Weakly-split spherical Tits systems in pseudo-reductive groups

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1. Introduction

1.1. In this paper, k will always be an infinite field, $\overline{k}$ a fixed algebraic closure of k and k_s the separable closure of k in $\overline{k}$. For some other notation see 2.1 below.

It is known that all maximal k-split tori in a smooth affine algebraic k-group $\mathbf{G}$ are $\mathbf{G}(k)$-conjugate ([6], Theorem C.2.3). The *k-rank* of such a $\mathbf{G}$ is by definition the dimension of a maximal k-split torus. We will say that a smooth affine algebraic k-group is *k-isotropic* if its k-rank is > 0. A smooth connected affine algebraic k-group will be called *k-pseudo-simple* (in the terminology of [6], *"pseudo-simple over k"*) if it is noncommutative and has no nontrivial smooth connected proper normal k-subgroups. A smooth affine algebraic k-group is said to be *pseudo-reductive* if it is connected and its k-unipotent radical, i.e., the maximal smooth connected unipotent normal k-subgroup, is trivial. Connected reductive k-groups and k-pseudo-simple groups are obviously pseudo-reductive. Over a perfect field, every pseudo-reductive group is reductive. For a comprehensive theory of pseudo-reductive groups, see [6].

In the rest of the paper, $\mathbf{G}$ will be a pseudo-reductive k-group. We will denote the derived subgroup $(\mathbf{G}, \mathbf{G})$ of $\mathbf{G}$ by $\mathscr{D}(\mathbf{G})$. It is known that $\mathscr{D}(\mathbf{G})$ is perfect ([6], Proposition 1.2.6). Any k-pseudo-simple subgroup of $\mathbf{G}$, being perfect, is contained in $\mathscr{D}(\mathbf{G})$, and the latter is a product of all the k-pseudo-simple normal subgroups of $\mathbf{G}$ ([6], 3.1.8). There is a natural bijective correspondence between the set of k-isotropic k-pseudo-simple normal subgroups of $\mathbf{G}$ and the set of irreducible components of the k-root system of $\mathbf{G}$ with respect to a maximal k-split torus; see Proposition C.2.22 in [7].

1.2. The standard Tits systems in $G := \mathbf{G}(k)$. Let $\mathbf{S}$ be a maximal k-split torus in $\mathbf{G}$ and $\mathbf{P}$ a minimal pseudo-parabolic k-subgroup of $\mathbf{G}$ containing $\mathbf{S}$ (for definition of pseudo-parabolic k-subgroups, see [6], 2.2; in a reductive group the pseudo-parabolic subgroups are the parabolic subgroups, see Proposition 2.2.9 of [6]). Then $\mathbf{P}$ contains the centralizer $Z_{\mathbf{G}}(\mathbf{S})$ of $\mathbf{S}$ in $\mathbf{G}$ ([6], Proposition C.2.4). Let $N_{\mathbf{G}}(\mathbf{S})$ be the normalizer of $\mathbf{S}$ in $\mathbf{G}$. Then the finite group $W := N_{\mathbf{G}}(\mathbf{S})(k)/Z_{\mathbf{G}}(\mathbf{S})(k)$ is called the *k-Weyl group* of $\mathbf{G}$. Let $B := \mathbf{P}(k)$ and $N := N_{\mathbf{G}}(\mathbf{S})(k)$. It has been shown in [7], Theorem C.2.15, that (B, N) is a Tits system in $G = \mathbf{G}(k)$ with Weyl group W. We call this Tits system a *standard Tits system in G*. Conjugacy of maximal k-split tori and of minimal pseudo-parabolic k-subgroups (see Theorems C.2.3 and C.2.5 of [6]) implies that any two standard Tits systems in G are conjugate to each other under an element of G.

It turns out, rather surprisingly, that any Tits system in G satisfying some natural conditions is a standard Tits system. For a precise statement see Theorem B below.

1.3. Let (B, N) be an arbitrary Tits system in G. Let $H = B \cap N$. Then the *Weyl group* of this Tits system is N/H, we will denote it by W^T in the sequel. Let S be the distinguished set of involutive generators of W^T. For $s \in S$, let $G_s = B \cup BsB$. Then G_s is a subgroup of G for every $s \in S$.

We will say that the Tits system (B, N) is *weakly-split* if there exists a nilpotent normal subgroup U of B such that $B = HU$. We will say that the Tits system is *split* if it is *saturated*, that is, $H = \bigcap_{n \in N} nBn^{-1}$, and there exists a nilpotent normal subgroup U of B such that $B = H \ltimes U$. The standard Tits system in G with $B = \mathbf{P}(k)$ and $N = N_{\mathbf{G}}(\mathbf{S})(k)$ is split with $H = B \cap N = Z_{\mathbf{G}}(\mathbf{S})(k)$ and $U = \mathscr{R}_{us,k}(\mathbf{P})(k)$ (see Remark C.2.17 in [7]).

In this paper, we will only work with *spherical* Tits systems in G (i.e., Tits systems with finite Weyl group). The Tits systems being considered here will often be either weakly-split or split, and U will be a nilpotent normal subgroup of B as in 1.3.

The purpose of this paper is to prove the following two theorems.

Theorem A. *Assume that $\mathscr{D}(\mathbf{G})$ is anisotropic over k (i.e., it does not contain a nontrivial k-split torus) and the Tits system (B, N) is weakly-split. Then B is of finite index in G. If either $\mathbf{G}$ is perfect or reductive, or if G is dense in $\mathbf{G}$ in the Zariski-topology and the Tits system (B, N) is split, then $B = G$.*

Theorem B. *Assume that the Tits system (B, N) is weakly-split, and for every $s \in S$, the index of B in G_s is infinite (or, equivalently, the index of $B \cap sBs^{-1}$ in B is infinite). Then*

(1) There exists a pseudo-parabolic k-subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $B = \mathbf{P}(k)$.

(2) If the Tits system (B, N) is saturated and B does not contain the group of k-rational points of any k-isotropic k-pseudo-simple normal subgroup of $\mathbf{G}$, then (B, N) is a standard Tits system, namely it is one of the Tits systems given in 1.2 above. In particular, the Tits system is split.

The above theorems hold in particular for $\mathbf{G}$ connected reductive, and appear to be new and interesting even for these groups. Since there are spherical Moufang buildings which arise from nonreductive pseudo-reductive groups (see, for example, [11], 10.3.2, or [12], 41.20; note that the groups appearing in 10.3.2 of [11] are the group of k-rational points of "exotic" pseudo-reductive groups described in [6], Ch. 7), we have chosen to work in the more general set-up of pseudo-reductive groups.

We will prove Theorem A in §3 and prove Theorem B, which is the main result of this paper, in §4 for Tits systems of rank 1 and in §5 for Tits systems of arbitrary rank.

Special cases of Theorem A were proven earlier in [5] and [1]. Caprace and Marquis proved in [5] that if $\mathbf{G}$ is semi-simple and k-anisotropic, then B is of finite index in G in case k is either perfect or it is a nondiscrete locally compact field. That paper inspired us to write this paper. In [1], Abramenko and Zaremsky have proved Theorem A, by an entirely different argument, for some classes of semi-simple groups.

Remark 1. Let $\mathbf{G}'$ be a k-isotropic k-pseudo-simple group and $G' = \mathbf{G}'(k)$. Let (B, N) be a weakly-split Tits system in $G = \mathbf{G}(k)$. Then $(B \times G', N \times G')$ is a weakly-split Tits system in $G \times G'$. This shows that the condition imposed on B in Theorem B(2) is necessary.

We owe the following remark to Richard Weiss.

Remark 2. Let (B, N) be a weakly-split spherical Tits system in an abstract group G, and let Ω be the building associated to this Tits system. We assume that (i) B acts faithfully on Ω, (ii) the Weyl group of the Tits system is irreducible, and (iii) the rank of the Tits system is at least 2. Let $B = HU$, with $H = B \cap N$ and U a nilpotent normal subgroup of B. Then the building Ω is Moufang and U is precisely the subgroup generated by the "root groups", of the group $\mathrm{Aut}(\Omega)$ of type-preserving automorphisms of Ω, contained in B. This has been proved by De Medts, Haot, Tent and Van Maldeghem, see the corollary in §2 of [9].

In the early 1970s, Tits proved that every irreducible thick spherical building of rank $\geqslant 3$ is Moufang; see [11], Addenda. Given a semi-simple k-group $\mathbf{G}$, in [11], 11.14, there is an example of a spherical Tits system (B, N) in a free group F with infinitely many generators such that B does not contain any nontrivial normal subgroups of F and the building associated to this Tits system is the building of the standard Tits system in $\mathbf{G}(k)$. As subgroups of F are free, it is obvious that no Tits system in F can be weakly-split. Thus spherical Moufang buildings can arise from non-weakly-split Tits systems.

