its numerator for none of the pairs $(k, \ell)$ turned out to be a power of 2. So by Proposition 1, Case (c) does not give rise to any arithmetic fake compact hermitian symmetric spaces.

| $D_k$ | $D_\ell$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ | $\zeta_k(-5)$ | $\zeta_k(-7)$ | $\zeta_{\ell\vert k}(-4)$ |
|---|---|---|---|---|---|---|
| 12 | 144 | 1/6 | 23/60 | 1681/126 | 257543/120 | 5/3 |
| 8 | 256 | 1/12 | 11/120 | 361/252 | 24611/240 | 285/2 |
| 8 | 576 | 1/12 | 11/120 | 361/252 | 24611/240 | 15940/3. |

For $k = \mathbb{Q}(\sqrt{5})$, $\zeta_k(-1) = 1/30$, $\zeta_k(-3) = 1/60$, $\zeta_k(-5) = 67/630$, $\zeta_k(-7) = 361/120$, and

| $D_\ell$ : | 125 | 225 | 400 | 1025 | 1225 |
|---|---|---|---|---|---|
| $\zeta_{\ell\vert k}(-4)$ : | 1172/25 | 1984/3 | 8805 | 608320 | 1355904 |

| $D_\ell$ : | 1525 | 1600 | 2725 | 3025 | 3625 |
|---|---|---|---|---|---|
| $\zeta_{\ell\vert k}(-4)$ : | 3628740 | 4505394 | 49421124 | 872059200/11 | 178910784. |

## Case (d)

**5.10.** We shall finally consider triality forms of type $D_4$. So assume now that $G$ is a triality form over a totally real number field $k \neq \mathbb{Q}$. For such a $G$, $\mathfrak{s} = 7$, $s = 4$, $\dim G = 28$, and $\ell$ is a totally real cubic extension of $k$ such that over the normal closure of $\ell/k$, $G$ is an inner form of a split group.

The exponents of the Weyl group of $G$ are 1, 3, 3, and 5 (3 has multiplicity 2). The value of $\mathscr{R}$ in this case, provided by equation (8), is

$$\mathscr{R} = 2^{-4d}|\zeta_k(-1)\zeta_{\ell|k}(-3)\zeta_k(-5)|.$$

Letting $A = (2\pi)^{16}/4320$ and using bounds (13), (16) and (22), and Zimmert's lower bound for the regulator, we conclude that

$$D_k^{1/d} < d_1(d, h_{\ell,2}) := \left[(4A)^d h_{\ell,2}\right]^{1/14d}, \tag{37}$$

$$D_k^{1/d} < d_2(d,\delta) := [(50\delta(1+\delta))\Big(A\frac{\{\zeta(1+\delta)\Gamma(\frac{1+\delta}{2})\}^3}{2e^{1.38}\pi^{\frac{3}{2}(1+\delta)}}\Big)^d]^{\frac{2}{d(25-3\delta)}}, \tag{38}$$

$$D_\ell^{1/3d} < \mathfrak{z}_1(d, D_k, h_{\ell,2}) := ((4A)^d D_k^{-\frac{13}{2}} h_{\ell,2})^{\frac{2}{15d}}, \tag{39}$$

$$D_\ell^{1/3d} < \mathfrak{z}_2(d, D_k, \delta) := [50\delta(1+\delta)D_k^{-\frac{13}{2}}\Big(A\frac{\{\zeta(1+\delta)\Gamma(\frac{1+\delta}{2})\}^3}{2e^{1.38}\pi^{\frac{3}{2}(1+\delta)}}\Big)^d]^{\frac{2}{3d(4-\delta)}}. \tag{40}$$

Note that for a fixed value of $\delta \geqslant 0.02$, all the expressions on the right hand side of the above bounds are decreasing in $d$. By a direct computation we find that $D_k^{1/d} < d_2(4, 1.6) < 5.03$. Using Proposition 2 we conclude from this that $d < 4$.

Consider now $d = 3$. As the smallest discriminant of a totally real cubic is 49, and $D_\ell^{1/9} < \mathfrak{z}_2(3, 49, 1) < 10$. But $M_r(9) > 11.8$ (see Table IV in [Mart]). Hence $d$ cannot be 3.

Consider now $d = 2$. As the smallest discriminant of a totally real quadratic field is 5, and $D_\ell^{1/6} < \mathfrak{z}_2(2, 5, 0.7) < 22.2$. But $M_r(21) > 22.3$ (Table IV in [Mart]). So by considering the Hilbert class field of $\ell$, we conclude that $h_\ell \leqslant \lfloor 20/6 \rfloor = 3$, and hence, $h_{\ell,2} \leqslant 2$. But then $D_\ell^{1/6} < \mathfrak{z}_1(2,5,2) < 10.4$. According to Proposition 2, $M_r(8) > 11.38$, so considering again the Hilbert class field of $\ell$, we infer that $h_\ell \leqslant \lfloor 7/6 \rfloor = 1$. Now since $D_\ell^{1/6} < \mathfrak{z}_1(2,5,1) < 9.896$, it follows that $D_\ell \leqslant \lfloor 9.896^6 \rfloor = 939200$.

Suppose $D_k \geqslant 8$, then $D_\ell^{1/6} < \mathfrak{z}_1(2,8,1) < 8.1$, which is smaller than the lower bound for $M_r(6)$ given by Proposition 2. Hence $D_k$ can only be 5.

For $D_k = 5$, i.e., $k = \mathbb{Q}(\sqrt{5})$, we find from table t66.001 of [1] that there are 11 totally real sextics with discriminant bounded as above. For $\ell$ to be an extension of degree three of $k$, it is necessary that $D_\ell/D_k^3$ is an integer. Going through the list of the 11 sextics, we are left with four possibilities for $D_\ell$, these are 300125, 485125, 722000 and 820125. Among these four sextics, only the one with $D_\ell = 300125$ contains $\mathbb{Q}(\sqrt{5})$ as a subfield. This $\ell$ is given by $x^6 - x^5 - 7x^4 + 2x^3 + 7x^2 - 2x - 1$. The values of $\zeta_k(-1)$, $\zeta_{\ell|k}(-3)$, $\zeta_k(-5)$ and $\mathscr{R}$ are given below.

| $\zeta_k(-1)$ | $\zeta_k(-5)$ | $\zeta_{\ell\vert k}(-3)$ | $\mathscr{R}$ |
|---|---|---|---|
| 1/30 | 67/630 | 1295932432/7 | 5426717059/33075. |

As the numerator of $\mathscr{R}$ is not a power of 2, from Proposition 1 we conclude that arithmetic fake compact hermitian symmetric spaces of type $D_4$ cannot arise from triality forms.

**5.11.** In conclusion, there does not exist an arithmetic fake compact hermitian symmetric space of type $D_n$, $n \geqslant 4$.

## 6. $G$ of type ${}^2E_6$

**6.1.** In this section $G$ is of type ${}^2E_6$. Its dimension is 78 and the complex dimension of the symmetric space of $\mathscr{G} = \prod_{j=1}^r G(k_{v_j})$ is $16r$. The exponents of the Weyl group of $G$ are 1, 4, 5, 7, 8 and 11, $\mathfrak{s} = 26$, and $s = 3$. Let $\mathscr{T}$ be as in 2.10. The bound (13) in the present case is

$$(D_k D_\ell)^{13}\Big(\frac{4!5!7!8!11!}{(2\pi)^{42}}\Big)^d 3^{\#\mathscr{T}} < [\Gamma : \Lambda]/3^r. \tag{41}$$

**6.2.** The center $C$ of $G$ is $k$-isomorphic to the kernel of the norm map $N_{\ell/k} : R_{\ell/k}(\mu_3) \to \mu_3$. As this map is onto, the Galois cohomology group $H^1(k, C)$ is isomorphic to the kernel of the homomorphism $\ell^\times/\ell^{\times 3} \to k^\times/k^{\times 3}$ induced by the norm map. We shall denote this kernel by $(\ell^\times/\ell^{\times 3})_\bullet$.

By Dirichlet's unit theorem, $U_k \cong \{\pm 1\} \times \mathbb{Z}^{d-1}$, and $U_\ell \cong \mu(\ell) \times \mathbb{Z}^{d-1}$, where $\mu(\ell)$ is the finite cyclic group of roots of unity in $\ell$. Hence, $U_k/U_k^3 \cong (\mathbb{Z}/3\mathbb{Z})^{d-1}$, and $U_\ell/U_\ell^3 \cong \mu(\ell)_3 \times (\mathbb{Z}/3\mathbb{Z})^{d-1}$, where $\mu(\ell)_3$ is the group of cube-roots of unity in $\ell$. Now we observe that $N_{\ell/k}(U_\ell) \supset N_{\ell/k}(U_k) = U_k^2$, which implies that the homomorphism $U_\ell/U_\ell^3 \to U_k/U_k^3$, induced by the norm map, is onto. Therefore, the order of the kernel $(U_\ell/U_\ell^3)_\bullet$ of this homomorphism equals $\#\mu(\ell)_3$.

