NUMBER-THEORETIC TECHNIQUES IN THE THEORY OF LIE GROUPS AND DIFFERENTIAL GEOMETRY

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The aim of this article is to give a brief survey of the results obtained in the series of papers [17]–[21]. These papers deal with a variety of problems, but have a common feature: they all rely in a very essential way on number-theoretic techniques (including p-adic techniques), and use results from algebraic and transcendental number theory. The fact that number-theoretic techniques turned out to be crucial for tackling certain problems originating in the theory of (real) Lie groups and differential geometry was very exciting. We hope that these techniques will become an integral part of the repertoire of mathematicians working in these areas.

To keep the size of this article within a reasonable limit, we will focus primarily on the paper [21], and briefly mention the results of [17]–[20] and some other related results in the last section. The work in [21], which was originally motivated by questions in differential geometry dealing with *length-commensurable* and *isospectral* locally symmetric spaces (cf. §1), led us to define a new relationship between Zariski-dense subgroups of a simple (or semi-simple) algebraic group which we call *weak commensurability* (cf. §2). The results of [21] give an almost complete characterization of weakly commensurable arithmetic groups, but there remain quite a few natural questions (some of which are mentined below) for general Zariski-dense subgroups. We hope that the notion of weak commensurability will be useful in investigation of (discrete) subgroups of Lie groups, geometry and ergodic theory.

1. Length-commensurable and isospectral manifolds

Let M be a Riemannian manifold. In differential geometry, one associates to M the following sets of data: the *length spectrum* $\mathcal{L}(M)$ (the set of lengths of all closed geodesics *with* multiplicities), the *weak length spectrum* $L(M)$ (the set of lengths of all closed geodesics *without* multiplicities), the *spectrum of the Laplace operator* $\mathcal{E}(M)$ (the set of eigenvalues of the Laplacian Δ_M with multiplicities). The fundamental question is to what extent do $L(M)$, $\mathcal{L}(M)$ and $\mathcal{E}(M)$ determine M?

In analyzing this question, the following terminology will be used: two Riemannian manifolds M_1 and M_2 are said to be *isospectral* if $\mathcal{E}(M_1) = \mathcal{E}(M_2)$, and *iso-length-spectral* if $\mathcal{L}(M_1) = \mathcal{L}(M_2)$.

First, it should be pointed out that the conditions like isospectrality, iso-length-spectrality are related to each other. For example, for compact hyperbolic 2-manifolds M_1 and M_2, we have $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ if and only if $\mathcal{E}(M_1) = \mathcal{E}(M_2)$ (cf. [10]), and two hyperbolic 3-manifolds are isospectral if and only if they have the same *complex-length* spectrum (for its definition see the footnote later in this section), cf. [5] or [7]. Furthermore, for compact locally symmetric spaces M_1 and M_2 of nonpositive curvature, if $\mathcal{E}(M_1) = \mathcal{E}(M_2)$, then $L(M_1) = L(M_2)$ (see [21], Theorem 10.1). (Notice that all these results rely on some kind of trace formula.)

Second, neither of $\mathcal{L}(M)$, $L(M)$ or $\mathcal{E}(M)$ determines M up to isometry. In fact, in 1980, Vignéras [27] constructed examples of isospectral, but nonisometric, hyperbolic 2 and 3-manifolds. This construction relied on arithmetic properties of orders in a quaternion algebra D. More precisely, her crucial observation was that it is possible to choose D so that it contains orders $\mathcal{O}_1$ and $\mathcal{O}_2$ with the property that the corresponding groups $\mathcal{O}_1^{(1)}$ and $\mathcal{O}_2^{(1)}$ of elements with reduced norm one are not conjugate, but their closures in the completions are conjugate, for all nonarchimedean places of the center. Five years later, Sunada [26] gave a very general, and purely group-theoretic, method for constructing isospectral, but nonisometric, manifolds. His construction goes as follows: *Let M be a Riemannian manifold with the fundamental group $\Gamma := \pi_1(M)$. Assume that Γ has a finite quotient G with the following property: there are subgroups H_1, H_2 of G such that $|C \cap H_1| = |C \cap H_2|$ for all conjugacy classes C of G. Let M_i be the finite-sheeted cover of M corresponding to the pull-back of H_i in Γ. Then (under appropriate assumptions), M_1 and M_2 are nonisometric isospectral (or iso-length-spectral) manifolds.*

Since its inception, Sunada's method and its variants have been used to construct examples of nonisometric manifolds with same invariants. In particular, Alan Reid [25] constructed examples of nonisometric iso-length-spectral hyperbolic 3-manifolds, and last year, in a joint paper [8], Leninger, McReynolds, Neumann and Reid gave examples of hyperbolic manifolds with the same weak length spectrum, but different volumes. These, and other examples, demonstrate that it is not possible to characterize Riemannian manifolds (even hyperbolic ones) up to isometry by their spectrum or length spectrum. On the other hand, it is worth noting that the manifolds furnished by Vignéras, and the ones obtained using Sunada's method are always *commensurable,* i.e., have a common finite-sheeted covering. This suggests that the following is perhaps a more reasonable question.

Question 1: *Let M_1 and M_2 be two (hyperbolic) manifolds (of finite volume or even compact). Suppose $L(M_1) = L(M_2)$. Are M_1 and M_2 necessarily commensurable?*

(Of course, the same question can be asked for other classes of manifolds, e.g. for general locally symmetric spaces of finite volume.)

The answer even to this modified question turns out to be "no" in general: Lubotzky, Samuels and Vishne [9] have given examples of isospectral (hence, with same weak length spectrum) compact locally symmetric spaces that are *not* commensurable. At the same time, some positive results have emerged. Namely, Reid [25] and Chinburg, Hamilton, Long and Reid [6] have given a positive answer to Question 1 for arithmetically defined hyperbolic 2- and 3-manifolds, respectively. Our results in [21] provide an almost complete answer to Question 1 for arithmetically defined locally symmetric spaces of arbitrary absolutely simple Lie groups. In fact, in [21] we analyze when two locally symmetric spaces are commensurable given that they satisfy a much weaker condition than iso-length-spectrality, which we termed *length-commensurability.* We observe that not only does the use of this condition produce stronger results, but the condition itself is more suitable for analyzing Question 1 as it allows one to replace the manifolds under consideration with commensurable manifolds.