2. Preliminaries

2.1. As in §1, $\mathbf{G}$ is a pseudo-reductive k-group. For a k-subgroup $\mathbf{H}$ of $\mathbf{G}$, $\mathbf{H}_{\overline{k}}$ will denote the $\overline{k}$-group obtained from $\mathbf{H}$ by extension of scalars $k \hookrightarrow \overline{k}$, and $\mathscr{R}_u(\mathbf{H}_{\overline{k}})$ will denote the unipotent radical of $\mathbf{H}_{\overline{k}}$, i.e., the maximal smooth connected unipotent normal $\overline{k}$-subgroup of $\mathbf{H}_{\overline{k}}$. If $\mathbf{H}$ is a smooth connected affine k-group, $\mathscr{R}_{u,k}(\mathbf{H})$ and $\mathscr{R}_{us,k}(\mathbf{H})$ will denote the k-unipotent radical (i.e., the maximal smooth connected unipotent normal k-subgroup) and the k-split unipotent radical (i.e., the maximal k-split smooth connected unipotent normal k-subgroup) of $\mathbf{H}$ respectively. Let $\mathbf{G}' = \mathbf{G}_{\overline{k}}/\mathscr{R}_u(\mathbf{G}_{\overline{k}})$ be the maximal reductive quotient of $\mathbf{G}_{\overline{k}}$ and $\pi : \mathbf{G}_{\overline{k}} \to \mathbf{G}'$ be the quotient map. For a k-subgroup $\mathbf{H}$ of $\mathbf{G}$, we shall denote the image $\pi(\mathbf{H}_{\overline{k}})$ $(\subseteq \mathbf{G}')$ by $\mathbf{H}'$.

2.2. A smooth connected unipotent k-group $\mathbf{U}$ is said to be k-*wound* if every map of k-schemes $\mathbf{A}^1_k \to \mathbf{U}$ is a constant map to a point in $\mathbf{U}(k)$. It follows from Theorem B.3.4 of [6] that $\mathbf{U}$ is k-wound if and only if it does not contain any nontrivial smooth connected k-split subgroups. It is known that action by a k-torus on a k-wound $\mathbf{U}$ must be trivial (see [6], B.4.4; for a simpler proof of this result see B.4.4 of [7]). Therefore, if $\mathbf{U}$ is a smooth connected normal k-wound unipotent k-subgroup of a k-group $\mathbf{B}$, then every k-torus of $\mathbf{B}$ commutes with $\mathbf{U}$.

The following theorem will be crucial in this paper.

Theorem 1. *Let* $\mathbf{B}$ *be a smooth connected* k*-subgroup of a pseudo-reductive group* $\mathbf{G}$ *such that some* $\{\mathbf{B}, \mathbf{B}\}$*-double coset in* $\mathbf{G}$ *is open. Let* $\mathbf{U}$ *be a nontrivial smooth connected nilpotent normal* k*-subgroup of* $\mathbf{B}$*. Then* *(i) The unique maximal k-torus of* $\mathbf{U}$ *is contained in the center of* $\mathbf{G}$*.*

(ii) If $\mathbf{U}$ *is contained in* $\mathscr{D}(\mathbf{G})$*, then it is unipotent.*

(iii) If $\mathbf{U}$ *is not central in* $\mathbf{G}$*, then the commutator subgroup* $(\mathbf{B}, \mathbf{U})$ *contains a nontrivial smooth connected k-split unipotent k-subgroup. In particular, then* $\mathscr{R}_{us,k}(\mathbf{B})$ *is nontrivial.*

Proof. Let $\mathbf{T}$ be the unique maximal k-torus of the smooth connected nilpotent k-subgroup $\mathbf{U}$. Then $\mathbf{T}$ is normal, and hence central, in the connected group $\mathbf{B}$. This implies that $\mathbf{B}$ is contained in the centralizer $\mathbf{M} := Z_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$. But then $\mathbf{B}' \subseteq \mathbf{M}' = Z_{\mathbf{G}'}(\mathbf{T}')$. Note that $\mathbf{M}'$, being the centralizer of a torus in the connected reductive group $\mathbf{G}'$, is a connected reductive subgroup. As some $\{\mathbf{B}', \mathbf{B}'\}$-double coset in $\mathbf{G}'$ is open, some $\{\mathbf{M}', \mathbf{M}'\}$-double coset in the reductive group $\mathbf{G}'$ is open. By Proposition 1.6 of [4] then $\mathbf{M}' = \mathbf{G}'$, i.e., $\mathbf{T}'$ is central in $\mathbf{G}'$. Then by Lemma 1.2.5(i) of [6], $\mathbf{T}$ is contained in the center of $\mathbf{G}$. This proves the first assertion of the theorem.

If $\mathbf{U} \subset \mathscr{D}(\mathbf{G})$, then $\mathbf{T}' \subset \mathscr{D}(\mathbf{G}')$. But $\mathscr{D}(\mathbf{G}')$ is a semi-simple group, so it does not contain any nontrivial central tori. This implies that $\mathbf{T}' = \pi(\mathbf{T}_{\overline{k}})$ is trivial. As the kernel of π is a unipotent group, we conclude that $\mathbf{T}$ is trivial and hence $\mathbf{U}$ is unipotent. This proves the second assertion.

Now we will prove the last assertion. We assume that $\mathbf{U}$ is not central in $\mathbf{G}$. Then it follows from Lemma 1.2.5(i) of [6] that $\mathbf{U}'$ is not central in $\mathbf{G}'$ (hence, in particular, $\mathbf{U}'$ is nontrivial). Since $\mathbf{T}'$ is central in $\mathbf{G}'$, but $\mathbf{U}'$ is not, we conclude that $\mathbf{T}' \neq \mathbf{U}'$, and hence the unipotent radical $\mathscr{R}_u(\mathbf{U}')$ of $\mathbf{U}'$ is nontrivial. If $\mathbf{U}$ is central in $\mathbf{B}$, then $\mathbf{U}'$, and so also $\mathscr{R}_u(\mathbf{U}')$, is central in $\mathbf{B}'$. We assert that $\mathbf{B}'$ cannot contain a nontrivial smooth unipotent central subgroup. For otherwise, $\mathbf{B}'(\overline{k})$ will contain a nontrivial unipotent element z which is central (in $\mathbf{B}'$). Then the centralizer $\mathbf{C}'_z$ of z in $\mathbf{G}'$ contains $\mathbf{B}'$. As some $\{\mathbf{B}', \mathbf{B}'\}$-double coset in $\mathbf{G}'$ is open, some $\{\mathbf{C}'_z, \mathbf{C}'_z\}$-double coset is open too. But it has been shown by Martin Liebeck that for no nontrivial unipotent element $z \in \mathbf{G}'(\overline{k})$, a $\{\mathbf{C}'_z, \mathbf{C}'_z\}$-double coset can be open in $\mathbf{G}'$ ([8], Ch. 1, Cor. 8). This proves our assertion. So we conclude that $\mathbf{U}$ cannot be central in $\mathbf{B}$. Then the commutator subgroup $(\mathbf{B}, \mathbf{U})$ is a nontrivial smooth connected nilpotent normal k-subgroup of $\mathbf{U}$. As $(\mathbf{B}, \mathbf{U})$ is contained in $\mathscr{D}(\mathbf{G})$, it is unipotent by (ii). We will prove that $(\mathbf{B}, \mathbf{U})$ contains a nontrivial k-split smooth connected k-subgroup $\mathbf{V}$. Then the conjugates $b\mathbf{V}b^{-1}$, $b \in \mathbf{B}(k_s)$, generate a smooth connected k-subgroup which, being contained in $\mathbf{U}$, is unipotent; it is clearly a normal subgroup of $\mathbf{B}$ and Theorem B.3.4 of [6] implies that it is k-split, hence $\mathscr{R}_{us,k}(\mathbf{B})$ is nontrivial.

As we will not need to work with the original $\mathbf{U}$ anymore, for simplicity we will denote the smooth connected unipotent normal k-subgroup $(\mathbf{B}, \mathbf{U})$ of $\mathbf{B}$ by $\mathbf{U}$ in the rest of this proof. Let us assume, if possible, that $\mathbf{U}$ does not contain any nontrivial smooth connected k-split subgroup. Then $\mathbf{U}$ is k-wound, and hence every k-torus of $\mathbf{B}$ commutes with it. This implies that the subgroup $\mathbf{B}_t$ of $\mathbf{B}$ generated by k-tori commutes with $\mathbf{U}$. But then every $\overline{k}$-torus of $\mathbf{B}_{\overline{k}}$ commutes with $\mathbf{U}_{\overline{k}}$ since according to Proposition A.2.11 of [6], $(\mathbf{B}_t)_{\overline{k}}$ is the subgroup of $\mathbf{B}_{\overline{k}}$ generated by all the $\overline{k}$-tori in $\mathbf{B}_{\overline{k}}$.