The short exact sequence (4) in the proof of Proposition 0.12 of [BP] gives us the following exact sequence:

$$1 \to (U_\ell/U_\ell^3)_\bullet \to (\ell_3/\ell^{\times 3})_\bullet \to (\mathscr{P} \cap \mathscr{I}^3)/\mathscr{P}^3,$$

where $(\ell_3/\ell^{\times 3})_\bullet = (\ell_3/\ell^{\times 3}) \cap (\ell^\times/\ell^{\times 3})_\bullet$, $\mathscr{P}$ is the group of all fractional principal ideals of $\ell$, and $\mathscr{I}$ the group of all fractional ideals (we use multiplicative notation for the group operation in both $\mathscr{I}$ and $\mathscr{P}$). Since the order of the last group of the above exact sequence is $h_{\ell,3}$, see (5) in the proof of Proposition 0.12 of [BP], we conclude that

$$\#(\ell_3/\ell^{\times 3})_\bullet \leqslant \#\mu(\ell)_3 \cdot h_{\ell,3}.$$

Now we note that the order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G' = G$ and $S = V_\infty$, is $3^r/\#\mu(\ell)_3$.

Using the above observations, together with Proposition 2.9 and Lemma 5.4 of [BP], and a close look at the arguments in 5.3 and 5.5 of [BP] for $S = V_\infty$ and $G$ as above, we can derive the following upper bound:

$$[\Gamma : \Lambda] \leqslant h_{\ell,3} 3^{r+\#\mathscr{T}}. \tag{42}$$

This, together with (41) leads to the following bound:

$$(D_k D_\ell)^{13} < \Big(\frac{(2\pi)^{42}}{4!5!7!8!11!}\Big)^d h_\ell. \tag{43}$$

**6.3.** Let

$$A = \frac{(2\pi)^{42}}{4!5!7!8!11!}.$$

From bound (42), using (30), we obtain

$$(D_k D_\ell)^{13} < h_\ell A^d \leqslant A^d \frac{\delta(1+\delta)\Gamma(1+\delta)^d D_\ell^{(1+\delta)/2} \zeta_\ell(1+\delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.$$

Hence,

$$D_k^{13} D_\ell^{13-\frac{1+\delta}{2}} < A^d \frac{\delta(1+\delta)\Gamma(1+\delta)^d \zeta_\ell(1+\delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}. \tag{44}$$

As $D_k^2 \leqslant D_\ell$, and $\zeta_\ell(1+\delta) \leqslant \zeta(1+\delta)^{2d}$, we conclude that

$$D_k^{38-\delta} < A^d \frac{\delta(1+\delta)\Gamma(1+\delta)^d \zeta(1+\delta)^{2d}}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.$$

Therefore,

$$D_k^{1/d} < \Big[\{A\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}}\} \cdot \{\frac{\delta(1+\delta)}{R_\ell/w_\ell}\}^{1/d}\Big]^{1/(38-\delta)}. \tag{45}$$

Using the lower bound $R_\ell \geqslant 0.02 w_\ell e^{0.1d}$ due to R. Zimmert [Z], we obtain from this the following:

$$D_k^{1/d} < f(d,\delta) := \Big[\{A\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\} \cdot \{50\delta(1+\delta)\}^{1/d}\Big]^{1/(38-\delta)}. \tag{46}$$

From bound (43) we also obtain,

$$D_\ell/D_k^2 < \left[A^d D_k^{-39} h_\ell\right]^{1/13}. \tag{47}$$

Furthermore, using (44) and Zimmert's bound $R_\ell \geqslant 0.02 w_\ell e^{0.1d}$, we get the following:

$$D_\ell/D_k^2 < \mathfrak{p}(d, D_k, \delta) := \Big[\{A\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\}\cdot\{\frac{50\delta(1+\delta)}{D_k^{38-\delta}}\}^{1/d}\Big]^{2d/(25-\delta)}. \tag{48}$$

**6.4.** For a fixed $\delta \geqslant 0.02$, $f(d,\delta)$ clearly decreases as $d$ increases. For $d \geqslant 2$, $D_k^{1/d} < f(d,2) \leqslant f(2,2) < 2.3$. We conclude now from Proposition 2 that $d \leqslant 2$, and for $d = 2$, $D_k \leqslant 5$. Then $D_k = 5$. It follows from bound (48) that $D_\ell/D_k^2 < \mathfrak{p}(2,5,2) < 2$. Hence, $D_\ell/D_k^2 = 1$ and $D_\ell = 25$, which contradicts the bound given by Proposition 2. We conclude that $d = 1$, i.e., $k = \mathbb{Q}$.

It is known, and follows, for example, from Proposition 7.1 of [PR], that a $\mathbb{Q}$-group $G$ of type ${}^2E_6$, which at the unique real place of $\mathbb{Q}$ is the outer form of rank 2 (this is the form ${}^2E_{6,2}^{16'}$ which gives rise to a hermitian symmetric space), is isotropic over $\mathbb{Q}$. This contradicts the fact that $G$ is anisotropic over $\mathbb{Q}$ (1.5), and hence we conclude that groups of type ${}^2E_6$ do not give rise to arithmetic fake compact hermitian symmetric spaces.

## 7. $G$ of type $E_7$

**7.1.** In this section $G$ is assumed to be of type $E_7$. The dimension of $G$ is 133, the exponents of its Weyl group are 1, 5, 7, 9, 11, 13 and 17; and $s = 2$. The dimension of the symmetric space $X$ of $\mathscr{G} = \prod_{j=1}^r G(k_{v_j})$ is $27r$. Let $\mathscr{T}$ be as in 2.10. The bound (13) in this case gives us the following:

$$D_k^{133/2} < \frac{[\Gamma:\Lambda]}{2^{r+\#\mathscr{T}}}\cdot\Big(\frac{(2\pi)^{70}}{5!7!9!11!13!17!}\Big)^d. \tag{49}$$

The center $C$ of $G$ is $k$-isomorphic to $\mu_2$. The Galois cohomology group $H^1(k,C)$ is isomorphic to $k^\times/k^{\times 2}$. The order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G' = G$ and $S = V_\infty$, is $2^{r-1}$. From the proof of Proposition 0.12 of [BP], we easily conclude that $\#k_2/k^{\times 2} \leqslant h_{k,2}2^d$. We can adapt the argument used to prove Proposition 5.1 in [BP], and the argument in 5.5, of [BP], for $S = V_\infty$ and $G' = G$, to derive the following bound:

$$[\Gamma:\Lambda] \leqslant h_{k,2}2^{d+r-1+\#T}. \tag{50}$$

Combining (49) and (50) we obtain the following bound:

$$D_k^{133/2} < 2^{d-1}\Big(\frac{(2\pi)^{70}}{5!7!9!11!13!17!}\Big)^d h_{k,2}. \tag{51}$$

**7.2.** Let

$$B = \frac{(2\pi)^{70}}{5!7!9!11!13!17!}.$$

From (51) we obtain the following:

$$D_k^{1/d} < \left[2B(h_{k,2}/2)^{1/d}\right]^{2/133}.$$

Using the Brauer-Siegel bound (16) for totally real number fields, and the obvious bound $\zeta_k(1+\delta) \leqslant \zeta(1+\delta)^d$, we obtain

$$D_k^{1/d} < \left[\{B\frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2}}\}\cdot\{\frac{\delta(1+\delta)}{R_k}\}^{1/d}\right]^{2/(132-\delta)}. \tag{52}$$

Now using the lower bound $R_k \geqslant 0.04\, e^{0.46d}$ due to R. Zimmert [Z] again, we get

$$D_k^{1/d} < \phi(d,\delta) := \left[\{B\frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2}e^{0.46}}\}\cdot\{25\delta(1+\delta)\}^{1/d}\right]^{2/(132-\delta)}. \tag{53}$$

**7.3.** For a fixed $\delta \geqslant 0.04$, $\phi(d,\delta)$ clearly decreases as $d$ increases. By a direct computation we see that $\phi(2,4) < 2$, and hence for all totally real number field $k$ of degree $d \geqslant 2$,

$$D_k^{1/d} < \phi(d,4) \leqslant \phi(2,4) < 2.$$

From this bound and Proposition 2 we conclude that $d$ can only be 1, i.e., $k = \mathbb{Q}$. But then $r = 1$ and the complex dimension of the associated symmetric space $X$ is 27. Then the Euler-Poincaré characteristic of any quotient of $X$ by a cocompact torsion-free discrete subgroup of $\overline{\mathscr{G}}$ is negative (1.3), and hence it cannot be a fake compact hermitian symmetric space. Another way to eliminate this case is to observe that an absolutely simple $\mathbb{Q}$-group of type $E_7$ is isotropic if it is isotropic over $\mathbb{R}$ (this result follows, for example, from Proposition 7.1 of [PR]).