Definition. Two Riemannian manifolds M_1 and M_2 are said to be *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Now, we are in a position to formulate precisely the question which is central to [21].

Question 2: *Suppose M_1 and M_2 are length-commensurable. Are they commensurable?*

In [21], we have been able to answer this question for arithmetically defined locally symmetric spaces of absolutely simple Lie groups. The precise formulations will be given in §3, after introducing appropriate definitions. The following theorem, however, is fully representative of these results.

Theorem. (1) *Let M_1 and M_2 be two* arithmetically defined hyperbolic *manifolds of* even *dimension. If M_1 and M_2 are not commensurable, then, after a possible interchange of M_1 and M_2, there exists $\lambda_1 \in$*

$L(M_1)$ *such that for* any $\lambda_2 \in L(M_2)$, *the ratio* λ_1/λ_2 *is* transcendental. *In particular,* M_1 *and* M_2 *are not length-commensurable.*

(2) *For any dimension* $d \equiv 1 (\mathrm{mod}\, 4)$, *there exist length-commensurable, but not commensurable, arithmetically defined hyperbolic d-manifolds.*

We have proved similar results for arithmetically defined locally symmetric spaces of absolutely simple real Lie groups of all types; see [21] and [22]. For example, for hyperbolic spaces modeled on Hamiltonian quaternions we have an assertion similar to (1) (i.e., Question 2 has an affirmative answer); but for complex hyperbolic spaces we have an assertion similar to (2) (i.e., Question 2 has a negative answer).

The key ingredient of our approach is the new notion of *weak commensurability* of Zariski-dense subgroups of an algebraic group, and the relationship between the length-commensurability of locally symmetric spaces and the weak commensurability of their fundamental groups. To motivate the definition of weak commensurability, we consider the following simple example.

Let $\mathfrak{H} = \{x + iy \mid y > 0\}$ be the upper half-plane with the standard hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$. Then $t \mapsto ie^t$ is a geodesic in $\mathfrak{H}$, whose piece $\tilde{c}$ connecting i to ai, where $a > 1$, has length $\ell(\tilde{c}) = \log a$. Now, let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete torsion-free subgroup, and $\pi \colon \mathfrak{H} \to \mathfrak{H}/\Gamma$ be the canonical projection. If $c := \pi(\tilde{c})$ is a closed geodesic in $\mathfrak{H}/\Gamma$ (traced once), then it is not difficult to see that for $\lambda = \sqrt{a}$, the element

$$\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

lies in Γ. Then the length $\ell(c)$ of c equals $\log a = 2 \log \lambda$. This shows that the lengths of closed geodesics in the hyperbolic 2-manifold $\mathfrak{H}/\Gamma$ are (multiples of) the logarithms of the eigenvalues of semi-simple elements of the fundamental group Γ (cf. §3 below, and Proposition 8.2 in [21] for a general statement that applies to arbitrary locally symmetric spaces)[1]. Furthermore, let c_i for $i = 1, 2$, be closed geodesics in $\mathfrak{H}/\Gamma$ that in the above notation correspond to semi-simple elements $\gamma_i \in \Gamma$ having the eigenvalue $\lambda_i > 1$. Then

$$\ell(c_1)/\ell(c_2) = m/n \quad \Leftrightarrow \quad \lambda_1^n = \lambda_2^m.$$

Notice that the condition on the right-hand side can be reformulated as follows: *If* T_i *is a torus of* SL_2 *such that* $\gamma_i \in T_i(\mathbb{R})$, *then there exist*

[1] In the above construction if we replace $\mathrm{SL}_2(\mathbb{R})$ with $\mathrm{SL}_2(\mathbb{C})$, then the collection of (principal values) of the logarithms of the eigenvalues of semi-simple elements is known as the *complex length spectrum* of the corresponding hyperbolic 3-manifold.

$\chi_i \in X(T_i)$ *with*

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

The above discussion suggests the following.

Definition. Let G be a semi-simple algebraic group defined over a field F. Two semi-simple elements $\gamma_1, \gamma_2 \in G(F)$ are *weakly commensurable* if, for $i = 1, 2$, there exist maximal F-tori T_i, and characters $\chi_i \in X(T_i)$, such that $\gamma_i \in T_i(F)$, and

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

As we have seen, weak commensurability adequately reflects length-commensurability of hyperbolic 2-manifolds. In fact, it remains relevant for length-commensurability of arbitrary locally symmetric spaces. This is easy to see for rank one spaces but is less obvious for higher rank spaces - cf. §3 below, and [21], §8.

2. Weakly commensurable arithmetic subgroups

We observe that for $G \neq \mathrm{SL}_2$, weak commensurability of $\gamma_1, \gamma_2 \in G(F)$ may not relate these elements to each other in a significant way (in particular, F-tori T_i of G containing these elements may be very different). So, to get meaningful consequences of weak commensurability, one needs to extend this notion from individual elements to "large" (in particular, Zariski-dense) subgroups.

Definition. Two (Zariski-dense) subgroups Γ_1, Γ_2 of $G(F)$ are *weakly commensurable* if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

It was discovered in [21] that weak commensurability has some important consequences even for completely general finitely generated Zariski-dense subgroups. For simplicity, we will assume henceforth that all our fields are of characteristic zero. To formulate our first result, we need one additional notation: given a subgroup Γ of $G(F)$, where G is an absolutely simple algebraic F-group, we let K_Γ denote the subfield of F generated by the traces $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$ for all $\gamma \in \Gamma$, where Ad denotes the adjoint representation of G. We recall that according to a result of Vinberg [28], for a Zariski-dense subgroup Γ of G, the field K_Γ is precisely the field of definition of $\mathrm{Ad}\,\Gamma$, i.e., it is the minimal subfield K of F such that all elements of $\mathrm{Ad}\,\Gamma$ can be represented simultaneously by matrices with entries in K, in a certain basis of the Lie algebra $\mathfrak{g}$ of G.