We will now show that $\mathbf{B}'$ contains a nontrivial smooth connected unipotent subgroup in its center. Let us fix a Borel subgroup $\mathbf{S}' \ltimes \mathbf{V}'$ of $\mathbf{B}'$, with $\mathbf{S}'$ a maximal $\overline{k}$-torus and $\mathbf{V}'$ the unipotent radical of the Borel subgroup. Note that $\mathbf{U}'$ is a nontrivial connected normal subgroup of $\mathbf{V}'$. Now we inductively define, for each positive integer i, the normal subgroups $\mathbf{V}'_i$ of $\mathbf{V}'$ contained in $\mathbf{U}'$ as follows. Let $\mathbf{V}'_1 = \mathbf{U}'$, and having defined $\mathbf{V}'_i$, let $\mathbf{V}'_{i+1}$ be the commutator subgroup $(\mathbf{V}', \mathbf{V}'_i)$. Let n be the largest integer such that $\mathbf{V}'_n$ is nontrivial. Then $\mathbf{V}'_n$ is a nontrivial smooth connected subgroup of $\mathbf{U}'$ which commutes with $\mathbf{V}'$. Since every $\overline{k}$-torus of $\mathbf{B}_{\overline{k}}$ commutes with $\mathbf{U}_{\overline{k}}$, the $\overline{k}$-torus $\mathbf{S}'$ of $\mathbf{B}'$ commutes with $\mathbf{U}'$, and hence also with the subgroup $\mathbf{V}'_n$. Thus the Borel subgroup $\mathbf{S}' \ltimes \mathbf{V}'$ of $\mathbf{B}'$ commutes with the subgroup $\mathbf{V}'_n$, which implies that $\mathbf{V}'_n$ is central in $\mathbf{B}'$ (by Corollary 11.4 of [2]). But we saw above that $\mathbf{B}'$ cannot contain any nontrivial smooth unipotent central subgroups. Thus we have arrived at a contradiction. Hence, $\mathbf{U}$ does contain a nontrivial smooth connected k-split unipotent k-subgroup. □

2.3. The Zariski-closure of $G = \mathbf{G}(k)$: Let $\mathbf{G}^\natural$ be the identity component of the Zariski-closure of G in $\mathbf{G}$. It is known that if either $\mathbf{G}$ is perfect or it is reductive, then G is Zariski-dense in $\mathbf{G}$ ([7], Proposition A.2.11, and [2], 18.3), so in these cases, $\mathbf{G}^\natural = \mathbf{G}$. As the derived subgroup $\mathscr{D}(\mathbf{G})$ of $\mathbf{G}$ is perfect (Proposition 1.2.6 of [6]), $\mathscr{D}(\mathbf{G})(k)$ is dense in $\mathscr{D}(\mathbf{G})$ in the Zariski-topology, and hence, $\mathbf{G}^\natural \supset \mathscr{D}(\mathbf{G})$. This implies that $\mathbf{G}^\natural$ is a smooth connected normal k-subgroup of (the pseudo-reductive k-group) $\mathbf{G}$ and therefore it is pseudo-reductive. For any k-torus $\mathbf{T}$ of $\mathbf{G}$ and any smooth connected k-split unipotent k-subgroup $\mathbf{U}$, as $\mathbf{T}(k)$ and $\mathbf{U}(k)$ are Zariski-dense in $\mathbf{T}$ and $\mathbf{U}$ respectively, we conclude that $\mathbf{G}^\natural \supset \mathbf{T}, \mathbf{U}$. Thus $\mathbf{G}/\mathbf{G}^\natural$ is a commutative unipotent group. Hence, the root systems of $\mathbf{G}$ and $\mathbf{G}^\natural$ with respect to any maximal k-split torus $\mathbf{S}$ are the same, and so the natural homomorphism

$$N_{\mathbf{G}^\natural}(\mathbf{S})(k)/Z_{\mathbf{G}^\natural}(\mathbf{S})(k) \longrightarrow N_{\mathbf{G}}(\mathbf{S})(k)/Z_{\mathbf{G}}(\mathbf{S})(k),$$

between the k-Weyl groups of $\mathbf{G}^\natural$ and $\mathbf{G}$, is an isomorphism. Moreover, the root groups ([7], C.2.16) of $\mathbf{G}$ and $\mathbf{G}^\natural$ with respect to $\mathbf{S}$ are identical.

The group $G^\natural := \mathbf{G}^\natural(k)$ is a normal subgroup of G of finite index, and so it is dense in $\mathbf{G}^\natural$ in the Zariski-topology.

2.4. As in 2.1, let $\pi : \mathbf{G}_{\overline{k}} \to \mathbf{G}' = \mathbf{G}_{\overline{k}}/\mathscr{R}_u(\mathbf{G}_{\overline{k}})$ be the quotient map. Since $\mathbf{G}/\mathbf{G}^\natural$ is a unipotent group, $\pi(\mathbf{G}^\natural_{\overline{k}}) = \mathbf{G}'$. Using this it easily follows from Lemma 1.2.5(i) of [6] that any smooth connected k-subgroup of $\mathbf{G}$ which commutes with $\mathbf{G}^\natural$ is central in $\mathbf{G}$. In particular, every smooth connected central k-subgroup of $\mathbf{G}^\natural$ is central in $\mathbf{G}$.

2.5. By definition, any pseudo-parabolic k-subgroup $\mathbf{P}$ of the pseudo-reductive group $\mathbf{G}$ equals $P_{\mathbf{G}}(\lambda)$ (in the notation of [6], 2.1 and 2.2, which we will use here and in the sequel) for a 1-parameter subgroup $\lambda : \mathrm{GL}_1 \to \mathbf{G}$, and as $\lambda(\mathrm{GL}_1) \subset \mathbf{G}^\natural$, we see that $\mathbf{P}^\natural := \mathbf{P} \cap \mathbf{G}^\natural = P_{\mathbf{G}^\natural}(\lambda)$ is a pseudo-parabolic k-subgroup of $\mathbf{G}^\natural$. Now let $\mathbf{P}$ and $\mathbf{Q}$ be two pseudo-parabolic k-subgroups of $\mathbf{G}$ and $\mathbf{P}^\natural := \mathbf{P} \cap \mathbf{G}^\natural$, and $\mathbf{Q}^\natural := \mathbf{Q} \cap \mathbf{G}^\natural$. According to Proposition C.2.7 of [6], there exists a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ such that $\mathbf{P} \cap \mathbf{Q}$ contains the centralizer $Z_{\mathbf{G}}(\mathbf{S})$ of $\mathbf{S}$. Since $\mathbf{G}^\natural \supset \mathscr{D}(\mathbf{G})$, $\mathbf{G} = Z_{\mathbf{G}}(\mathbf{S})\mathbf{G}^\natural$ (cf. Proposition 1.2.6 of [6]), which implies that $\mathbf{P} = Z_{\mathbf{G}}(\mathbf{S})\mathbf{P}^\natural$ and $\mathbf{Q} = Z_{\mathbf{G}}(\mathbf{S})\mathbf{Q}^\natural$. From this we conclude that if $\mathbf{P}^\natural = \mathbf{Q}^\natural$, then $\mathbf{P} = \mathbf{Q}$. In particular, letting $\mathbf{P} = \mathbf{G}$, we see that if $\mathbf{Q}^\natural = \mathbf{G}^\natural$, then $\mathbf{Q} = \mathbf{G}$.

2.6. Since the multiplication map

$$U_{\mathbf{G}^\natural}(-\lambda) \times Z_{\mathbf{G}^\natural}(\lambda) \times U_{\mathbf{G}^\natural}(\lambda) \longrightarrow \mathbf{G}^\natural$$

is an open immersion (Proposition 2.1.8 of [6]), the Zariski-density of $\mathbf{G}^\natural(k)$ in $\mathbf{G}^\natural$ implies that $Z_{\mathbf{G}^\natural}(\lambda)(k)$ and $\mathbf{P}^\natural(k)\,(= P_{\mathbf{G}^\natural}(\lambda)(k))$ are Zariski-dense in $Z_{\mathbf{G}^\natural}(\lambda)$ and $\mathbf{P}^\natural\,(= P_{\mathbf{G}^\natural}(\lambda) = Z_{\mathbf{G}^\natural}(\lambda) \ltimes U_{\mathbf{G}^\natural}(\lambda))$ respectively.