## 8. $G$ of type ${}^2A_n$ with $n$ odd

**8.1.** We shall assume from now on that $G$ is an absolutely simple simply connected $k$-group of type ${}^2A_n$ with $n > 1$ odd. We retain the notation introduced in §§1, 2. In particular, $\ell$ is the totally complex quadratic extension of $k$ over which $G$ is an inner form, $d = [k : \mathbb{Q}]$, $s = n+1$; $\Gamma$, $\Lambda$, for $v \in V_f$, the parahoric subgroups $P_v$ of $G(k_v)$ are as in 2.1, and $\mathscr{T}$ is as in 2.10. We recall that for every nonarchimedean $v \notin \mathscr{T}$, $\Xi_{\Theta_v}$ is trivial. For all $v \in \mathscr{T}$, $\#\Xi_{\Theta_v}|(n+1)$ and $e(P_v) > e'(P_v) > n+1$. We also recall from 2.1 that $\mu(\mathscr{G}/\Gamma)$ is a submultiple of $1/(n+1)^r$, hence, $(n+1)^r\mu(\mathscr{G}/\Gamma) \leqslant 1$.

The center $C$ of $G$ is the kernel of the norm map $N_{\ell/k} : R_{\ell/k}(\mu_{n+1}) \to \mu_{n+1}$. Therefore, we get the following exact sequence:

$$(*) \qquad 1 \to \mu_{n+1}(k)/N_{\ell/k}(\mu_{n+1}(\ell)) \to H^1(k,C) \to (\ell^\times/\ell^{\times n+1})_\bullet \to 1,$$

where $(\ell^\times/\ell^{\times n+1})_\bullet$ is the kernel of the homomorphism $\ell^\times/\ell^{\times n+1} \to k^\times/k^{\times n+1}$ induced by the norm map $N_{\ell/k} : \ell^\times \to k^\times$. By Dirichlet's unit theorem, $U_k \cong \{\pm 1\}\times\mathbb{Z}^{d-1}$ and $U_\ell \cong \mu(\ell)\times\mathbb{Z}^{d-1}$, and hence, $U_k/U_k^{n+1} \cong \{\pm1\}\times(\mathbb{Z}/(n+1)\mathbb{Z})^{d-1}$ and $U_\ell/U_\ell^{n+1} \cong \mu_{n+1}(\ell)\times(\mathbb{Z}/(n+1)\mathbb{Z})^{d-1}$. Since $N_{\ell/k}(U_\ell) \supset N_{\ell/k}(U_k) = U_k^2$, the image of the homomorphism $U_\ell/U_\ell^{n+1} \to U_k/U_k^{n+1}$ induced by the norm map $N_{\ell/k}$ contains $U_k^2/U_k^{n+1}$ ($\cong (2\mathbb{Z}/(n+1)\mathbb{Z})^{d-1}$), and hence the kernel

$(U_\ell/U_\ell^{n+1})_\bullet$ of this homomorphism is of order at most $\#\mu_{n+1}(\ell)\cdot 2^{d-1}$. The short exact sequence (4) in the proof of Proposition 0.12 of [BP] gives us the following exact sequence:

$$1 \to (U_\ell/U_\ell^{n+1})_\bullet \to (\ell_{n+1}/\ell^{\times n+1})_\bullet \to (\mathscr{P}\cap\mathscr{I}^{n+1})/\mathscr{P}^{n+1},$$

where $\ell_{n+1}$ is the subgroup of $\ell^\times$ consisting of all $x$ such that for every normalized nonarchimedean valuation $v$ of $\ell$, $v(x)\in(n+1)\mathbb{Z}$, $(\ell_{n+1}/\ell^{\times n+1})_\bullet = (\ell_{n+1}/\ell^{\times n+1})\cap(\ell^\times/\ell^{\times n+1})_\bullet$, $\mathscr{P}$ is the group of all fractional principal ideals of $\ell$, and $\mathscr{I}$ the group of all fractional ideals (we use multiplicative notation for the group operation in both $\mathscr{I}$ and $\mathscr{P}$). Since the order of the last group of the above exact sequence is $h_{\ell,n+1}$, see (5) in the proof of Proposition 0.12 of [BP], we conclude that $\#(\ell_{n+1}/\ell^{\times n+1})_\bullet \leqslant \#\mu_{n+1}(\ell)\cdot 2^{d-1}h_{\ell,n+1}$.

Let $c$ be the order of the kernel of the norm map $N_{\ell/k}:\mu_{n+1}(\ell)\to\mu_{n+1}(k)=\{\pm1\}$. Then the order of the first term of $(*)$ is $2c/\#\mu_{n+1}(\ell)$, whereas the order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G'=G$ and $S=V_\infty$, is $(n+1)^r/c$. Now from Lemma 5.4 of [BP] and the arguments given in 5.3 and 5.5 of that paper we obtain the following upper bound (note that we need to replace "$n$" in 5.3 and 5.5 of [BP] with "$n+1$" since the group $G$ in this and the next section is of type ${}^2A_n$):

$$[\Gamma:\Lambda]\leqslant h_{\ell,n+1}2^d(n+1)^{r+\#\mathscr{T}}. \tag{54}$$

For the group $G$ under consideration here, $\dim G = n^2+2n$, the exponent $m_j = j$ and $\mathfrak{s}=(n-1)(n+2)/2$, so the volume formula (3) gives us the following:

$$\mu(\mathscr{G}/\Lambda) = D_k^{\frac12(n^2+2n)}(D_\ell/D_k^2)^{\frac14(n-1)(n+2)}\Big(\prod_{j=1}^{n}\frac{j!}{(2\pi)^{j+1}}\Big)^d\prod_{v\in V_f}e(P_v). \tag{55}$$

For $v\in\mathscr{T}$, as $e(P_v)>(n+1)$, and moreover for all $v\in V_f$, $e(P_v)>1$, using (54) and (55) we find that

$$1\geqslant(n+1)^r\mu(\mathscr{G}/\Gamma) > D_k^{\frac12(n^2+2n)}(D_\ell/D_k^2)^{\frac14(n-1)(n+2)}\Big(\prod_{j=1}^{n}\frac{j!}{(2\pi)^{j+1}}\Big)^d\frac{1}{2^d h_{\ell,n+1}}, \tag{56}$$

As $D_k^2|D_\ell$, from (56) we obtain the following bound for $D_k$:

$$D_k^{1/d} < f_1(n,d,h_{\ell,n+1}) := \Big[\{2\prod_{j=1}^{n}\frac{(2\pi)^{j+1}}{j!}\}^d\cdot h_{\ell,n+1}\Big]^{\frac{2}{d(n^2+2n)}}. \tag{57}$$

Since $\zeta_\ell(1+\delta)\leqslant\zeta(1+\delta)^{2d}$, for $\delta>0$, we obtain the following bound from (57) and (30)

$$\begin{aligned} D_k^{1/d} &< f_2(n,d,R_\ell/w_\ell,\delta)\\ &:= \Big[\{2\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}}\prod_{j=1}^{n}\frac{(2\pi)^{j+1}}{j!}\}\cdot\{\frac{\delta(1+\delta)}{(R_\ell/w_\ell)}\}^{1/d}\Big]^{\frac{2}{(n^2+2n-2\delta-2)}}. \end{aligned} \tag{58}$$

Using the lower bound $R_\ell\geqslant 0.02w_\ell\, e^{0.1d}$ due to Zimmert, we obtain the following from (30)

$$\frac{1}{h_{\ell,n+1}}\geqslant\frac{1}{h_\ell}\geqslant\frac{0.02}{\delta(1+\delta)}\Big(\frac{(2\pi)^{1+\delta}e^{0.1}}{\Gamma(1+\delta)}\Big)^d\frac{1}{D_\ell^{(1+\delta)/2}\zeta_\ell(1+\delta)}. \tag{59}$$

Since $D_\ell \geqslant D_k^2$, and $\zeta_\ell(1+\delta) \leqslant \zeta(1+\delta)^{2d}$, we get the following for all $\delta$ in the interval $[0.02, 6.5]$.

$$D_k^{1/d} < f_3(n,d,\delta) := \Big[\{2\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\prod_{j=1}^{n}\frac{(2\pi)^{j+1}}{j!}\}\cdot\{50\delta(1+\delta)\}^{1/d}\Big]^{\frac{2}{(n^2+2n-2\delta-2)}}. \tag{60}$$

Now the following three bounds for the relative discriminant $D_\ell/D_k^2$ are obtained from (56), (59) and (30).

$$D_\ell/D_k^2 < \mathfrak{p}_1(n,d,D_k,h_{\ell,n+1}) := \Big[h_{\ell,n+1}\cdot\{2\prod_{j=1}^{n}\frac{(2\pi)^{j+1}}{j!}\}^d D_k^{-(n^2+2n)/2}\Big]^{\frac{4}{(n-1)(n+2)}}. \tag{61}$$

$$\begin{aligned} D_\ell/D_k^2 &< \mathfrak{p}_2(n,d,D_k,R_\ell/w_\ell,\delta) \\ &:= \Big[\frac{\delta(1+\delta)}{(R_\ell/w_\ell)D_k^{(n^2+2n-2\delta-2)/2}}\{\frac{2\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}}\prod_{j=1}^{n}\frac{(2\pi)^{j+1}}{j!}\}^d\Big]^{\frac{4}{(n^2+n-2\delta-4)}}. \end{aligned} \tag{62}$$

$$\begin{aligned} D_\ell/D_k^2 &< \mathfrak{p}_3(n,d,D_k,\delta) \\ &:= \Big[\frac{50\delta(1+\delta)}{D_k^{(n^2+2n-2\delta-2)/2}}\cdot\{\frac{2\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\prod_{j=1}^{n}\frac{(2\pi)^{j+1}}{j!}\}^d\Big]^{\frac{4}{(n^2+n-2\delta-4)}}. \end{aligned} \tag{63}$$

We also get the following bound for $D_\ell$ from (56).

$$D_\ell^{1/2d} < \mathfrak{q}_1(n,d,D_k,h_{\ell,n+1}) := \Big[\frac{h_{\ell,n+1}}{D_k^{(n+2)/2}}\cdot\{2\prod_{j=1}^{n}\frac{(2\pi)^{j+1}}{j!}\}^d\Big]^{\frac{2}{d(n-1)(n+2)}}, \tag{64}$$

which in turn provides the following bound using (30) and (59)

$$\begin{aligned} D_\ell^{1/2d} &< \mathfrak{q}_2(n,d,D_k,R_\ell/w_\ell,\delta) \\ &:= \Big[\frac{\delta(1+\delta)}{(R_\ell/w_\ell)D_k^{(n+2)/2}}\cdot\{\frac{2\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}}\cdot\prod_{j=1}^{n}\frac{(2\pi)^{j+1}}{j!}\}^d\Big]^{\frac{2}{d(n^2+n-2\delta-4)}}. \end{aligned} \tag{65}$$

We now state the following simple lemma.