Theorem A. *Let Γ_1 and Γ_2 be two finitely generated Zariski-dense subgroups of $G(F)$. If Γ_1 and Γ_2 are weakly commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

Much stronger results are available for the case of arithmetic subgroups. To formulate these, we need to describe the terminology we use regarding arithmetic subgroups. Let G be a semi-simple algebraic group over a field F of characteristics zero. Suppose we are given:

• a number field K contained in F;

• a (finite) subset S of places of K containing all the archimedean places;

• a K-form G_0 of G, i.e., a group G_0 defined over K such that ${}_FG_0 \stackrel{\iota}{\simeq} G$ over F.

Then subgroups of $G(F)$ commensurable with the image of the natural embedding $G_0(\mathcal{O}_K(S)) \hookrightarrow G(F)$ induced by ι, where $\mathcal{O}_K(S)$ is the ring of S-integers in K, are by definition (G_0, K, S)-*arithmetic* subgroups. Notice that in this definition we *do* fix an embedding of K into F (in other words, isomorphic, but distinct, subfields of F are treated as different fields), but we *do not* fix an F-isomorphism ι, so by varying it we generate a class of subgroups *invariant under F-automorphisms.* For this reason, by "commensurability" we will mean "commensurability *up to F-isomorphism,*" i.e., two subgroups Γ_1 and Γ_2 of $G(F)$ will be called commensurable if there exists an F-automorphism φ of G such that $\varphi(\Gamma_1)$ and Γ_2 are commensurable in the usual sense, viz. their intersection has finite index in both of them. Another convention is that S will be assumed to contain no nonarchimedean places v such that G_0 is K_v-anisotropic (this assumption enables us to recover S uniquely from a given S-arithmetic subgroup).

The group G in Theorems B–F is assumed to be absolutely simple.

Theorem B. *Let Γ_i be a Zariski-dense (G_i, K_i, S_i)-arithmetic subgroup of $G(F)$ for $i = 1, 2$. If Γ_1 and Γ_2 are weakly commensurable, then $K_1 = K_2$ and $S_1 = S_2$.*

One shows that Γ_1 and Γ_2 as in Theorem B are commensurable if and only if $K_1 = K_2$, $S_1 = S_2$ and $G_1 \simeq G_2$ over $K := K_1 = K_2$ (cf. Proposition 2.5 in [21]). So, according to Theorem B, the weak commensurability of Γ_1 and Γ_2 implies that the first two of these three conditions do hold. In general, however, G_1 and G_2 do not have to be K-isomorphic. Our next theorem describes the situations where it can be inferred that G_1 and G_2 are K-isomorphic.

Theorem C. *Suppose G is not of type A_n $(n > 1)$, D_{2n+1} $(n \geqslant 1)$ or E_6. If $G(F)$ contains Zariski-dense weakly commensurable (G_i, K, S)-arithmetic subgroups Γ_i for $i = 1, 2$, then $G_1 \simeq G_2$ over K, and hence Γ_1 and Γ_2 are commensurable up to an F-automorphism of G.*

In the general case, we have the following finiteness result.

Theorem D. *Let Γ_1 be a Zariski-dense (G_1, K, S)-arithmetic subgroup of $G(F)$. Then the set of K-isomorphism classes of K-forms G_2 of G such that $G(F)$ contains a Zariski-dense (G_2, K, S)-arithmetic subgroup weakly commensurable to Γ_1 is finite. In other words, the set of all (K, S)-arithmetic subgroups of $G(F)$ which are weakly commensurable to a given (K, S)-arithmetic subgroup is a union of finitely many commensurability classes.*

Note that for the types A_n $(n > 1)$, D_{2n+1} and E_6 excluded in Theorem C, the number of commensurability classes in Theorem D may not be bounded by an absolute constant depending, say, on G, K and S : as one varies Γ_1 (or, equivalently, G_1), this number changes and typically grows to infinity. To explain what happens for groups of these types, let us consider the following example.

Fix any $n > 1$ and pick four nonarchimedean places v_1, v_2, v_3, $v_4 \in V^K$. Next, consider central division K-algebras D_1 and D_2 of degree $d = n + 1 > 2$ with local invariants $(\in \mathbb{Q}/\mathbb{Z})$:

$$n_v^{(1)} = \begin{cases} 0 & , \quad v \neq v_i, i \leqslant 4 \\ 1/d & , \quad v = v_1 \text{ or } v_2 \\ -1/d & , \quad v = v_3 \text{ or } v_4 \end{cases}$$

and

$$n_v^{(2)} = \begin{cases} 0 & , \quad v \neq v_i, i \leqslant 4 \\ 1/d & , \quad v = v_1 \text{ or } v_3 \\ -1/d & , \quad v = v_2 \text{ or } v_4. \end{cases}$$

Then the algebras D_1 and D_2 are neither isomorphic nor anti-isomorphic, implying that the algebraic groups $G_1 = \mathrm{SL}_{1,D_1}$ and $G_2 = \mathrm{SL}_{1,D_2}$ (which are anisotropic inner forms of type A_n) are not K-isomorphic. On the other hand, D_1 and D_2 have exactly the same maximal subfields, which means that G_1 and G_2 have the same maximal K-tori. It follows that for any S, the corresponding S-arithmetic subgroups are weakly commensurable, but not commensurable. Furthermore, by increasing the number of places in this construction, one can construct an arbitrarily large number of central division K-algebras of degree d with the above properties. Then the associated S-arithmetic groups will all be weakly commensurable, but will constitute an arbitrarily large number of commensurability classes.