Since $\mathbf{P}^\natural = \mathbf{P} \cap \mathbf{G}^\natural$, we have $\mathbf{P}^\natural(k) = \mathbf{P}(k) \cap \mathbf{G}^\natural(k)$. But as $G^\natural := \mathbf{G}^\natural(k)$ is a subgroup of $G = \mathbf{G}(k)$ of finite index, we see that $\mathbf{P}^\natural(k)$ is a subgroup of finite index in $\mathbf{P}(k)$. Now the Zariski-density of $\mathbf{P}^\natural(k)$ in $\mathbf{P}^\natural$ implies that the identity component of the Zariski-closure of $\mathbf{P}(k)$ is $\mathbf{P}^\natural$.

Let $\mathbf{Q}$ be a pseudo-parabolic k-subgroup of $\mathbf{G}$ such that $\mathbf{Q}(k) = G$. Then $\mathbf{Q}^\natural := \mathbf{Q} \cap \mathbf{G}^\natural$ equals $\mathbf{G}^\natural$, and so $\mathbf{Q} = \mathbf{G}$ (2.5).

2.7. We observe here for later use that if $\mathbf{P}$ is a pseudo-parabolic k-subgroup of $\mathbf{G}$ such that $\mathbf{P}(k)$ is a maximal proper subgroup of $G = \mathbf{G}(k)$, then $\mathbf{P}$ is a maximal proper pseudo-parabolic k-subgroup of $\mathbf{G}$. For, if there is a proper pseudo-parabolic k-subgroup $\mathbf{Q}$ of $\mathbf{G}$ properly containing $\mathbf{P}$, then the pseudo-parabolic k-subgroup $\mathbf{Q}^\natural := \mathbf{Q} \cap \mathbf{G}^\natural$ of $\mathbf{G}^\natural$ would properly contain the pseudo-parabolic k-subgroup $\mathbf{P}^\natural := \mathbf{P} \cap \mathbf{G}^\natural$ of $\mathbf{G}^\natural$ (2.5). Now as the identity component of the Zariski-closure of $\mathbf{P}(k)$ (resp. $\mathbf{Q}(k)$) is $\mathbf{P}^\natural$ (resp. $\mathbf{Q}^\natural$), we conclude that $\mathbf{P}(k) \neq \mathbf{Q}(k)$. Hence, $\mathbf{Q}(k) = G$, which implies that $\mathbf{Q} = \mathbf{G}$, a contradiction.

3. Proof of Theorem A

3.1. As in §1, let $\mathbf{G}$ be a pseudo-reductive k-group, $G = \mathbf{G}(k)$. Let (B, N) be a weakly-split Tits system in G, with $B = HU$, $H = B \cap N$. Let $W^T = N/H$ be the Weyl group of this Tits system. We assume that the Tits system is spherical, i.e., W^T is finite. Let S be the distinguished set of involutive generators of W^T. For a subset X of S, we will denote by W^T_X the subgroup of W^T generated by the elements in X, and by G_X the subgroup BW^T_XB of G. It is known that B equals its normalizer in G, so in particular, it contains the center of G. (If the Tits system (B, N) is saturated, then the center of G is contained in H.) Any subgroup of G containing B equals G_X for a unique subset X of S. For these basic results on Tits systems, see [3], Ch. IV.

If U is *virtually central* in G, i.e., if a subgroup of U of finite index is central in G, then from the Bruhat decomposition $G = \bigcup_{w\in W^T} BwB = \bigcup_{w\in W^T} UwB$, we see at once that B is of finite index in G.

3.2. Notation. As in 2.3, we will denote the identity component of the Zariski-closure of G in $\mathbf{G}$ by $\mathbf{G}^\natural$. Recall that $\mathbf{G}^\natural$ is a pseudo-reductive normal subgroup of $\mathbf{G}$ since it contains $\mathscr{D}(\mathbf{G})$. The identity component of the Zariski-closure of B (resp., U) will be denoted by $\mathbf{B}$ (resp., $\mathbf{U}$); $\mathbf{U}$ is a connected nilpotent normal k-subgroup of $\mathbf{B}$. Let $\mathbf{G}_X$ denote the

Zariski-closure of G_X in $\mathbf{G}$. In particular, since $B = G_\emptyset$, its Zariski-closure is $\mathbf{G}_\emptyset$ and $\mathbf{B}$ is the identity component of $\mathbf{G}_\emptyset$. The groups $\mathbf{B}$ and $\mathbf{U}$ are contained in $\mathbf{G}^\natural$.

Let $B_0 = B \cap \mathbf{B}(k)$. Then B_0 is a normal subgroup of B of finite index. Since G is the union of finitely many double cosets BwB, $w \in W^T$, finitely many $\{B_0, B_0\}$-double cosets cover G, which implies that some $\{\mathbf{B}, \mathbf{B}\}$-double coset in $\mathbf{G}^\natural$ is dense (and hence open) in $\mathbf{G}^\natural$, so Theorem 1 is applicable with $\mathbf{G}$ replaced by $\mathbf{G}^\natural$.

We will now prove the following result about split Tits systems in G.

Proposition 1. *We assume that G is dense in $\mathbf{G}$ in the Zariski-topology and the Tits system (B, N) is split. Then $G_X = \mathbf{G}_X(k)$. In particular, since any subgroup of G of finite index is dense in $\mathbf{G}$ in the Zariski-topology, if B is of finite index in G, then $B = G$.*

Proof. We first observe that it suffices to prove the assertion for maximal proper subsets X of S. In fact, assuming the assertion for maximal proper subsets of S, if X is an arbitrary proper subset of S, then as $\mathbf{G}_X(k)$ contains G_X, it equals G_Y for some subset Y of S containing X. If $Y \neq X$, let us fix $y \in Y - X$. Then $X' = S - \{y\}$ is a maximal proper subset containing X, and hence $\mathbf{G}_{X'}(k) = G_{X'}$. But $\mathbf{G}_X(k)$ contains G_Y, and hence $\mathbf{G}_{X'}(k)$ contains the subgroup generated by G_Y and $G_{X'}$, which is all of G, a contradiction.

We will now prove the assertion for maximal proper subsets of S. Let X be a maximal proper subset of S, and assume, if possible, that $\mathbf{G}_X(k) \neq G_X$. Then $\mathbf{G}_X(k) = G = \mathbf{G}(k)$, and hence $\mathbf{G}_X = \mathbf{G}$ since G has been assumed to be Zariski-dense in $\mathbf{G}$. Let

$$V = \bigcap_{g \in G_X} gUg^{-1}.$$

Then V ($\subseteq U$) is a normal subgroup of G_X. Let $\mathbf{V}$ be the identity component of its Zariski-closure. Then $\mathbf{V}(\subseteq \mathbf{U})$ is a smooth connected nilpotent k-subgroup normalized by the Zariski-closure $\mathbf{G}_X = \mathbf{G}$ of G_X, i.e., it is a nilpotent normal subgroup of $\mathbf{G}$. We will now show that $\mathbf{V}$ is trivial. We assert that $\mathbf{V}$ is contained in the center of $\mathbf{G}$. For otherwise, according to Theorem 1(*iii*), it contains a nontrivial smooth connected k-split unipotent k-subgroup. Therefore, if $\mathbf{V}$ is not central, then $\mathscr{R}_{us,k}(\mathbf{V})$, which is a normal subgroup of $\mathbf{G}$, is nontrivial. But as $\mathbf{G}$ is a pseudo-reductive group, it cannot contain a nontrivial smooth connected normal unipotent k-subgroup. We conclude that $\mathbf{V}$ is central. Then $V_0 := U \cap \mathbf{V}(k)$, which is Zariski-dense in $\mathbf{V}$, is contained in the center of G, and hence $V_0 \subseteq H$. But $V_0 \subseteq U$, and $H \cap U$ is trivial, so V_0, and hence, $\mathbf{V}$ is trivial. Therefore, V is finite. So it is normalized by $\mathbf{G}_X = \mathbf{G}$, and hence it is contained in the center of G. We conclude from this that V acts trivially on the spherical building associated with the above Tits system. But according to Proposition 3.3 and Lemma 3.5 of [5], V is a nontrivial subgroup of G which acts faithfully on the building. Thus we have arrived at a contradiction. □

Proof of Theorem A. We begin by observing that since in Theorem A, $\mathscr{D}(\mathbf{G})$ has been assumed to be anisotropic over k, it does not contain a proper pseudo-parabolic k-subgroup, hence neither does $\mathbf{G}$ (see page 375 of [6]).