**Lemma 2.** *Let $\delta \in [0.02, 6.5]$. For fixed values of $n$ and $\delta$, $f_3(n,d,\delta)$ decreases as $d$ increases. Furthermore, for fixed values of $d$ and $\delta$, if $n \geqslant 7$, then $f_3(n+1,d,\delta) <$* $\max(1, f_3(n,d,\delta))$.

**8.2.** Let us begin determination of the totally real number field $k$. Let $f_3(n,d,\delta)$ be as in (60). By a direct computation we obtain the following upper bound for the value of $f_3(n,2,3)$ for small $n$.

| $n:$ | 13 | 11 |
|---|---|---|
| $f_3(n,2,3) <$ | 2.1 | 2.4. |

Hence for $n \geqslant 13$ and $d \geqslant 2$, $f_3(n, d, 3) \leqslant f_3(13, 2, 3) < 2.1$, which in view of Proposition 2 implies that $k = \mathbb{Q}$.

**8.3.** Now we will determine the degrees $d$ of possible $k$ for $n \leqslant 11$ using (60). We get the following table by evaluating $f_3(n, d, \delta)$, with $n$ given in the first column, $d$ given in the second column, and $\delta$ given in the third column

| $n$ | $d$ | $\delta$ | $f_3(n, d, \delta) <$ |
|---|---|---|---|
| 11 | 3 | 2 | 2.4 |
| 9 | 3 | 1.9 | 2.9 |
| 7 | 3 | 1.6 | 3.63 |
| 5 | 4 | 1.3 | 5.12 |
| 3 | 7 | 1 | 10. |

Taking into account the upper bound in the last column of the above table, Proposition 2 implies the following bound for $d$ for each odd integer $n$ between 3 and 11.

| $n:$ | 11 | 9 | 7 | 5 | 3 |
|---|---|---|---|---|---|
| $d \leqslant$ | 2 | 2 | 2 | 3 | 6. |

**8.4.** We will now narrow down the possibilities for $d$ further. We begin with larger values of $n$.

For $n = 11, 9$ and 7, we know that $d \leqslant 2$.

For $n = 11$ and $d = 2$, $D_k^{1/2} \leqslant f_3(11, 2, 2) \leqslant 2.5$, so $D_k = 5$. Then $D_\ell/D_k^2 \leqslant \lfloor \mathfrak{p}_3(11, 2, 5, 2)\rfloor = 1$. Hence $D_\ell = 25$, but there is no such $\ell$. This implies that if $n = 11$, then $k = \mathbb{Q}$.

For $n = 9$ and $d = 2$, $D_k^{1/2} \leqslant f_3(9, 2, 2)) < 3$. Hence, $D_k = 5$ or 8. As $\lfloor \mathfrak{p}_3(9, 2, 5, 2)\rfloor = 3$ and $\lfloor \mathfrak{p}_3(9, 2, 8, 1.6)\rfloor = 1$. So $D_\ell \leqslant 75$, but there is no $\ell$ of degree 4 for which this bound holds, and we conclude that if $n = 9$, then again $k = \mathbb{Q}$.

**8.5.** We shall now consider the case $n = 7$ and $d = 2$. As $D_k^{1/2} \leqslant f_3(7, 2, 1) < 3.8$, $D_k = 5, 8, 12$ or 13. Computations give that $\lfloor \mathfrak{p}_3(7, 2, 5, 1.3)\rfloor = 11$, $\lfloor \mathfrak{p}_3(7, 2, 8, 1.3)\rfloor = 3$, $\lfloor \mathfrak{p}_3(7, 2, 12, 1.3)\rfloor = 1$ and $\lfloor \mathfrak{p}_3(7, 2, 13, 1)\rfloor = 1$ Hence $D_\ell$ is bounded from above by $\max(5^2 \cdot 11, 8^2 \cdot 3, 12^2, 13^2)$. From the list of number fields given in [1], we conclude that the class number of all these totally complex quartic $\ell$ is 1. Hence the pairs $(k, \ell)$ belong to the list of [PY1], 8.2 (see also [PY1], 7.10). Also, the bound for the relative discriminant $D_\ell/D_k^2$ can be improved to $\lfloor \mathfrak{p}_1(7, 2, 5, 1)\rfloor = 8$, and $\lfloor \mathfrak{p}_1(7, 2, 8, 1)\rfloor = 2$ in the first two cases. Now checking against the list of [PY1], 8.2, we conclude that the following are the only possible pairs $(k, \ell)$.

$$\mathcal{C}_1,\ \mathcal{C}_{11}.$$

We eliminate these pairs by computing $\mathscr{R}$ and then using Proposition 1. The values of $\zeta_k$ and $\zeta_{\ell|k}$ required for the computation of $\mathscr{R}$ are given below.

| $(k, \ell)$ | $\zeta_k(-1)$ | $\zeta_{\ell\vert k}(-2)$ | $\zeta_k(-3)$ | $\zeta_{\ell\vert k}(-4)$ | $\zeta_k(-5)$ | $\zeta_{\ell\vert k}(-6)$ | $\zeta_k(-7)$ |
|---|---|---|---|---|---|---|---|
| $\mathcal{C}_1$ | 1/30 | 4/5 | 1/60 | 1172/25 | 67/630 | 84676/5 | 361/120 |
| $\mathcal{C}_{11}$ | 1/6 | 1/9 | 23/60 | 5/3 | 1681/126 | 427/3 | 257543/120. |

**8.6.** Consider now $n = 5$. We know that $d \leqslant 3$. Assume, if possible, that $d = 3$. As $D_k^{1/3} < f_3(5,3,1) < 5.3$, we see from the table of totally real cubics given in [C] that $D_k$ is either 49 or 81. On the other hand, $D_\ell/D_k^2 \leqslant \lfloor \mathfrak{p}_3(5,3,49,1)\rfloor = 16$ and $D_\ell/D_k^2 \leqslant \lfloor \mathfrak{p}_3(5,3,81,1)\rfloor = 4$ for the two cases respectively. So $D_\ell \leqslant \max(49^2 \cdot 16, 81^2 \cdot 4) = 49^2 \cdot 16$. The class number of all totally complex sextic fields $\ell$ with $D_\ell \leqslant 49^2 \cdot 16$ is 1. Now the bound for the relative discriminant $D_\ell/D_k^2$ can be improved to $\lfloor \mathfrak{p}_1(5,3,49,1)\rfloor = 6$, and $\lfloor \mathfrak{p}_1(5,3,81,1)\rfloor = 1$ in the two cases. Among the pairs $(k,\ell)$ listed in [PY1], 8.2, none satisfy these conditions. Hence, $d < 3$.