In [21], Example 6.6, we described how a similar construction can be given for *some* outer form of type A_n (i.e., for special unitary groups), at least when $d = n + 1$ is odd. The restriction on d is due to the fact that our argument relies on a local-global principle for embedding fields with an involutive automorphisms into an algebra with an involution of the second kind (Proposition A.2 in [16]), which involves some additional assumptions. Recently, we have been able to remove any restrictions in the local-global principle (unpublished), so the construction can in fact be implemented for all d.

However, no construction of nonisomorphic K-groups with the same K-tori was known for types D_n and E_6. We have given a construction, using Galois cohomology, which works uniformly for types A_n, D_{2n+1} and E_6 (cf. [21], §9). Towards this end, we established a new local-global principle for the existence of an embedding of a given K-torus as a maximal torus in a given absolutely simple simply connected K-group. This construction, of course, allows one to produce examples of noncommensurable weakly commensurable S-arithmetic subgroups in groups of types A_n, D_{2n+1} and E_6, and in fact, show that the number of commensurability classes is unbounded. This construction may also be useful elsewhere, for example, in the Langlands program.

Even though the definition of weak commensurability involves only semi-simple elements, it detects the presence of unipotent elements; in fact it detects K-rank.

Theorem E. *Assume that $G(F)$ contains Zariski-dense weakly commensurable (G_1, K, S)- and (G_2, K, S)-arithmetic subgroups. Then the Tits indices of G_1/K and G_2/K are isomorphic. In particular,* $\mathrm{rk}_K G_1 = \mathrm{rk}_K\, G_2$.

The above results provide an almost complete picture of weak commensurability among S-arithmetic subgroups. In view of the connection of weak commensurability with length-commensurability of locally symmetric spaces (cf. §3), one would like to extend these results to not necessarily arithmetic Zariski-dense subgroups. We conclude this section with an arithmeticity theorem in which only one subgroup is assumed to be arithmetic, and a discussion of some open questions.

Theorem F. *Let G be an absolutely simple algebraic group over a nondiscrete locally compact field F. Let Γ_1 and Γ_2 be two lattices in $G(F)$, with Γ_1 (K, S)-arithmetic. If Γ_1 and Γ_2 are weakly commensurable, then Γ_2 is also (K, S)-arithmetic.*

Remarks: (i) The assumption that *both* Γ_1 and Γ_2 be lattices cannot be omitted. For example, let $\Gamma \subset SL_2(\mathbb{Z})$ be a torsion-free subgroup of finite index, and Γ^n be the subgroup generated by the n-th powers of elements in Γ. Then Γ^n is weakly commensurable with Γ for all n. On the other hand, $[\Gamma : \Gamma^n] = \infty$ for all sufficiently large n, and then Γ^n is not arithmetic. The same remark applies to all hyperbolic groups. However, we do not know what happens in the higher rank situation.

(ii) The case of lattices in products of real and p-adic groups has not been fully investigated.

(iii) Yet another interesting open question is whether or not the discreteness of one of the two weakly commensurable subgroups Γ_1, Γ_2 of $G(F)$ implies the discreteness of the other (here F is a locally compact nondiscrete field).

Further analysis of weak commensurability of general Zariski-dense subgroups of $G(F)$ for an arbitrary field F would require information about classification of forms of G over general fields, which is not yet available. For example, even the following basic question seems to be open.

Question 3: *Let D_1 and D_2 be two quaternion division algebras over a finitely generated field K. Assume that D_1 and D_2 have the* same maximal subfields. *Are they isomorphic?*

M. Rost has informed us that over large fields (like those used in the proof of the Merkurjev-Suslin theorem) the answer can be "no" (apparently, the same observation was independently made by A. Wadsworth and some other people). But for finitely generated fields (note that the fields arising in the investigation of weakly commensurable finitely generated subgroups are finitely generated), the answer is unknown. Furthermore, if the answer turns out to be "no", we would like to know if the number of isomorphism classes of quaternion algebras over a given finitely generated field, and containing the same maximal subfields is finite (this may be useful for extending the finiteness result of Theorem D to such nonarithmetic subgroups as the fundamental groups of general compact Riemann surfaces).

3. Length-commensurable locally symmetric spaces

Let G be a connected semi-simple algebraic $\mathbb{R}$-group, $\mathcal{G} = G(\mathbb{R})$. We let $\mathcal{K}$ denote a maximal compact subgroup of $\mathcal{G}$, and let $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ be the corresponding symmetric space of $\mathcal{G}$. For a discrete torsion-free subgroup Γ of $\mathcal{G}$, we let $\mathfrak{X}_\Gamma := \mathfrak{X}/\Gamma$ denote the locally symmetric space with the fundamental group Γ. We say that $\mathfrak{X}_\Gamma$ is *arithmetically defined* if Γ is arithmetic (with S the set of archimedean places of K) in the sense specified in §2. Notice that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable as manifolds (i.e., have a common finite-sheeted cover) if and only if Γ_1 and Γ_2 are commensurable up to $\mathbb{R}$-automorphism of G.

Our goal now is to relate length-commensurability of locally symmetric spaces to weak commensurability of their fundamental groups. We need to recall some basic facts about closed geodesics in $\mathfrak{X}_\Gamma$ (cf. [20], or [21], §8). The closed geodesics on $\mathfrak{X}/\Gamma$ correspond to semi-simple elements of Γ. For a semi-simple element $\gamma \in \Gamma$, let c_γ be the closed geodesic corresponding to γ. Its length is given by the following formula (see [21], Proposition 8.2).

$$\ell_\Gamma(c_\gamma)^2 = (1/n_\gamma^2)\left(\sum(\log|\alpha(\gamma)|)^2\right), \tag{1}$$

where n_γ is an integer, and the sum is over all roots α of G with respect to a maximal $\mathbb{R}$-torus T such that $\gamma \in T(\mathbb{R})$. (We notice that for the upper half-plane $\mathfrak{H} = \mathrm{SO}_2\backslash\mathrm{SL}_2(\mathbb{R})$ this metric differs from the standard hyperbolic metric, considered in §1, by a factor of $\sqrt{2}$, which, of course, does not affect length commensurability.)