From the fact that every smooth connected central k-subgroup of $\mathbf{G}^\natural$ is central in $\mathbf{G}$ (2.4), it follows at once that U is virtually central in G if and only if $\mathbf{U}$ is central in $\mathbf{G}^\natural$. We assert that U is virtually central in G and hence B is of finite index in G (3.1). For otherwise, $\mathbf{U}$ is not central in $\mathbf{G}^\natural$ and then by Theorem 1(iii), $\mathscr{R}_{us,k}(\mathbf{B})$ is nontrivial. But since $\mathbf{G}$ does not contain a proper pseudo-parabolic k-subgroup, it follows from Theorem C.3.8 of [6] that it cannot contain any nontrivial smooth connected k-split unipotent subgroups. Therefore, $\mathbf{U}$ must be central in $\mathbf{G}^\natural$, and hence U is virtually central in G.

Now in case G is Zariski-dense in $\mathbf{G}$ and the Tits system is split, we can apply Proposition 1 to conclude that $B = G$ since B is of finite index in G.

The centrality of $\mathbf{U}$ in $\mathbf{G}^\natural$ implies that $\mathbf{U}$ is central in $\mathbf{G}$ (2.4). We will assume now that either $\mathbf{G}$ is perfect, i.e., $\mathscr{D}(\mathbf{G}) = \mathbf{G}$, or it is reductive. In both the cases, G is Zariski-dense in $\mathbf{G}$. As B is of finite index in G, B is also Zariski-dense in $\mathbf{G}$. We will prove that $B = G$. We will first show that U is central in G. If $\mathbf{G}$ is perfect, by Theorem 1(ii), $\mathbf{U}$ is unipotent, thus it is a smooth connected unipotent central k-subgroup of $\mathbf{G}$. Since $\mathbf{G}$ is pseudo-reductive, such a subgroup is necessarily trivial. This implies that $\mathbf{U}$ is trivial and hence U is finite if $\mathbf{G}$ is perfect. Now as the finite subgroup U is a normal subgroup of B, and B is Zariski-dense in $\mathbf{G}$, U is central (in G). On the other hand, if $\mathbf{G}$ is reductive, the centrality of U (in G) follows from Lemma 5.3.2 of [6] using the centrality of $\mathbf{U}$ in $\mathbf{G}$. Now as N normalizes H and U is central, $B = HU$ is normalized by N and hence by G which is generated by $B \cup N$. Therefore, B is a normal subgroup of G. But the normalizer of B equals B which implies that $B = G$. □

3.3. A Tits system of rank 1 in $G := \mathrm{SL}_1(D)$. Let F be a field. Then $\mathsf{F}^\times$ acts on F by multiplication, and we form the semi-direct product $\mathscr{G} := \mathsf{F} \rtimes \mathsf{F}^\times$. The solvable group $\mathscr{G}$ admits a split Tits system (B, N) of rank 1 with $B = \{(0, x) |\, x \in \mathsf{F}^\times\}$ and $N = \{(-1, -1), (0, 1)\}$.

Now let D be the division algebra with center the 2-adic field $\mathbf{Q}_2$ and of dimension d^2 over $\mathbf{Q}_2$. Let F be the field extension of degree d of the field with 2 elements. Let $\mathbf{G} = \mathrm{SL}_{1,D}$. Then $\mathbf{G}$ is an absolutely simple simply connected algebraic group defined and anisotropic over $\mathbf{Q}_2$, and $G := \mathbf{G}(\mathbf{Q}_2)$ is the subgroup $\mathrm{SL}_1(D)$ of $D^\times$ consisting of elements of reduced norm 1. Let G_2 be the "second congruence subgroup" of G as in [10], 1.1. Then $G/G_2 \simeq \mathscr{G} = \mathsf{F} \rtimes \mathsf{F}^\times$ ([10], §1). Hence, G admits a spherical Tits system of rank 1. Such a Tits system cannot be weakly-split (cf. Theorem A).

4. Proof of Theorem B when the Tits system is of rank 1

4.1. Since Theorem B is of considerable interest in case the Tits system is of rank 1, and the proof is simpler than in the general case, we first prove it for Tits systems of rank 1. The proof of Theorem B for Tits systems of arbitrary rank will be given in the next section; it will use results proved in this section.

We will assume in this section that the Tits system (B, N) in $G = \mathbf{G}(k)$ is weakly-split, and of rank 1 (i.e., $S = \{s\}$) with, as before, $B = HU$. We will also assume that the index of B in G is infinite. To prove Theorem B (for Tits systems of rank 1), we need to prove that (1) there exists a proper pseudo-parabolic k-subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $B = \mathbf{P}(k)$, and (2) if the Tits system (B, N) is saturated and B does not contain the group of k-rational points of any k-isotropic k-pseudo-simple normal subgroup of $\mathbf{G}$, then the Tits system is a standard Tits system, i.e., it is as in 1.2. Hence the Tits system is split (possibly, in terms of a different decomposition of B), see Remark C.2.17 in [7].

Let $\mathbf{U}$ be the identity component of the Zariski-closure of U in $\mathbf{G}$. Since the index of B in G is infinite, U is not virtually central in G, or, equivalently, $\mathbf{U}$ is not contained in the center of $\mathbf{G}^\natural$ (see 3.1 and 2.4). Therefore, by Theorem 1(iii) applied to $\mathbf{G}^\natural$ in place of $\mathbf{G}$, $\mathscr{R}_{us,k}(\mathbf{B})$ is nontrivial. By Theorem C.3.8 of [7] there is a proper pseudo-parabolic k-subgroup $\mathbf{P}$ of $\mathbf{G}$ such that (a) the k-split unipotent radical $\mathscr{R}_{us,k}(\mathbf{P})$ of $\mathbf{P}$ contains $\mathscr{R}_{us,k}(\mathbf{B})$ and (b) $B \subseteq P := \mathbf{P}(k)$. Since $\mathbf{P}$ is a proper pseudo-parabolic k-subgroup of $\mathbf{G}$, P is a proper subgroup of G (2.6). As the Tits system is of rank 1, no proper subgroup of G properly contains B. Hence, $P = B$, and $\mathbf{P}$ is a maximal proper pseudo-parabolic k-subgroup of $\mathbf{G}$ (2.7). This proves assertion (1).

The proof of assertion (2) requires Proposition 2 whose proof in turn requires the following lemma. We first fix some notation. Let Ψ be an irreducible root system given with a basis Δ. Let W be the Weyl group of Ψ and $R = \{r_a \mid a \in \Delta\}$, where for a root a, r_a is the reflection in a; R is a set of involutive generators of W. We now fix a root $a \in \Delta$. Let Ψ' be the root subsystem of Ψ spanned by $\Delta - \{a\}$ and $W'(\subset W)$ be the Weyl group of Ψ'; W' is generated by the subset $R - \{r_a\}$. The length of an element $w \in W$, in terms of the generating set R of W, will be denoted by $\ell(w)$. Let w_0 and w_0' be the longest elements of W and W' respectively. Both these elements are of order 2.

Lemma 1. *Let us assume that the root system Ψ is of rank $m > 1$. Then $\#W'\backslash W/W' > 2$ unless the root system is of type A_m and a is one of the two end roots of Δ.*

Proof. If the root system is not reduced (then it is of type BC_m), the subset $\Psi^\bullet$ consisting of nondivisible roots of Ψ is a root system of type B_m with Weyl group W, and Δ is a basis of $\Psi^\bullet$. Therefore, to prove the lemma, we may (and do) replace Ψ by $\Psi^\bullet$ and assume that Ψ is reduced (and irreducible). We will denote by Ψ^+ the positive system of roots determined by the basis Δ.

Let us first consider the case where $w_0 = -1$ (this is the case unless the root system Ψ is of type $A_m\,(m > 1)$, D_m, m odd, or E_6) and assume for the sake of contradiction that $\#W'\backslash W/W' \leqslant 2$. Then, clearly, $W = W' \cup W'r_aW'$, and we conclude that $w_0 = -1 = w_1 r_a w_2$, with w_1, $w_2 \in W'$. Then $-r_a = w_1^{-1}{w_2}^{-1}$. As a is the only positive root which is transformed into a negative root by r_a, we see that $w_1^{-1}{w_2}^{-1}\,(\in W')$ takes all the roots in ${\Psi'}^+ := \Psi' \cap \Psi^+$ into negative roots. Therefore, $(w_2 w_1)^{-1} = w_0'$. Hence, $w_2 = w_0' w_1^{-1}$, and so $\ell(w_2) = \ell(w_0') - \ell(w_1)$. But then $\ell(w_0) = \ell(w_1 r_a w_2) \leqslant \ell(w_1) + 1 + \ell(w_2) = \ell(w_0') + 1$. Since $2\ell(w_0) = \#\Psi$ and $2\ell(w_0') = \#\Psi'$, we obtain the bound $\#\Psi \leqslant \#\Psi' + 2$. But it is easily seen that this bound does not hold.