Assume now $d = 2$. As $D_k^{1/2} < f_3(5,2,1) < 5.54$, $D_k \leqslant 30$. From Friedman [F] we know that $R_\ell/w_\ell \geqslant 1/8$ except when $D_\ell = 117, 125$ and 144. Therefore, apart from the three exceptional cases, we conclude that $D_\ell/D_k^2 \leqslant \lfloor \mathfrak{p}_2(5,2,5,1/8,1)\rfloor = 82$. Since the discriminant in each of the three exceptional cases is smaller than $82 \cdot 5^2$, we conclude that the bound $D_\ell/D_k^2 \leqslant 82$ always holds. So $D_\ell \leqslant 30^2 \cdot 82 = 73800$. From the list in [1] of totally complex quartics $\ell$ with $D_\ell \leqslant 73800$, we see that $h_\ell \leqslant 15$ and hence $h_{\ell,6} \leqslant 12$. Then $D_k^{1/2} < f_1(5,2,12) < 5.1$, and so $D_k \leqslant 24$. We know that either $D_k = 5$ or $D_k \geqslant 8$. In the latter case, as $\lfloor \mathfrak{p}_1(5,2,8,12)\rfloor = 17$. Thus $D_\ell \leqslant \max(5^2 \cdot 82, 24^2 \cdot 17) = 9792$. By checking the list of totally complex quartic number fields in [1] again, we conclude that $h_\ell \leqslant 5$ and hence $h_{\ell,6} \leqslant 4$. Then $D_k^{1/2} < f_1(5,2,4) < 4.87$, so $D_k \leqslant 21$. We now compute $\lfloor \mathfrak{p}_1(5,2,D_k,4)\rfloor$ for $5 \leqslant D_k \leqslant 21$ to get the following bound for $D_\ell$:

| | | | | | | |
|---|---|---|---|---|---|---|
| $D_k$ : | 5 | 8 | 12 | 13 | 17 | 21 |
| $\lfloor \mathfrak{p}_1(5,2,D_k,4)\rfloor$ : | 48 | 15 | 5 | 4 | 2 | 1 |
| $D_\ell \leqslant$ | 1200 | 960 | 720 | 676 | 578 | 441. |

The list of number fields satisfying the above constraint was provided by Malle using the tables in [1]. It turns out that all the number fields involved are listed in the tables in [PY1], 8.2. Moreover, all have class number 1. It follows that $D_\ell$ is bounded by $\lfloor \mathfrak{p}_1(5,2,D_k,1)\rfloor$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $D_k$ : | 5 | 8 | 12 | 13 | 17 | 21 |
| $\lfloor \mathfrak{p}_1(5,2,D_k,1)\rfloor$ : | 40 | 12 | 4 | 3 | 1 | 1 |
| $D_\ell \leqslant$ | 1000 | 768 | 576 | 507 | 289 | 441. |

From the table in [PY1], 8.2, we conclude that the following are the ony possibilities for the pair $(k,\ell)$.

$$\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{C}_3,\ \mathcal{C}_8,\ \mathcal{C}_9,\ \mathcal{C}_{11},\ \mathcal{C}_{17}.$$

We eliminate each of the above pairs by computing $\mathscr{R}$, using the following values of $\zeta_k$ and $\zeta_{\ell|k}$, and then use Proposition 1.

| $(k,\ell)$ | $\zeta_k(-1)$ | $\zeta_{\ell\|k}(-2)$ | $\zeta_k(-3)$ | $\zeta_{\ell\|k}(-4)$ | $\zeta_k(-5)$ |
|---|---|---|---|---|---|
| $\mathcal{C}_1$ | 1/30 | 4/5 | 1/60 | 1172/25 | 67/630 |
| $\mathcal{C}_2$ | 1/30 | 32/9 | 1/60 | 1984/3 | 67/630 |
| $\mathcal{C}_3$ | 1/30 | 15 | 1/60 | 8805 | 67/630 |
| $\mathcal{C}_8$ | 1/12 | 3/2 | 11/120 | 285/2 | 361/252 |
| $\mathcal{C}_9$ | 1/12 | 92/9 | 11/120 | 15940/3 | 361/252 |
| $\mathcal{C}_{11}$ | 1/6 | 1/9 | 23/60 | 5/3 | 1681/126 |
| $\mathcal{C}_{17}$ | 1/3 | 32/63 | 77/30 | 64/3 | 17971/63. |

**8.7.** The case $n = 3$ requires more detailed considerations.

• Again we are considering totally real $k$ with $d > 1$ in this section. We know from 8.3 that $d \leqslant 6$. Consider first $d = 6$. Then $D_k \geqslant 300125$ (see §3). Hence, $D_\ell^{1/12} \leqslant \mathfrak{q}_2(3, 6, 300125, 1/8, 1) < 12$. According to Table IV of [Mart], $M_c(32) > 12$, so considering the Hilbert class field of $\ell$ which is an extension of degree $h_\ell$ of $\ell$, we infer that $h_\ell \leqslant \lfloor 31/12 \rfloor = 2$. Hence $h_{\ell,n+1} \leqslant 2$. Now applying bound (64) we obtain $D_k^{1/6} \leqslant D_\ell^{1/12} \leqslant \mathfrak{q}_1(3, 6, 300125, 2) < 7$, which contradicts Proposition 2.

• Consider now $d = 5$. In this case $D_k \geqslant 14641$. Hence, $D_\ell^{1/10} \leqslant \mathfrak{q}_2(3, 5, 14641, 1/8, 1) < 14$. According to Table IV of [Mart], $M_c(52) > 14.1$. Using again the Hilbert class field of $\ell$ we conclude that $h_\ell \leqslant \lfloor 51/10 \rfloor = 5$, and hence $h_{\ell,4} \leqslant 4$. Then $D_\ell^{1/10} \leqslant \mathfrak{q}_1(3, 5, 14641, 4) < 7.74$ So $D_\ell < 7.74^{10} < 7.72 \times 10^8$. On the other hand, Schehrazad Selmane [Sel] has shown that the totally complex number field of degree 10, containing a totally real quintic field, with smallest absolute discriminant is the cyclotomic field $\mathbb{Q}(\zeta_{11})$ generated by a primitive 11-th root $\zeta_{11}$ of unity. This field has absolute discriminant $11^9$. Since $11^9 > 7.72 \times 10^8$, we conclude that $d \neq 5$.

• Consider now $d = 4$. In this case $D_k \geqslant 725$. Hence $D_\ell^{1/8} \leqslant \mathfrak{q}_2(3, 4, 725, 1/8, 0.86) < 17.43$. According to Table IV of [Mart], $M_c(140) > 17.49$, so considering the Hilbert class field of $\ell$ we find that $h_\ell \leqslant \lfloor 139/8 \rfloor = 17$. So $h_{\ell,4} \leqslant 16$ and then $D_\ell^{1/8} \leqslant \mathfrak{q}_1(3, 4, 725, 16) < 9.7$. According to Table IV of [Mart], $M_c(20) > 9.8$ which by considering the Hilbert class field of $\ell$ implies that $h_\ell \leqslant \lfloor 19/8 \rfloor = 2$ and $h_{\ell,4} \leqslant 2$. It follows that $D_\ell^{1/8} \leqslant \mathfrak{q}_1(3, 4, 725, 2) < 8.7$. According to Table IV of [Mart], $M_c(16) > 8.7$. Hence, again by considering the Hilbert class field of $\ell$ implies that $h_\ell \leqslant \lfloor 15/8 \rfloor = 1$. But then $D_\ell^{1/8} \leqslant \mathfrak{q}_1(3, 4, 725, 1) < 8.386$. So $D_\ell \leqslant \lfloor 8.386^8 \rfloor < 2.45 \times 10^7$. Also, $D_k^{1/4} < f_1(3, 4, 1) < 7.146$. Hence, $D_k \leqslant 2607$. We also know that $D_\ell / D_k^2 \leqslant \lfloor \mathfrak{p}_1(3, 4, 725, 1) \rfloor = 46$. Any such pair $(k, \ell)$ lies in the list of pairs tabulated in [PY1], 8.2. We find that the possible pairs are $\mathcal{C}_{34} - \mathcal{C}_{37}$ in the notation used in [PY1], 8.2. Again, we eliminate each of the pairs by computing $\mathscr{R}$ using the following zeta values and applying Proposition 1.

| $(k,\ell)$ | $\zeta_k(-1)$ | $\zeta_{\ell\vert k}(-2)$ | $\zeta_k(-3)$ |
|---|---|---|---|
| $\mathcal{C}_{34}$ | 4/15 | 128/45 | 2522/15 |
| $\mathcal{C}_{35}$ | 2/3 | 12 | 3793/3 |
| $\mathcal{C}_{36}$ | 5/6 | 411 | 87439/60 |
| $\mathcal{C}_{37}$ | 1 | 46/3 | 22011/10. |

• Consider now $d = 3$. The three smallest absolute discrimants of totally real cubic fields are 49, 81 and 148. Let us consider first the totally real cubic fields $k$ with $D_k \geqslant 148$. Note that $D_\ell^{1/6} < \mathfrak{q}_2(3, 3, 148, 1/8, 0.7) < 18.1$, since $R_\ell / w_\ell \geqslant 1/8$ except for the six sextics whose discriminants are listed in [PY1], 7.3. The root discriminant of these six sextics clearly satisfy the above bound. We see from Table IV in [Mart] that $M_c(180) > 18.1$. Hence, considering the Hilbert class field of $\ell$, we conclude that $h_\ell \leqslant \lfloor 179/6 \rfloor = 29$ and so $h_{\ell,4} \leqslant 16$. Then $D_\ell^{1/6} < \mathfrak{q}_1(3, 3, 148, 16) < 10$. Again from Table IV of [Mart] we find that $M_c(22) > 10.25$, and as before considering the Hilbert class field of $\ell$, we conclude that $h_\ell \leqslant \lfloor 21/6 \rfloor = 3$. So $h_{\ell,4} \leqslant 2$. Then $D_\ell^{1/6} < \mathfrak{q}_1(3, 3, 148, 2) < 8.7$. Furthermore, $D_k^{1/3} < f_1(3, 3, 2) < 7.37$. Hence $D_\ell \leqslant \lfloor 8.7^6 \rfloor = 433626$ and $148 \leqslant D_k \leqslant \lfloor 7.37^3 \rfloor = 400$. There are only three pair of number fields $(k, \ell)$ satisfying the above bounds and $h_\ell = 1$ for each of the $\ell$ occurring in these three pairs from which we conclude that $D_\ell^{1/6} < \mathfrak{q}_1(3, 3, 148, 1) < 8.31$. Hence, $D_\ell < 329311$. But there are no pairs $(k, \ell)$ of totally real cubic $k$, and totally complex quadratic extension $\ell$ of $k$ with $148 \leqslant D_k \leqslant 400$ and $D_\ell < 329311$.