For our purposes, we need to recast (1) using the notion of a *positive real character*. Given a real torus T, a real character χ of T is called *positive* if $\chi(t) > 0$ for all $t \in T(\mathbb{R})$. We notice that for any character χ of T we have

$$|\chi(t)|^2 = \chi(t)\cdot\overline{\chi(t)} = (\chi + \overline{\chi})(t) = \chi_0(t),$$

where χ_0 is a positive real character. Hence,

$$\ell_\Gamma(c_\gamma)^2 = (1/n_\gamma^2)\sum_{i=1}^{p} s_i(\log\chi^{(i)}(\gamma))^2, \tag{2}$$

where $s_i \in \mathbb{Q}$, and $\chi^{(i)}$ are positive real characters.

The right-hand side of (2) is easiest to analyze when $\mathrm{rk}_{\mathbb{R}} G = 1$, which we will now assume. Let χ be a generator of the group of positive real characters of a maximal $\mathbb{R}$-torus T containing γ. Then

$$\ell_\Gamma(c_\gamma) = (s/n_\gamma)\cdot|\log\chi(\gamma)|,$$

where s is independent of γ and T (because any two maximal $\mathbb{R}$-tori of real rank one are conjugate to each other by an element of $G(\mathbb{R})$). So, if $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are *not* weakly commensurable, then

$$\ell_{\Gamma_1}(c_{\gamma_1})/\ell_{\Gamma_2}(c_{\gamma_2}) = (n_{\gamma_2}/n_{\gamma_1})\cdot\left(\pm\frac{\log\chi_1(\gamma_1)}{\log\chi_2(\gamma_2)}\right) \notin \mathbb{Q}. \tag{3}$$

Therefore, if Γ_1 and Γ_2 are *not weakly commensurable*, then $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are *not length-commensurable.* Thus, the connection noted in §1 for hyperbolic 2-manifolds remains valid for arbitrary locally symmetric spaces of rank one. In fact, one can make a stronger statement assuming that Γ_1 and Γ_2 are arithmetic (or, more generally, can be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$). Then $\chi_i(\gamma_i) \in \overline{\mathbb{Q}}^{\times}$ for $i = 1, 2$. But according to a theorem proved independently by Gelfond and Schneider in 1934, if α and β are algebraic numbers such that $\log\alpha/\log\beta$ is irrational, then it is transcendental over $\mathbb{Q}$ (cf. [3]). So, it follows from (3) that if Γ_1 and Γ_2 are as above, and $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are not weakly commensurable, then

$$\ell_{\Gamma_1}(c_{\gamma_1})/\ell_{\Gamma_2}(c_{\gamma_2})$$

is transcendental over $\mathbb{Q}$.

To relate length-commensurability of locally symmetric spaces of higher rank with the notion of weak commensurability of their fundamental groups, we need to invoke the Schanuel's Conjecture from transcendental number theory (cf. [3]).

Schanuel's conjecture. *If $z_1, \ldots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree over $\mathbb{Q}$ of the field generated by*

$$z_1, \ldots, z_n;\ e^{z_1}, \ldots, e^{z_n}$$

is $\geqslant n$.

What we need is the following corollary of Schanuel's conjecture. Let $\alpha_1, \ldots, \alpha_n$ be nonzero *algebraic* numbers, and set $z_i = \log\alpha_i$. Applying Schanuel's conjecture, we obtain that $\log\alpha_1, \ldots, \log\alpha_n$ are *algebraically independent* as soon as they are *linearly independent* (over $\mathbb{Q}$), i.e., whenever $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent.

Before we proceed, we would like to point out that our results for locally symmetric spaces of rank > 1 depend on the truth of Schanuel's conjecture (hence are *conditional*). Analyzing the right hand side of equation (2) with the help of the above consequence of Schanuel's conjecture, we show that if both Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$, for non-weakly commensurable $\gamma_i \in \Gamma_i$, $\ell_{\Gamma_1}(c_{\gamma_1})$ and $\ell_{\Gamma_2}(c_{\gamma_2})$ are algebraically independent over $\mathbb{Q}$. Thus, we obtain the following.

Proposition. *Let Γ_1 and Γ_2 be discrete torsion-free subgroups of $\mathcal{G} = G(\mathbb{R})$, where G is an absolutely simple $\mathbb{R}$-subgroup of SL_n. In the case $\mathrm{rk}_{\mathbb{R}}\, G > 1$, assume that Schanuel's conjecture holds and both*

Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$*. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then the subgroups Γ_1 and Γ_2 are weakly commensurable.*

We recall that if Γ is a lattice in $\mathcal{G} = G(\mathbb{R})$, where G is an absolutely simple real algebraic group, not isogenous to SL_2, then there exists a real number field K such that G is defined over K and $\Gamma \subset G(K)$, see [23], Proposition 6.6. In particular, if $\mathrm{rk}_{\mathbb{R}}\, G > 1$ and Γ is a lattice in $\mathcal{G}$ (or, equivalently, $\mathfrak{X}_\Gamma$ has finite volume), then Γ can always be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$, so the corresponding assumption in the proposition is redundant. Theorem A now implies

Theorem 1. *Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are of finite volume, and let K_{Γ_i} denote the field generated by the traces* $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$ *for $\gamma \in \Gamma_i$. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

We now turn to arithmetically defined locally symmetric spaces. Combining Theorems C and D with the above proposition, we obtain the following.