The following argument to prove the lemma in case $w_0 \neq -1$, and more generally if the root system Φ is simply laced, has kindly been provided by John Stembridge.

We will assume now that the root system Ψ is simply laced, is of rank $m > 1$, and if it is of type A_m then a is not an end root of Δ. Then either a is a node of degree 2, or there is a node of degree > 2 in the Coxeter graph of Δ. We index the elements of Δ so that there is a path $a_1 = a, a_2, \ldots, a_i$ in the graph such that $r_1r_2, r_2r_3, \ldots, r_{i-1}r_i, r_ir_{i+1}$, and r_ir_{i+2} all have order 3. Here, $r_j = r_{a_j}$ and the case $i = 1$ occurs when the node $a = a_1$ has degree 2.

Consider the element $w = r_1r_2\cdots r_i \cdot r_{i+1}r_{i+2} \cdot r_i \cdots r_2r_1 \in W$. Here, there is at most one opportunity to apply a braid relation, and it occurs only if r_{i+1} and r_{i+2} happen to commute. So from Lemma 4 and Proposition 5 of §1.5, Ch. IV in [3], we see that w has at most two reduced expressions, and both of them start and end with r_1. Thus it is the shortest element in its $\{W', W'\}$-double coset. Hence, we have at least three distinct double cosets: W', $W'r_1W'$ and $W'wW'$. □

In the next proposition, which will be used again in the next section, we assume that the Tits system (B, N) is weakly-split and of rank 1, but do *not* assume that it is saturated. Let $\mathbf{P}$ be as above.

Proposition 2. *The pseudo-parabolic k-subgroup $\mathbf{P}$ contains all k-pseudo-simple normal subgroups of $\mathbf{G}$ except one. The one not contained in $\mathbf{P}$ is of k-rank 1.*

Proof. As in 2.3 let $\mathbf{G}^\natural$ be the identity component of the Zariski-closure of $G = \mathbf{G}(k)$ in $\mathbf{G}$. It is a pseudo-reductive normal k-subgroup of $\mathbf{G}$, $\mathbf{P}^\natural = \mathbf{P} \cap \mathbf{G}^\natural$ is a pseudo-parabolic k-subgroup of $\mathbf{G}^\natural$, and $G^\natural := \mathbf{G}^\natural(k)$ is of finite index in G. Since $\mathbf{G}^\natural$ contains $\mathscr{D}(\mathbf{G})$, any k-pseudo-simple subgroup of $\mathbf{G}$ is contained in $\mathbf{G}^\natural$.

We fix a minimal pseudo-parabolic k-subgroup $\mathbf{P}_0$ contained in $\mathbf{P}$ and a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ contained in $\mathbf{P}_0$. Let Φ be the set of k-roots of $\mathbf{G}$ with respect to the maximal k-split torus $\mathbf{S}$, $\Phi^+(\subset \Phi)$ be the positive system of roots, and Δ the set of simple roots, determined by the minimal pseudo-parabolic k-subgroup $\mathbf{P}_0$. We will denote the k-Weyl group $N_{\mathbf{G}}(\mathbf{S})(k)/Z_{\mathbf{G}}(\mathbf{S})(k)$ of the pseudo-reductive algebraic k-group $\mathbf{G}$ by W. Let $R = \{r_a \mid a \in \Delta\}$, where for $a \in \Delta$, $r_a \in W$ is the reflection in a. Then R is a set of involutive generators of W. Let w_0 be the longest element of W. For a subset Z of R, we will denote by W_Z the subgroup of W generated by the elements in Z. As pointed-out in 2.3, the natural homomorphism

$$N_{\mathbf{G}^\natural}(\mathbf{S})(k)/Z_{\mathbf{G}^\natural}(\mathbf{S})(k) \longrightarrow N_{\mathbf{G}}(\mathbf{S})(k)/Z_{\mathbf{G}}(\mathbf{S})(k),$$

between the k-Weyl groups of $\mathbf{G}^\natural$ and $\mathbf{G}$, is an isomorphism. Therefore, the natural homomorphism $N_{\mathbf{G}^\natural}(\mathbf{S})(k)\,(\subset N_{\mathbf{G}}(\mathbf{S})(k)) \to W$ is surjective.

Let $P_0 = \mathbf{P}_0(k)$ and X be the subset of R such that $P = P_0W_XP_0$. Then since $\mathbf{P}$ is a maximal proper pseudo-parabolic k-subgroup of $\mathbf{G}$, $X = R - \{r_a\}$, for an $a \in \Delta$. Let Δ_a be the connected component of Δ containing a, and Ψ the irreducible component of Φ spanned by Δ_a. As the set of roots of $\mathbf{P}$ (with respect to $\mathbf{S}$) contains all the connected components of Φ except Ψ, it is clear that $\mathbf{P}$ contains all k-pseudo-simple normal subgroups of $\mathbf{G}$ except the one whose root system with respect to $\mathbf{S}$ is Ψ. Let Y be the set of reflections in the roots contained in Δ_a, and $Y' = Y - \{r_a\}$; W_Y is clearly the Weyl group of the root system Ψ. As Δ_a is orthogonal to $\Delta - \Delta_a$, $W = W_Y \times W_{R-Y}$. Since $G = B \cup BsB$ and $B = P$, G is the union of two $\{P, P\}$-double cosets, we infer using the Bruhat decomposition (Theorem C.2.8 of [6]) that W is the union of two $\{W_X, W_X\}$-double cosets, and hence W_Y is the union of two $\{W_{Y'}, W_{Y'}\}$-cosets. (Therefore, $W_Y = W_{Y'} \cup W_{Y'} r_a W_{Y'}$.)

We shall now show that Ψ is of rank 1 using the fact that the Tits system under consideration is weakly-split. Assume, if possible, that Ψ is of rank > 1. As W_Y is the union of two $\{W_{Y'}, W_{Y'}\}$-cosets, from Lemma 1 we conclude that Ψ is of type A_m, for some $m > 1$, and a is one of the two end roots of the basis Δ_a of this root system. Since $W = W_Y \times W_{R-Y}$ and $W_Y = W_{Y'} \cup W_{Y'} r_a W_{Y'}$, we obtain that $W = W_X \cup W_X r_a W_X$, and then w_0 will have to belong to $W_X r_a W_X$ which implies that $W_X r_a W_X = W_X w_0 W_X$. Hence, $G = P \cup P w_0 P$. As the Weyl group of the Tits system $(B = P, N)$ is $\{1, s\}$, $H \subseteq B \cap sBs^{-1} = P \cap gPg^{-1}$ for some $g \in P w_0 P$. Therefore, H is contained in the conjugate of $P \cap w_0 P w_0^{-1}$ under an element of P. We will show below that if Ψ is the root system of type A_m, with $m > 1$, and a is an end root of its basis Δ_a, then there cannot exist a nilpotent normal subgroup U of P such that $P = B = HU$. This will prove the proposition.

Suppose there is a nilpotent normal subgroup U of P such that $P = HU$. Let $\mathbf{U}$ be the identity component of the Zariski-closure of U. Then as $\mathbf{P}^\natural$ is the identity component of the Zariski-closure of P (2.6) and U is a normal subgroup of the latter, we see that $\mathbf{U}$ is a normal subgroup of $\mathbf{P}^\natural$. Since H is contained in the conjugate of $P \cap w_0 P w_0^{-1}$ under an element of P, we see that $P = (P \cap w_0 P w_0^{-1})U$. From this we obtain, using again the fact that $\mathbf{P}^\natural$ is the identity component of the Zariski-closure of P, that $\mathbf{P}^\natural = (\mathbf{P}^\natural \cap w_0 \mathbf{P}^\natural w_0^{-1})\mathbf{U}$. According to Proposition 3.5.12 of [6], $\mathbf{Q}^\natural := \mathbf{P}^\natural \cap w_0 \mathbf{P}^\natural w_0^{-1}$ is a smooth connected k-subgroup. It clearly contains $Z_{\mathbf{G}^\natural}(\mathbf{S})$. Hence, as the root system Ψ is reduced, we see from Proposition 3.3.5 of [7] that if an element of Ψ is a root of $\mathbf{P}^\natural$ (with respect to $\mathbf{S}$), then it is a root of either $\mathbf{Q}^\natural$ or $\mathbf{U}$.