We will consider now the unique totally real cubic field $k_1$ with $D_{k_1} = 81$. Note that $k_1 = \mathbb{Q}[x]/(x^3 - 3x - 1)$, the regulator $R_{k_1} \geqslant 0.849$ according to [C]. Now by listing $m$ such that the value $\phi(m)$ of the Euler function $\phi$ is a divisor of 6, we see that unless $\ell$ is $\mathbb{Q}(\zeta_{18})$, $w_\ell = 2$, 4 or 6 (note that $\mathbb{Q}(\zeta_{14})$ does not contain $k_1$). As $\ell$ is a CM field which is a quadratic extension of $k_1$, $R_\ell = 2^2 R_{k_1}/Q$, where $Q = 1$ or 2 (cf. [W]), hence unless $\ell$ is $\mathbb{Q}(\zeta_{18})$, $R_\ell/w_\ell \geqslant 2 \times 0.849/6 = 0.283$. So either $\ell = \mathbb{Q}(\zeta_{18})$ or $D_\ell^{1/6} < \mathfrak{q}_2(3, 3, 81, 0.283, 0.66) < 19.4$. From Table IV of [Mart] we find that $M_c(340) > 19.4$. Hence, by considering the Hilbert class field of $\ell$, we successively get the following improved bounds for $h_{\ell,4}$: $h_\ell \leqslant \lfloor 339/6 \rfloor = 56$, so $h_{\ell,4} \leqslant 32$. Therefore, $D_\ell^{1/6} < \mathfrak{q}_1(3, 3, 81, 32) < 11.6$. Again in Table IV of [Mart] we see that $M_c(30) > 11.6$. So $h_\ell \leqslant \lfloor 29/6 \rfloor = 4$, and $D_\ell^{1/6} < \mathfrak{q}_1(3, 3, 81, 4) < 10.1$. But according to Table IV of [Mart], $M_c(22) > 10.2$. Hence, $h_\ell \leqslant \lfloor 21/6 \rfloor = 3$, and $h_{\ell,4} \leqslant 2$. It follows that $D_\ell/D_{k_1}^2 \leqslant \lfloor \mathfrak{p}_1(3, 3, 81, 2) \rfloor = 120$, from which we conclude that $D_\ell \leqslant 120 \cdot 81^2 = 787320$. Malle provided us a complete list of totally complex quadratic extensions $\ell$ of the above $k_1$ with $D_\ell \leqslant 787320$. This list consists of three fields whose absolute discriminant, defining monic polynomial and the value of $\zeta_{\ell|k_1}(-2)$ are given below.

| $D_\ell$ | $\ell$ | $\zeta_{\ell\vert k_1}(-2)$ |
|---|---|---|
| 19683 | $x^6 - x^3 + 1$ | $-104/27$ |
| 419904 | $x^6 + 6x^4 + 9x^2 + 1$ | $-7826$ |
| 465831 | $x^6 - 3x^5 + 9x^4 - 13x^3 + 15x^2 - 9x + 3$ | $-10944.$ |

The first of these fields is $\mathbb{Q}(\zeta_{18})$. We shall denote the three pairs $(k_1, \ell)$ with $\ell$ from this list, and $k_1 = \mathbb{Q}[x]/(x^3 - 3x - 1)$, $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ respectively.

Let us now consider the unique totally real cubic field $k_2$ with $D_{k_2} = 49$. Note that $k_2 = \mathbb{Q}[x]/(x^3 - x^2 - 2x + 1)$, and from [C] we find that its regulator is larger than 0.525. Hence, as for $k_1$, we see that except for the cyclotomic field $\mathbb{Q}(\zeta_{14})$ which has class number 1 and discriminant $7^5 = 16807$, $w_\ell = 2$, 4 or 6, and for the noncyclotomic $\ell$, $R_\ell/w_\ell \leqslant 2 \times 0.525/6 = 0.175$. Therefore, $D_\ell/D_{k_2}^2 \leqslant \lfloor \mathfrak{p}_2(3, 3, 49, 0.175, 0.64) \rfloor = 62697$, and hence $D_\ell \leqslant 62697 \cdot 49^2$. Malle provided the authors a list of totally complex quadratic extensions $\ell$ of $k_2$ for which this bound holds. For every $\ell$ in this list, $h_\ell \leqslant 30$, and hence, $h_{\ell,4} \leqslant 16$. Therefore, $D_\ell^{1/6} < \mathfrak{q}_1(3, 3, 49, 16) < 12.02$. From Table IV in [Mart] we see that $M_c(34) \geqslant 12.4$. We conclude, as before, by considering the Hilbert class field of $\ell$, that $h_\ell \leqslant \lfloor 33/6 \rfloor = 5$, so $h_{\ell,4} \leqslant 4$. Then $D_\ell^{1/6} < \mathfrak{q}_1(3, 3, 49, 4) < 10.96$. Hence $D_\ell < 1.74 \times 10^6$. From the list provided by Malle, we see that there are eleven candidates for $\ell$, each with class number $h_\ell \leqslant 2$. Therefore, $D_\ell < \lfloor \mathfrak{p}_1(3, 3, 49, 2) \rfloor \times 49^2 = 1306144$. For all totally complex quadratic extension $\ell$ of $k_2$ satisfying this bound, $h_\ell = 1$ and hence we conclude that $D_\ell < \lfloor \mathfrak{p}_1(3, 3, 49, 1) \rfloor \times 49^2 =$

991613. From the list provided by Malle, we see that the possible $\ell$ are:

| $D_\ell$ | $\ell$ | $\zeta_{\ell\|k_2}(-2)$ |
|---|---|---|
| 16807 | $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$ | $-64/7$ |
| 64827 | $x^6 - x^5 + 3x^4 + 5x^2 - 2x + 1$ | $-2408/9$ |
| 153664 | $x^6 + 5x^4 + 6x^2 + 1$ | $-2306$ |
| 400967 | $x^6 - 2x^5 + 5x^4 - 7x^3 + 10x^2 - 8x + 8$ | $-25536$ |
| 573839 | $x^6 - x^5 + 4x^4 - 3x^3 + 8x^2 - 4x + 8$ | $-62208$ |
| 602651 | $x^6 - 3x^5 + 10x^4 - 15x^3 + 21x^2 - 14x + 7$ | $-70392$ |
| 909979 | $x^6 - 2x^5 + 7x^4 - 12x^3 + 21x^2 - 15x + 13$ | $-196216.$ |

We shall denote the seven pairs $(k_2, \ell)$ with $k_2 = \mathbb{Q}[x]/(x^3 - x^2 - 2x + 1)$, and $\ell$ one of the fields from the above list, by $\mathscr{E}_j$, $4 \leqslant j \leqslant 10$. Note that three pairs belonging to the above two lists coincide with pairs of number fields in [PY1], §8: $\mathscr{E}_4 = C_{31}$, $\mathscr{E}_5 = C_{32}$ and $\mathscr{E}_1 = C_{33}$.

Using the value of $\zeta_{\ell|k}(-2)$ given in the last column of the above two tables and the values of $\zeta_k(-1)$ and $\zeta_k(-3)$ given below for $k = k_1$ and $k_2$, we compute $\mathscr{R} = 2^{-9}\zeta_k(-1))\zeta_{\ell|k}(-2)\zeta_k(-3)$ for each of the ten pairs $\mathscr{E}_j$, $j \leqslant 10$. We find that the numerator of none of them is a power of 2. Proposition 1 then implies that $d$ cannot be 3 either.

$$\zeta_{k_1}(-1) = -1/9, \quad \zeta_{k_1}(-3) = 199/90; \quad \zeta_{k_2}(-1) = -1/21, \quad \zeta_{k_2}(-3) = 79/210.$$

The following is a summary of what we have proved above.