Theorem 2. *Each class of length-commensurable arithmetically defined locally symmetric spaces of $\mathcal{G} = G(\mathbb{R})$ is a union of finitely many commensurability classes. It in fact consists of a single commensurability class if G is not of type A_n $(n > 1)$, D_{2n+1} $(n \geqslant 1)$, or E_6.*

Next, Theorems E and F imply

Theorem 3. *Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are of finite volume, and at least one of them is arithmetically defined. If they are length-commensurable then both are arithmetically defined and compactness of one of them implies the compactness of the other.*

We now recall that isospectral compact locally symmetric spaces have same weak length spectrum ([21], Theorem 10.1). Combining this fact with Theorems 2 and 3, we obtain the following results, which apparently do not follow directly from the spectral theory.

Theorem 4. *Any two arithmetically defined compact isospectral locally symmetric spaces of an absolutely simple real Lie group of type other than A_n $(n > 1)$, D_{2n+1} $(n \geqslant 1)$, or E_6, are commensurable to each other.*

Theorem 5. *If two compact locally symmetric spaces of an absolutely simple Lie group are isospectral, and one of them is arithmetically defined, then the other is also arithmetically defined.*

Finally, assuming Schanuel's conjecture we can obtain the following (unpublished) strengthening of the result of [25] for hyperbolic 2-manifolds.

Theorem 6. *Let M_1 and M_2 be arithmetically defined compact hyperbolic 2-manifolds which are not commensurable. Let $\mathcal{L}_i$ denote the subfield of $\mathbb{R}$ generated (over $\mathbb{Q}$) by $L(M_i)$. Then $\mathcal{L} := \mathcal{L}_1\mathcal{L}_2$ has infinite transcendence degree over either $\mathcal{L}_1$ or $\mathcal{L}_2$.*

It would be interesting to show that a similar statement holds for arbitrary locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ assuming that they are not length-commensurable.

4. Proofs: p-adic techniques

Given two arithmetic subgroups, or, more generally, two Zariski-dense subgroups, the proofs of Theorems A–F ultimately rely on the possibility of constructing semi-simple elements in one subgroup whose spectra are quite different from the spectra of *all* semi-simple in the other subgroup unless certain strong conditions relating these subgroups hold. These results fit into a broader project of constructing elements with special properties in a given Zariski-dense subgroup dealt with in our papers [17], [19]-[20]. The starting point of this project was the following question asked independently by G.A. Margulis and R. Spatzier: *Let Γ be a Zariski-dense arithmetic subgroup of a simple algebraic group G. Does there exist a regular semisimple $\gamma \in \Gamma$ such that $\langle\gamma\rangle$ is Zariski-dense in $T := Z_G(\gamma)^\circ$?* It should be pointed out that the existence of such an element is by no means obvious. For example, if $\varepsilon \in \mathbb{C}^\times$ is any element of infinite order, then the subgroup $\langle\varepsilon\rangle \times \langle\varepsilon\rangle \subset \mathbb{C}^\times \times \mathbb{C}^\times$ is Zariski-dense, but it contains no Zariski-dense cyclic subgroup. Elaborating on this observation, one can construct a $\mathbb{Q}$-torus T such that $T(\mathbb{Z})$ is Zariski-dense in T, but no element of $T(\mathbb{Z})$ generates a Zariski-dense subgroup of T. Something similar may also happen in the semi-simple situation. Namely, let G be a simple $\mathbb{Q}$ group with $\mathrm{rk}_{\mathbb{R}}\, G = 1$. Then if a $\mathbb{Q}$-subtorus T of G has a nontrivial decomposition into an almost direct product $T = T_1 \cdot T_2$ over $\mathbb{Q}$ (and such a decomposition exists if T has a nontrivial $\mathbb{Q}$-subtorus), no element of $T(\mathbb{Z})$ generates a Zariski-dense subgroup of T. The latter example shows that the fact that a given torus contains a proper subtorus is an obstruction to the existence of an element with the desired property. So, in [17] we singled out tori which were called "irreducible", and used them to provide an affirmative answer to the question of Margulis and Spatzier.

Definition. A K-torus T is *K-irreducible* if it does not contain any proper K-subtori.

The point is that if T is K-irreducible then *any* $t \in T(K)$ of infinite order generates a Zariski-dense subgroup. So, given a simple group G defined over a number field K, to find a required element γ in a given S-arithmetic subgroup (assuming that the latter is Zariski-dense), it is enough to construct an irreducible maximal K-torus T of G such that the group $T(\mathcal{O}_K(S))$ is infinite.

We will now outline a general procedure for constructing irreducible tori. Let T be a K-torus, $\mathcal{G}_T = \mathrm{Gal}(K_T/K)$, where K_T is the splitting field of T. Then T is K-irreducible if and only if $\mathcal{G}_T$ acts irreducibly on $X(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. Now, if T is a maximal K-torus of G, then $\mathcal{G}_T$ acts faithfully on the root system $\Phi(G,T)$, which allows us to identify $\mathcal{G}_T$ with a subgroup of $\mathrm{Aut}(\Phi(G,T))$. If under this identification $\mathcal{G}_T$ contains the Weyl group $W(G,T)$, then T is K-irreducible. Therefore, it would suffice to find a way to construct maximal K-tori T of G such that $\mathcal{G}_T \supset W(G,T)$. For $G = \mathrm{SL}_n$, one simply needs to find a polynomial $f(t) \in K[t]$ of degree n with Galois group S_n; the existence of such polynomials is well-known. Apparently, similar constructions can be implemented to obtain irreducible K-tori in any other group of classical type, but an additional difficulty one has to deal with is that it needs to be shown that the torus one constructs admits a K-embedding into the group. For the groups of exceptional types, an explicit construction appears to be difficult. Our proof of the existence of required tori in [17] was based on so-called “generic tori”.