Now to determine the roots of $\mathbf{P}^\natural$ and $\mathbf{Q}^\natural$ (with respect to $\mathbf{S}$) contained in Ψ, we enumerate the roots in Δ_a as $\{a_1, a_2, \ldots, a_m\}$ so that $a_1 = a$ and for $i \leqslant m-1$, a_i is *not* orthogonal to a_{i+1}, i.e., the corresponding nodes are connected in the Coxeter graph. Then $w_0(a_i) = -a_{m-i}$ for all $i \leqslant m$. The roots of $\mathbf{P}^\natural$ which belong to Ψ are all the roots in $\Psi^+ := \Psi \cap \Phi^+$, and also all the negative roots contained in the span of $\{a_2, \ldots, a_m\}$. The roots of $w_0 \mathbf{P}^\natural w_0^{-1}$ contained in Ψ are all the roots in $-\Psi^+$, and also all the positive roots contained in the span of $\{a_1, \ldots, a_{m-1}\}$. From this we see that the roots of $\mathbf{P}^\natural$ which lie in Ψ, but are not roots of $\mathbf{Q}^\natural$, are the m roots $a_i + \cdots + a_m$, with $i \geqslant 1$. These m roots must therefore be roots of $\mathbf{U}$. Note that for $i \geqslant 2$, $-(a_i + \cdots + a_m)$ is a root of $\mathbf{Q}^\natural$. Now for a root $b \in \Psi$, $\mathbf{U}_b$ be the corresponding root group (see C.2.16 of [7]); $\mathbf{U}_b$ is the largest smooth connected unipotent k-subgroup of $\mathbf{G}^\natural$ normalized by $\mathbf{S}$ on whose Lie algebra the only weight of $\mathbf{S}$ under the adjoint action is b. Then if b is a root of $\mathbf{P}^\natural$ which is also a root of $\mathbf{Q}^\natural$, $\mathbf{U}_b \subset \mathbf{Q}^\natural$. On the other hand, if b is a root of $\mathbf{P}^\natural$ which is not a root of $\mathbf{Q}^\natural$, then it must be a root of $\mathbf{U}$ and the root group $\mathbf{U}_b$ is contained in $\mathbf{U}$. Let now $b = a_2 + \cdots + a_m$. Then $-b$ is a root of $\mathbf{Q}^\natural$, so $\mathbf{U}_{-b} \subset \mathbf{Q}^\natural$, and b is not a root of $\mathbf{Q}^\natural$ so it is a root of $\mathbf{U}$ and $\mathbf{U}_b \subset \mathbf{U}$. Since $\mathbf{U}$ is a (connected nilpotent) normal subgroup of $\mathbf{P}^\natural$, it is normalized by $\mathbf{U}_{-b}$, and hence $\mathbf{U}_{-b}\mathbf{U}$ is a solvable subgroup ($\mathbf{U}_{-b}\mathbf{U}$ is in fact nilpotent since according to Theorem 1(i) the unique maximal torus of the nilpotent group $\mathbf{U}$ is contained in the center of $\mathbf{G}^\natural$). But $\mathbf{U}_{-b}\mathbf{U}$ contains the subgroup $\mathbf{G}_b^\natural$ generated by $\mathbf{U}_{\pm b}$ which is a nonsolvable pseudo-reductive group (note that $\mathbf{G}_b^\natural$ is the derived group of the centralizer of the torus $\mathbf{S}_b = (\ker b)^\circ_{\mathrm{red}}$ in $\mathbf{G}^\natural$). This is impossible, so we are done. □

4.2. To prove assertion (2) of 4.1, we will assume now that the Tits system (B, N) is saturated and $B\,(= P)$ does not contain the group of k-rational points of any k-isotropic k-pseudo-simple normal subgroup of $\mathbf{G}$. Then it follows from Proposition 2 that $\mathbf{G}$ contains exactly one k-isotropic k-pseudo-simple normal subgroup and the k-rank of this subgroup is 1. Therefore, the k-rank of $\mathscr{D}(\mathbf{G})$ is 1 and hence $\mathbf{P}$ is a minimal pseudo-parabolic k-subgroup of $\mathbf{G}$. It is known that for $g \in G$, $\mathbf{P} \cap g\mathbf{P}g^{-1}$ contains the centralizer of a maximal k-split torus of $\mathbf{G}$ (Proposition C.2.7 of [6]); moreover, since $\mathscr{D}(\mathbf{G})$ is of k-rank 1, we easily see, using the Bruhat decomposition, that any two distinct minimal pseudo-parabolic k-subgroups of $\mathbf{G}$ are opposed to each other, therefore, for any $g \notin B = P$, $\mathbf{P} \cap g\mathbf{P}g^{-1}$ equals the centralizer (in $\mathbf{G}$) of a maximal k-split torus of $\mathbf{G}$. Then as the Tits system (B, N) is saturated, $H = P \cap sPs^{-1} = Z_{\mathbf{G}}(\mathbf{S})(k)$ for a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$. Now since N normalizes $H = Z_{\mathbf{G}}(\mathbf{S})(k)$ and contains it as a subgroup of index 2, whereas $N_{\mathbf{G}}(\mathbf{S})(k)$ is the unique such subgroup of G, we conclude that $N = N_{\mathbf{G}}(\mathbf{S})(k)$. *Thus the saturated Tits system (B, N) is a standard Tits system.* This proves assertion (2) of 4.1.

4.3. A Tits system of rank 1 in $G := \mathrm{SL}_{n+1}(k)$. Let B be the subgroup of matrices in G whose first column has all the entries zero except the top entry. Let B' be the subgroup of matrices in G whose last column has all the entries zero except the bottom entry. Let $H = B \cap B'$, and choose a $g \in G$ such that $gBg^{-1} = B'$, and let $N = H \cup gH$. Then N is a subgroup and H is a normal subgroup of N of index 2. It is easily checked that (B, N) is a spherical Tits system in G of rank 1. Note that B itself admits a Tits system of rank $n-1$ since it is isomorphic to $\mathrm{GL}_n(k) \ltimes k^n$.

It has been pointed out to me by Pierre-Emmanuel Caprace and Katrin Tent that giving a Tits system (B, N) of rank 1 in G is equivalent to providing a 2-transitive action of G on a set X. For a given pair $x \neq x'$ in X, let B be the stabilizer of x and N be the stabilizer of the subset $\{x, x'\}$. Now in the above example, the set X is the projective space $\mathbf{P}^n(k)$ with the natural action of $\mathrm{SL}_{n+1}(k)$.

5. Proof of Theorem B for Tits systems of arbitrary rank

5.1. We will retain most of our earlier notation. As before, let $\mathbf{G}$ be a pseudo-reductive k-group and $G = \mathbf{G}(k)$. Let (B, N) be a weakly-split Tits system in G with Weyl group W^T. We assume that W^T is finite, i.e., the Tits system is spherical, and let S be the distinguished set of involutive generators of W^T. Let $B = HU$, with $H = B \cap N$, and U a nilpotent normal subgroup of B. Let $\mathbf{U}$ be the identity component of the Zariski-closure of U in $\mathbf{G}$. For $s \in S$, let $G_s = B \cup BsB$. For a subset X of S, let W^T_X be the subgroup of W^T generated by the elements in X, and $G_X = BW^T_X B$. Then for each $s \in S$, and $X \subseteq S$, G_s and G_X are subgroups of G.

5.2. We will prove Theorem B by induction on the rank ($= \#S$) of the Tits system (B, N). If the rank is zero, i.e., if S is empty, $B = G$. Then assertion (1) of Theorem B holds if we take $\mathbf{P} = \mathbf{G}$. The hypothesis "B does not contain the group of k-rational points of any k-isotropic k-pseudo-simple normal subgroup of $\mathbf{G}$" in assertion (2) does not hold in this case if $\mathscr{D}(\mathbf{G})$ is k-isotropic. So to prove assertion (2) when S is empty, we assume that $\mathscr{D}(\mathbf{G})$ is anisotropic over k. If the Tits system (B, N), with $B = G$ is saturated, then $H := B \cap N = \bigcap_{n \in N} nGn^{-1} = G$, and hence, $N = G$. The Tits system $(B, N) = (G, G)$ is clearly the unique standard Tits system in G if $\mathscr{D}(\mathbf{G})$ is anisotropic over k.