**Proposition 3.** *(i) If $n \geqslant 5$, then $d = 1$, i.e., $k = \mathbb{Q}$.*
*(ii) If $n = 3$, then $d \leqslant 2$.*

## 9. $G$ of type ${}^2A_n$ with $n > 1$ odd and $k = \mathbb{Q}$

**9.1.** We shall assume in the sequel that $k = \mathbb{Q}$ which according to Proposition 3 is the case if $n \geqslant 5$. Then $r = 1$ and $\ell = \mathbb{Q}(\sqrt{-a})$ for some square-free positive integer $a$. By setting $d = 1$ and $D_k = 1$ in bounds (61) and (62) we obtain

$$\begin{aligned} D_\ell &\leqslant \kappa_1(n, h_{\ell,n+1}) := \lfloor \mathfrak{p}_1(n, 1, 1, h_{\ell,n+1}) \rfloor. \\ D_\ell &\leqslant \kappa_2(n, R_\ell/w_\ell, \delta) := \lfloor \mathfrak{p}_2(n, 1, 1, R_\ell/w_\ell, \delta) \rfloor. \end{aligned}$$

**9.2.** We easily see that for fixed $\delta\,(\geqslant 0.02)$ and $n$, $\kappa_2$ decreases as $R_\ell/w_\ell$ increases, and for fixed $\delta \geqslant$ (0.02) and $R_\ell/w_\ell$, $\kappa_2$ decreases as $n$ increases provided $n \geqslant 7$. Since the regulator of a complex quadratic field is 1, and $w_\ell = 2$ for any complex quadratic field $\ell$ different from $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-1})$, $R_\ell/w_\ell = 1/2$ for all complex quadratic $\ell$ with $D_\ell > 4$. Now for $n \geqslant 17$, as $\kappa_2(n, 1/8, 1.8) \leqslant \kappa_2(17, 1/8, 1.8) = 2$, and there is no complex quadratic number field with discriminant $\leqslant 2$, *we conclude that $n \leqslant 15$.* For $n = 15$, unless $\ell = \mathbb{Q}(\sqrt{-3})$, we know that $D_\ell \leqslant \kappa_2(15, 1/4, 1.6) = 3$. Hence, if $n = 15$, $\ell = \mathbb{Q}(\sqrt{-3})$. For odd integers $n$ between 3 and 13, unless $\ell = \mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-1})$, with $D_\ell = 3$ and 4 respectively, we can use the bound $D_\ell \leqslant \kappa_2(n, 1/2, \delta)$, with $\delta$ as indicated below, to obtain:

| $n$ : | 13 | 11 | 9 | 7 | 5 | 3 |
|---|---|---|---|---|---|---|
| $\delta$ : | 1.3 | 1 | 0.9 | 0.7 | 0.5 | 0.26 |
| $D_\ell \leqslant$ | 4 | 6 | 10 | 21 | 68 | 2874. |

**9.3.** We will now improve the bound for the discriminant $D_\ell$ in case $n = 3$. From the Table t20.001-t20.002 of [1] we see that the class number of every complex quadratic number field $\ell$ with $D_\ell \leqslant 2874$ is $\leqslant 76$, and hence, $h_{\ell,4} \leqslant 64$. So we obtain the bound $D_\ell \leqslant \kappa_1(3, 64) = 1926$. We can improve this bound further as follows. From Table t20.001 we see that $h_\ell \leqslant 52$, and hence $h_{\ell,4} \leqslant 32$, for all complex quadratic $\ell$ with $D_\ell \leqslant 1926$. Now we observe that, for $n = 3$ and $k = \mathbb{Q}$, combining equations (3), (4), and using the bounds (54), $(n+1)^r \mu(\mathscr{G}/\Gamma) \leqslant 1$, $e'(P_v) \geqslant 1$ for all $v \in V_f$, and $e'(P_v) > n + 1$ for all $v \in \mathscr{T}$ (2.10), we get the following upper bound for $D_\ell$:

$$D_\ell \quad < \quad \lfloor \Big[ h_{\ell,4} \cdot \{2 \prod_{j=1}^{3} \frac{(2\pi)^{j+1}}{j!}\} \cdot \frac{1}{\zeta_{\mathbb{Q}}(2)\zeta_{\ell|\mathbb{Q}}(3)\zeta_{\mathbb{Q}}(4)} \Big]^{\frac{2}{5}} \rfloor \tag{66}$$

$$\leqslant \quad \lfloor \Big[ h_{\ell,4} \cdot \{2 \prod_{j=1}^{3} \frac{(2\pi)^{j+1}}{j!}\} \cdot \frac{1}{\zeta_{\mathbb{Q}}(2)^{1/2}\zeta_{\mathbb{Q}}(4)} \Big]^{\frac{2}{5}} \rfloor =: \widetilde{\kappa}_1(h_{\ell,4}) \tag{67}$$

where we have used the fact that $\zeta_{\mathbb{Q}}(2)^{1/2}\zeta_{\ell|\mathbb{Q}}(3) > 1$ (see Lemma 1 in [PY2]). Hence we conclude that $D_\ell \leqslant \widetilde{\kappa}_1(32) \leqslant 1363$.

**9.4.** We can improve the bounds for $D_\ell$ for $15 > n \geqslant 5$ as follows. From the table of complex quadratics in [C], we know that $h_\ell \leqslant 5$ for $D_\ell \leqslant 68$. Hence $h_{\ell,n+1} \leqslant 5$ for $n \geqslant 5$.

We now compute the values of $\kappa_1(n, j)$ for $5 \leqslant n < 15$ and $1 \leqslant j \leqslant 5$.

| | $\kappa_1(n,1)$ | $\kappa_1(n,2)$ | $\kappa_1(n,3)$ | $\kappa_1(n,4)$ | $\kappa_1(n,5)$ |
|---|---|---|---|---|---|
| $n = 5$ | 47 | 52 | 55 | 57 | 59 |
| $n = 7$ | 18 | 19 | 20 | 20 | 20 |
| $n = 9$ | 10 | 10 | 10 | 10 | 10 |
| $n = 11$ | 6 | 6 | 6 | 6 | 6 |
| $n = 13$ | 4 | 4 | 4 | 4 | 4. |

Comparing the above table with the table of complex quadratic number fields (cf. [C]) in terms of discriminants and class number, we obtain the following possibilities for $D_\ell$ and $a$ (recall that $\ell = \mathbb{Q}(\sqrt{-a})$):

| $n$ | $D_\ell$ | $a$ |
|---|---|---|
| 15 | 3 | 3 |
| 13 | 3, 4 | 3, 1 |
| 11 | 3, 4 | 3, 1 |
| 9 | 3, 4, 7, 8 | 3, 1, 7, 2 |
| 7 | 3, 4, 7, 8, 11, 15 | 3, 1, 7, 2, 11, 15 |
| 5 | 3, 4, 7, 8, 11, 15, 19, 20, 23 | 3, 1, 7, 2, 11, 15, 19, 5, 23 |
| | 24, 31, 35, 39, 40, 43 | 6, 31, 35, 39, 10, 43 |
| | 47, 51, 52, 55, 56 | 47, 51, 13, 55, 14. |

In the above we have used the fact that for $\ell = \mathbb{Q}(\sqrt{-a})$ , where $a$ is a square-free positive integer, $D_\ell = a$ if $a \equiv 3 \pmod 4$, and $D_\ell = 4a$ otherwise.

**9.5.** To prove Theorem 2 (stated in the Introduction) we compute $\mathscr{R}$ in each of the cases occurring in the second table of 9.4 using the following values of $\zeta := \zeta_{\mathbb{Q}}$ and $\zeta_{\ell|\mathbb{Q}}$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $j$ : | $-1$ | $-3$ | $-5$ | $-7$ | $-9$ | $-11$ | $-13$ | $-15$ |
| $\zeta(j)$ : | $-1/12$ | $1/120$ | $-1/252$ | $1/240$ | $-1/132$ | $691/32760$ | $-1/12$ | $3617/8160$. |

Listed below are the values of $\zeta_{\ell|k}$, for $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-3}))$,

| $\zeta_{\ell\mid\mathbb{Q}}(-2)$ | $\zeta_{\ell\mid\mathbb{Q}}(-4)$ | $\zeta_{\ell\mid\mathbb{Q}}(-6)$ | $\zeta_{\ell\mid\mathbb{Q}}(-8)$ | $\zeta_{\ell\mid\mathbb{Q}}(-10)$ | $\zeta_{\ell\mid\mathbb{Q}}(-12)$ | $\zeta_{\ell\mid\mathbb{Q}}(-14)$ |
|---|---|---|---|---|---|---|
| $-2/9$ | $2/3$ | $-14/3$ | $1618/27$ | $-3694/3$ | $111202/3$ | $13842922/9$, |

and the values of $\zeta_{\mathbb{Q}(\sqrt{-a})|\mathbb{Q}}$ required to compute $\mathscr{R}$ for $5 \leqslant n \leqslant 13$, $a \neq 3$, are given below:

| $a$ | $\zeta_{\ell\mid\mathbb{Q}}(-2)$ | $\zeta_{\ell\mid\mathbb{Q}}(-4)$ | $\zeta_{\ell\mid\mathbb{Q}}(-6)$ | $\zeta_{\ell\mid\mathbb{Q}}(-8)$ | $\zeta_{\ell\mid\mathbb{Q}}(-10)$ | $\zeta_{\ell\mid\mathbb{Q}}(-12)$ |
|---|---|---|---|---|---|---|
| 1 | $-1/2$ | $5/2$ | $-61/2$ | $-1385/2$ | $-50521/2$ | $2702765/2$ |
| 7 | $-16/7$ | $32$ | $-1168$ | $565184/7$ | | |
| 2 | $-3$ | $57$ | $-2763$ | $250737$ | | |
| 11 | $-6$ | $2550/11$ | $-21726$ | | | |
| 15 | $-16$ | $992$ | $-165616$. | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a$ : | 19 | 5 | 23 | 6 | 31 | 35 | 39 |
| $\zeta_{\ell\mid\mathbb{Q}}(-2)$ : | $-22$ | $-30$ | $-48$ | $-46$ | $-96$ | $-108$ | $-176$ |
| $\zeta_{\ell\mid\mathbb{Q}}(-4)$ : | 2690 | 3522 | 6816 | 7970 | 25920 | 42372 | 73120. |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a$ : | 10 | 43 | 47 | 51 | 13 | 55 | 14 |
| $\zeta_{\ell\mid\mathbb{Q}}(-2)$ : | $-158$ | $-166$ | $-288$ | $-268$ | $-302$ | $-400$ | $-396$ |
| $\zeta_{\ell\mid\mathbb{Q}}(-4)$ : | 79042 | 106082 | 169920 | 229700 | 257314 | 341984 | 362340. |

Explicit computation of $\mathscr{R}$ in each of the above cases shows that for every odd integer $n > 7$, the numerator of $\mathscr{R}$ has a prime divisor which does not divide $n + 1$. In view of Propositions 1 and 3, this proves Theorem 2 for $n > 7$.