It was shown by V.E. Voskresenskii [29] that G has a maximal torus $\mathcal{T}$ defined over a purely transcendental extension $\mathcal{K} = K(x_1, \ldots, x_n)$ such that $\mathcal{G}_{\mathcal{T}} \supset W(G,\mathcal{T})$. Then, using Hilbert's Irreducibility Theorem, one can specialize parameters to get (plenty of) maximal K-tori T of G such that $\mathcal{G}_T \supset W(G,T)$. In fact, we can construct such tori with prescribed behavior at finitely many places of K, using which it is easy to ensure that for the resulting torus T the group $T(\mathcal{O}_K(S))$ is infinite, and then any element $\gamma \in T(\mathcal{O}_K(S))$ of infinite order has the desired property.

Some time later, G.A. Margulis and G.A. Soifer asked us a different version of the original question which arose in their joint work with H. Abels on the Auslander problem: *Let G be a simple real algebraic group, Γ be a finitely generated Zariski-dense subgroup of $G(\mathbb{R})$. Is there a regular semi-simple element γ in Γ which generates a Zariski-dense subgroup of $T = Z_G(\gamma)^\circ$ and which is also $\mathbb{R}$-regular?* We recall that $\gamma \in G(\mathbb{R})$ is $\mathbb{R}$-*regular* if the number of eigenvalues, counted with multiplicity, of modulus 1 of $\mathrm{Ad}\gamma$ is minimal possible (cf. [15]). It should be noted that even the existence of an $\mathbb{R}$-regular element without any additional requirement in an arbitrary Zariski-dense subgroup Γ is a nontrivial matter: this was established by Benoist and Labourie [4] using the multiplicative ergodic theorem, and then by Prasad [14] by a direct argument; we will not, however, discuss this aspect here. The real problem is that the above argument for the existence of a regular semisimle element in Γ which generates a Zariski-dense subgroup of its centralizer does not extend to the case where Γ is not arithmetic. More precisely, since Γ is finitely generated, we can choose a finitely generated subfield K of $\mathbb{R}$ such that G is defined over K and $\Gamma \subset G(K)$. Then we can construct a maximal K-torus T of G which is irreducible over K. However, it is not clear at all why $T(K)$ should contain an element of Γ of infinite order if the latter is not of "arithmetic type". Nevertheless, the answer to the question of Margulis and Soifer turned out to be in the affirmative.

Theorem 7 ([18]). *Let G be a connected semi-simple real algebraic group. Then any Zariski-dense subsemigroup Γ of $G(\mathbb{R})$ contains a regular $\mathbb{R}$-regular element γ such that the cyclic subgroup generated by it is a Zariski-dense subgroup of the maximal torus $T = C_G(\gamma)^{\circ}$.*

The proof of the theorem, which we will now sketch, used a rather interesting technique, viz. that of p-adic embeddings. We begin by recalling the following proposition.

Proposition ([19]). *Let $\mathcal{K}$ be a finitely generated field of characteristic zero, $\mathcal{R} \subset \mathcal{K}$ be a finitely generated ring. Then there exists an infinite set of primes Π such that for each $p \in \Pi$, there exists an embedding $\varepsilon_p \colon \mathcal{K} \hookrightarrow \mathbb{Q}_p$ with the property $\varepsilon_p(\mathcal{R}) \subset \mathbb{Z}_p$.*

We will only show that Γ contains an "irreducible" element γ, i.e., a regular semi-simple element whose centralizer T is a K-irreducible maximal torus of G. For this, we fix a matrix realization $G \hookrightarrow \mathrm{SL}_n$ and pick a finitely generated subring $\mathcal{R}$ of K so that $\Gamma \subset G(\mathcal{R}) := G(K) \cap \mathrm{SL}_n(\mathcal{R})$. We then choose a finitely generated field extension $\mathcal{K}$ of K over which G splits, and fix a $\mathcal{K}$-split maximal torus T_0 of G. We now let $C_1, \ldots, C_r$ denote the nontrivial conjugacy classes in the Weyl group $W(G, T_0)$. Using the above proposition, we pick r primes $p_1, \ldots, p_r$ such that for each p_i there is an embedding $\mathcal{K} \hookrightarrow \mathbb{Q}_{p_i}$ for which $\mathcal{R} \hookrightarrow \mathbb{Z}_{p_i}$. We then employ results on Galois cohomology of semi-simple groups over local field to construct, for each $i = 1, \ldots, r$, an open set $\Omega_{p_i}(C_i) \subset G(\mathbb{Q}_{p_i})$ such that any $\omega \in \Omega_{p_i}(C_i)$ is regular semi-simple and for $T_\omega = Z_G(\omega)^{\circ}$, the Galois group $\mathcal{G}_{T_\omega}$ contains an element from the image of C_i under the natural identification $[W(G,T_0)] \simeq [W(G,T_\omega)]$, where for a maximal torus T of G, $[W(G,T)]$ is the set of conjugacy classes in the Weyl group $W(G,T)$. To conclude the argument, we show that

$$\bigcap_{i=1}^{r} (\Gamma \cap \Omega_{p_i}(C_i)) \neq \emptyset,$$

and any element γ of this intersection has the property that for $T = Z_G(\gamma)^\circ$, the inclusion $\mathcal{G}_T \supset W(G,T)$ holds, as required.

Theorem 7 was already used in [1]. Furthermore, its suitable generalizations were instrumental in settling a number of questions about Zariski-dense subgroups of Lie groups posed by Y. Benoist, T.J. Hitchman and R. Spatzier (cf. [20]). As we already mentioned, the elements constructed in Theorem 7 play a crucial role in the proof of Theorems A–F.

We conclude this article with a brief survey of other applications of p-adic embeddings. To our knowledge, Platonov [12] was the first to use p-adic embeddings in the context of algebraic groups. He proved the following.