For $n = 7,\ 5$, we list below the value of $\mathscr{R}$ for those $a$ in the second table in 9.4 for which the prime divisors of the numerator of $\mathscr{R}$ divide $n + 1$.

| $n$ | $a$ | $\zeta_{\ell\mid\mathbb{Q}}(-2)$ | $\zeta_{\ell\mid\mathbb{Q}}(-4)$ | $\zeta_{\ell\mid\mathbb{Q}}(-6)$ | $\mathscr{R}$ |
|---|---|---|---|---|---|
| 7 | 3 | $-2/9$ | $2/3$ | $-14/3$ | $1/16124313600 = 1/(2^{15} \cdot 3^9 \cdot 5^2)$ |
| 5 | 3 | $-2/9$ | $2/3$ | | $1/78382080 = 1/(2^{10} \cdot 3^7 \cdot 5 \cdot 7)$ |
| 5 | 1 | $-1/2$ | $5/2$ | | $1/9289728 = 1/(2^{14} \cdot 3^4 \cdot 7)$ |
| 5 | 7 | $-16/7$ | $32$ | | $1/158760 = 1/(2^3 \cdot 3^4 \cdot 5 \cdot 7^2)$ |
| 5 | 31 | $-96$ | $25920$ | | $3/14$. |

We need to consider only the $a$ appearing in the above table.

**9.6.** In our treatment of groups of type ${}^2A_n$, with $n$ odd, we have not so far made use of the assumption that $\Gamma$ is cocompact, or, equivalently, $G$ is anisotropic over $k$, see 1.5. We will now use the fact that $G$ is anisotropic over $k = \mathbb{Q}$ to exclude $n = 7,\ 5$. This will complete our proof of Theorem 2.

From the well-known description of absolutely simple simply connected $\mathbb{Q}$-groups of type ${}^2A_n$ we know that there is a division algebra $\mathscr{D}$ with center $\ell$ and of degree $\mathfrak{d} = \sqrt{[\mathscr{D} : \ell]}$, $\mathfrak{d}|(n+1)$, $\mathscr{D}$ given with an involution $\sigma$ of the second kind, and a nondegenerate hermitian form $h$ on $\mathscr{D}^{(n+1)/\mathfrak{d}}$ defined in terms of the involution $\sigma$, so that $G$ is the special unitary group $\mathrm{SU}(h)$ of $h$.

If $\mathscr{D} = \ell$, then $h$ is an hermitian form on $\ell^{n+1}$ such that the quadratic form $q$ on the $2(n+1)$-dimensional $\mathbb{Q}$-vector space $V = \ell^{n+1}$ defined by

$$q(v) = h(v, v) \quad \text{for} \quad v \in V,$$

is isotropic over $\mathbb{R}$ (since $G$ is isotropic over $\mathbb{R}$, i.e., $G(\mathbb{R})$ is noncompact). Then as $n \geqslant 3$, $q$ is isotropic over $\mathbb{Q}$ by Meyer's theorem and hence G is isotropic over $\mathbb{Q}$. But this is not the case. Therefore, $\mathscr{D} \neq \ell$, i.e., $\mathscr{D}$ is a noncommutative division algebra of degree $\mathfrak{d} > 1$.

Using the structure of the Brauer group of a global field, we see that there exists at least one prime $p$ which splits over $\ell$ such that $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathscr{D} = (\mathbb{Q}_p \otimes_{\mathbb{Q}} \ell) \otimes_{\ell} \mathscr{D}$ is isomorphic to $M_m(\mathfrak{D}_p) \times M_m(\mathfrak{D}_p^o)$, where $\mathfrak{D}_p$ is a noncommutative central division algebra over $\mathbb{Q}_p$ of degree $\mathfrak{d}_p > 1$, $\mathfrak{D}_p^o$ is its opposite, and $m = (n+1)/\mathfrak{d}_p$. The involution $\sigma$ interchanges the two factors of $M_m(\mathfrak{D}_p) \times M_m(\mathfrak{D}_p^o)$, and hence, $G(\mathbb{Q}_p) \cong \mathrm{SL}_m(\mathfrak{D}_p)$.

In the rest of this section $n$ is either 7 or 5, $a$ and $\mathscr{R}$ are as in the last table of 9.5. The nonarchimedean place of $\mathbb{Q}$ corresponding to a prime $p$ will be denoted by $p$. Now let $p$ be a prime which splits in $\ell = \mathbb{Q}(\sqrt{-a})$ and $\mathfrak{D}_p$ is a noncommutative division algebra with center $\mathbb{Q}_p$. Then (see the computation in 2.3(ii) of [PY2]) $e'(P_p)$ is an integral multiple of $f_7(p) := (p-1)(p^3-1)(p^5-1)(p^7-1)$ if $n = 7$, and it is an integral multiple of either $f_5(p) := (p-1)(p^3-1)(p^5-1)$ or $g_5(p) := (p-1)(p^2-1)(p^4-1)(p^5-1)$ if $n = 5$.

Now let $\mathscr{T}$ be as in 2.10. Recall that for every prime $q$, $e'(P_q)$ is an integer, and for $q \in \mathscr{T}$, $e'(P_q) > n+1$. Also recall that $\mu(\mathscr{G}/\Gamma)$ is a submultiple of $1/(n+1)$, and

$$\mu(\mathscr{G}/\Gamma) = \frac{\mathscr{R} \prod e'(P_q)}{[\Gamma : \Lambda]} = \frac{\mathscr{R} e'(P_p) \prod_{q \neq p} e'(P_q)}{[\Gamma : \Lambda]}. \tag{68}$$

As every prime divisor of $[\Gamma : \Lambda]$ divides $n+1$, we conclude that every prime divisor of the numerator of $\mathscr{R} e'(P_p)$ divides $n+1$. Also since $[\Gamma : \Lambda] \leqslant 2h_{\ell,n+1}(n+1)^{1+\#\mathscr{T}}$ (cf. (54)), we see that

$$\frac{\mathscr{R} e'(P_p)}{2h_{\ell,n+1}(n+1)^2} \leqslant \mu(\mathscr{G}/\Gamma) \leqslant \frac{1}{(n+1)}, \tag{69}$$

and hence,

$$\mathscr{R} e'(P_p) \leqslant 2h_{\ell,n+1}(n+1). \tag{70}$$

Now we note that the class number of the complex quadratic field $\ell = \mathbb{Q}(\sqrt{-a})$, for $a = 3, 1, 7$ is 1, and for $a = 31$ the class number is 3. The first two primes $\{p_1, p_2\}$ which split in $\mathbb{Q}(\sqrt{-a})$ are $\{7, 13\}$, $\{5, 13\}$, $\{2, 11\}$ and $\{2, 5\}$ for $a = 3, 1, 7$ and 31 respectively. Let $\mathscr{R}$ be as in the last column of the last table of 9.5. By direct computations we see that if $n = 7$ and $a = 3$, $\mathscr{R} e'(P_p) \geqslant \mathscr{R} f_7(7) > 16$, and if $n = 5$ and $a = 31$, both $\mathscr{R} f_5(2)$ and $\mathscr{R} g_5(2)$ are larger than 36. On the other hand, if $n = 5$ and $a = 3, 1$ or 7, both $\mathscr{R} f_5(p_2)$ and $\mathscr{R} g_5(p_2)$ are larger than 12, and at least one prime divisor of the numerator of $\mathscr{R} f_5(p_1)$ and $\mathscr{R} g_5(p_1)$ is different from 2 and 3. We conclude from these observations that $n$ cannot be 5 or 7. Thus we have proved Theorem 2.

**Corrections in** [PY2]: (i) In line 11 on page 381 and in the last two lines on page 386, "$\chi(\Gamma)$" and "$\chi(\Lambda)$"should be replaced with "$|\chi(\Gamma)|$" and "$|\chi(\Lambda)|$" respectively. (ii) In the statement of Theorem 2 on page 402, "$\chi(X_u)/n$" should be replaced with "$\chi(X_u)$".

We note that a revised version of [PY1] which incorporates corrections and additions given in the "Addendum" has recently been posted on the arXiv.

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