Theorem 8 ([12]). *If $\pi\colon \widetilde{G} \to G$ is a nontrivial isogeny of connected semi-simple groups over a finitely generated field K of characteristic zero then $\pi(\widetilde{G}(K)) \neq G(K)$.*

It is enough to show that if $\pi\colon \widetilde{T} \to T$ is a nontrivial isogeny of K-tori then $\pi(\widetilde{T}(K)) \neq T(K)$. For this, we pick a finitely generated extension $\mathcal{K}$ of K so that $\widetilde{T}$ and T split over $\mathcal{K}$, and every element of $\mathrm{Ker}\, \pi$ is $\mathcal{K}$-rational. Then, using the above proposition, one finds an embedding $\mathcal{K} \hookrightarrow \mathbb{Q}_p$ for some p. To conclude the argument, one shows that $\pi(\widetilde{T}(\mathcal{K})) = T(\mathcal{K})$ would imply $\pi(\widetilde{T}(\mathbb{Q}_p)) = T(\mathbb{Q}_p)$, which is obviously false.

Another application is representation-theoretic rigidity of groups with bounded generation (cf. [24], and [13], Appendix A.2). We recall that an abstract group Γ has *bounded generation* if there are elements $\gamma_1, \ldots, \gamma_d \in \Gamma$ such that

$$\Gamma = \langle \gamma_1 \rangle \cdots \langle \gamma_d \rangle,$$

where $\langle \gamma_i \rangle$ is the cyclic subgroup generated by γ_i.

Theorem 9 ([24]). *Let Γ be a group with bounded generation satisfying the following condition*

$(*)$ $\Gamma'/[\Gamma', \Gamma']$ *is finite for every subgroup Γ' of Γ of finite index.*

Then for any $n \geqslant 1$, there are only finitely many inequivalent completely reducible representations $\rho\colon \Gamma \longrightarrow \mathrm{GL}_n(\mathbb{C})$.

The proof is based on the following strengthening of the above proposition: given $\mathcal{K}$ and $\mathcal{R}$ as above, there exists an infinite set of primes Π such that for each $p \in \Pi$ there are embeddings $\varepsilon_p^{(i)}\colon \mathcal{K} \to \mathbb{Q}_p$, where $i = 1, 2, \ldots,$ such that $\varepsilon_p^{(i)}(\mathcal{R}) \subset \mathbb{Z}_p$ for all i, and $\varepsilon_p^{(i)}(\mathcal{R}) \cap \varepsilon_p^{(j)}(\mathcal{R})$ consists of algebraic numbers for all $i \neq j$. The usual argument using representation varieties show that it is enough to show that for any $\rho\colon \Gamma \longrightarrow \mathrm{GL}_n(\mathbb{C})$, the traces Tr $\rho(\gamma)$ are algebraic numbers, for all $\gamma \in \Gamma$. For this we pick a finitely generated subring $\mathcal{R}$ of $\mathbb{C}$ for which $\rho(\Gamma) \subset \mathrm{GL}_n(\mathcal{R})$, and then fix a prime p for which there are embeddings $\varepsilon_p^{(i)}\colon \mathcal{R} \to \mathbb{Z}_p$ as above. Let $\rho^{(i)}\colon \Gamma \longrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ be the representation obtained by composing ρ with the embedding $\mathrm{GL}_n(\mathcal{R}) \to \mathrm{GL}_n(\mathbb{Z}_p)$ induced by $\varepsilon_p^{(i)}$. One then observes that bounded generation of Γ implies that for any subgroup Γ' of Γ of finite index, the pro-p completion Γ'_p of Γ' is a p-adic analytic group. Moreover, $(*)$ implies that the corresponding Lie algebra is a semi-direct product of a semi-simple algebra and a nilpotent one where the former acts on the latter without fixed point. Using the fact that a semi-simple algebra has only finitely many inequivalent representations in any dimension, one derives that there are $i \neq j$ such that Tr $\rho^{(i)}(\gamma) =$ Tr $\rho^{(j)}(\gamma)$ for all γ in a suitable subgroup Γ' of Γ of finite index. Then it follows from our construction that the traces Tr $\rho(\gamma)$ are algebraic for $\gamma \in \Gamma'$, and consequently all traces Tr $\rho(\gamma)$ for $\gamma \in \Gamma$ are algebraic.

Finally, we would like to mention the following theorem which provides a far-reaching generalization of the results of [2] and [11].

Theorem 10 ([18]). *Let G be a connected reductive group over an infinite field K. Then no noncentral subnormal subgroup of $G(K)$ can be contained in a finitely generated subgroup of $G(K)$.*

(In fact, a similar result is available in the situation where $G(K)$ is replaced with the group of points over a semi-local subring of K.) To avoid technicalities, let us assume that G is absolutely simple, and let N be a noncentral normal (rather than subnormal) subgroup of $G(K)$. Assume that N is contained in a finitely generated subgroup of $G(K)$. Then, after fixing a matrix realization $G \subset \mathrm{SL}_n$, one can pick a finitely generated subring $\mathcal{R}$ of K so that $N \subset G(\mathcal{R}) := G(K) \cap \mathrm{SL}_n(\mathcal{R})$. Let $\mathcal{K}$ be a finitely generated field that contains $\mathcal{R}$, and such that G is defined and split over $\mathcal{K}$. Now, choose an embedding $\varepsilon_p\colon \mathcal{K} \hookrightarrow \mathbb{Q}_p$ so that $\varepsilon_p(\mathcal{R}) \subset \mathbb{Z}_p$, and consider the closures $\Delta = \overline{N}$ and $\mathcal{G} = \overline{G(K)}$. Then

$\Delta \subset G(\mathbb{Z}_p)$, hence it is compact, and at the same time it is normal in $\mathcal{G}$. On the other hand, $\mathcal{G}$ is essentially $G(\mathbb{Q}_p)$. However, $G(\mathbb{Q}_p)$ does not have any noncentral compact normal subgroups (in fact, the subgroup $G(\mathbb{Q}_p)^+$ of $G(\mathbb{Q}_p)$ is a normal subgroup of finite index which does not contain *any* noncentral normal subgroups). A contradiction.

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