Now let us assume that $\#S > 0$. As for $s \in S$, the index of B in $G_s\,(\subset G)$ is infinite, the index of B in G is infinite. Therefore, U cannot be virtually central in G (3.1), and hence $\mathbf{U}$ is not central in $\mathbf{G}^\natural$ (2.4). By Theorem 1(iii) (applied to $\mathbf{G}^\natural$ in place of $\mathbf{G}$) then $\mathscr{R}_{us,k}(\mathbf{B})$ is nontrivial. Now according to Theorem C.3.8 of [7] there is a proper pseudo-parabolic k-subgroup $\mathbf{Q}$ of $\mathbf{G}$ such that $\mathscr{R}_{us,k}(\mathbf{B}) \subseteq \mathscr{R}_{us,k}(\mathbf{Q})$ and $B \subseteq \mathbf{Q}(k)$. If $B = \mathbf{Q}(k)$, then the first assertion holds if we take $\mathbf{P} = \mathbf{Q}$, so let us assume that $B \neq \mathbf{Q}(k)$. As $\mathbf{Q}(k)$ properly contains B, and is not equal to G (2.6), it equals G_X for a nonempty subset $X\,(\neq S)$ of S. Since $\mathscr{R}_{us,k}(\mathbf{Q}) = \mathscr{R}_{u,k}(\mathbf{Q})$ ([6], Corollary 2.2.5), the quotient $\overline{\mathbf{M}} := \mathbf{Q}/\mathscr{R}_{us,k}(\mathbf{Q})$ is a pseudo-reductive k-group. We consider the natural projection $\mathbf{Q} \to \overline{\mathbf{M}}$. As $\mathscr{R}_{us,k}(\mathbf{Q})$ is a smooth connected k-split unipotent group, the map $\mathbf{Q}(k) \to \overline{\mathbf{M}}(k) =: \overline{M}$ is surjective (cf. [2], Corollary 15.7; the surjectivity also follows from the fact that the Galois cohomology $H^1(k, \mathscr{R}_{us,k}(\mathbf{Q}))$ is trivial). Let $K = \mathscr{R}_{us,k}(\mathbf{Q})(k)$, then K is a normal subgroup of $G_X = \mathbf{Q}(k)$, and the projection map $G_X = \mathbf{Q}(k) \to \overline{M}$ induces an isomorphism of G_X/K with $\overline{M}$. We will use this isomorphism to identify G_X/K with $\overline{M}$.

We want to prove now that the unipotent normal subgroup K of G_X is contained in B. For this purpose, let us consider the subgroup BK. As it contains B, it equals G_Y for a subset Y of X. To prove that K is contained in B, we need to show that Y is empty. Let us assume that this is not the case, and fix a $s \in Y$. Let $K_s = K \cap G_s$. Then $BK_s = HUK_s = G_s$, and, in particular, K_s is not contained in B. The pair $(B, N_s = N \cap G_s)$ is a Tits system in G_s of rank 1. Let $\mathscr{B}$ be the building associated with this Tits system. $\mathscr{B}$ is just the infinite set G_s/B endowed with the left-multiplication action of G_s. Any pair of distinct points in this building constitute an apartment. Let x and x' be the points of $\mathscr{B}$ fixed by B and $B' = sBs^{-1}$ respectively. Then $H\,(\subseteq B \cap sBs^{-1})$ fixes both x and x' and s interchanges them. As $G_s(= K_sUH)$ operates transitively on the set of apartments of $\mathscr{B}$, we see that it acts 2-transitively on $\mathscr{B}$. Now since H fixes x and x', we conclude that the subgroup K_sU is 2-transitive on $\mathscr{B}$. Now we note that the smooth connected k-split unipotent k-subgroup $\mathscr{R}_{us,k}(\mathbf{Q})$ is contained in $\mathbf{G}^\natural$ (2.3). Moreover, according to Theorem 1(i), the unique maximal

k-torus of the connected nilpotent subgroup $\mathbf{U}$ is contained in the center of $\mathbf{G}^{\natural}$. Therefore, $\mathscr{R}_{us,k}(\mathbf{Q})\mathbf{U}$ is a nilpotent subgroup. Hence the subgroup KU, and so also the subgroup K_sU, is virtually nilpotent. By the following lemma, which was pointed out to the author by Katrin Tent, such a group cannot act 2-transitively on an infinite set. Hence, we have arrived at a contradiction. This proves that K is contained in B.

Lemma 2. *A virtually nilpotent group cannot act* 2*-transitively on an infinite set.*

Proof (by Katrin Tent). Let $\mathscr{G}$ be a virtually nilpotent group acting 2-transitively on an infinite set $\mathscr{X}$. After replacing $\mathscr{G}$ with its quotient by the kernel of the action, we may (and do) assume that the action is faithful. Then any nontrivial commutative normal subgroup of $\mathscr{G}$ acts simply transitively on $\mathscr{X}$. Let $\mathscr{N}$ be a nilpotent normal subgroup of $\mathscr{G}$ of finite index and $Z(\mathscr{N})$ be its center. Then $Z(\mathscr{N})$ is a nontrivial commutative normal subgroup of $\mathscr{G}$, so it acts simply transitively on $\mathscr{X}$, in particular, $Z(\mathscr{N})$ is infinite. Fix an $x \in \mathscr{X}$ and let $\mathscr{G}_x$ be the stabilizer of x in $\mathscr{G}$. We identify $\mathscr{X}$ with $Z(\mathscr{N})$ using the bijection $g \mapsto gx$. With this identification, the action of $\mathscr{G}_x$ on $\mathscr{X}$ is just the conjugation action (of $\mathscr{G}_x$) on $Z(\mathscr{N})$. As $\mathscr{G}_x$ acts transitively on $\mathscr{X} - \{x\}$, we conclude that the conjugation action of $\mathscr{G}_x$ on $Z(\mathscr{N})$ is transitive on the set of nontrivial elements of $Z(\mathscr{N})$. But since $\mathscr{N}$ is of finite index in $\mathscr{G}$, the orbit of any element of $Z(\mathscr{N})$ under $\mathscr{G}$, and hence under $\mathscr{G}_x$, is finite. This is a contradiction. ☐

5.3. Now since K is contained in B and (B, N_X), with $N_X = N \cap G_X$, is a Tits system in G_X, $(\overline{B}, \overline{N}_X)$, with $\overline{B} = B/K$ and $\overline{N}_X = N_XK/K$ is a Tits system in $\overline{M} = G_X/K$. Let $\overline{H}$ (resp., $\overline{U}$) be the image of H (resp., U) in $\overline{B}$. As $B = HU$, $\overline{B} = \overline{HU}$; moreover, since $B \cap N_XK = (B \cap N_X)K = HK$, $\overline{B} \cap \overline{N}_X = \overline{H}$, so the Tits system $(\overline{B}, \overline{N}_X)$ in $\overline{M} = \overline{\mathbf{M}}(k)$ is weakly-split. As $H \subseteq N_X \cap HK \subseteq N_X \cap B \subseteq H$, the natural homomorphism $W_X^T = N_X/H \to \overline{N}_X/\overline{H}$ is an isomorphism, and hence the Weyl group of the Tits system $(\overline{B}, \overline{N}_X)$ in $\overline{M}$ is W_X^T. For $x \in X$, $\overline{M}_x$ $(= \overline{B} \cup \overline{B}x\overline{B})$ is the image of G_x in $\overline{M}$ and the induced map $G_x/B \to \overline{M}_x/\overline{B}$ is bijective. Therefore, the index of $\overline{B}$ in $\overline{M}_x$ is infinite for all $x \in X$. Now since the rank of the Tits system $(\overline{B}, \overline{N}_X)$ is $\#X < \#S$, we conclude by induction on the rank of Tits systems that there exists a proper pseudo-parabolic k-subgroup $\overline{\mathbf{P}}$ of the pseudo-reductive k-group $\overline{\mathbf{M}} = \mathbf{Q}/\mathscr{R}_{us,k}(\mathbf{Q})$ such that $\overline{B} = \overline{\mathbf{P}}(k)$. Let $\mathbf{P}$ be the inverse image of $\overline{\mathbf{P}}$ in $\mathbf{Q}$. Then $\mathbf{P}$ is a pseudo-parabolic k-subgroup of $\mathbf{G}$, and $B \subseteq \mathbf{P}(k)$. Moreover, since B contains the kernel $K = \mathscr{R}_{us,k}(\mathbf{Q})(k)$ of the natural map $\mathbf{Q}(k) \to \overline{\mathbf{M}}(k)$, and its image $\overline{B}$ in $\overline{\mathbf{M}}(k)$ equals $\overline{\mathbf{P}}(k)$, we conclude that $B = \mathbf{P}(k) =: P$. This proves assertion (1) of Theorem B.

5.4. Now to prove assertion (2) of Theorem B, we will assume that the Tits system (B, N) is saturated and B does not contain the group of k-rational points of any k-isotropic k-pseudo-simple normal subgroup of $\mathbf{G}$. We wish to first find a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ such that $Z_{\mathbf{G}}(\mathbf{S})(k) \subseteq H = B \cap N$. For this purpose, and also for use later, we introduce the following notation: for an element $w \in W^T$, $\ell(w)$ will denote its length with respect to the set S of generators of the Weyl group W^T. The longest element of W^T will be denoted by

w_0. We will now show that $B \cap w_0 B w_0^{-1} = H$. For this, we will use Lemma 13.13 of [11]. This lemma says that $B \cap ww'B(ww')^{-1} \subseteq B \cap wBw^{-1}$ if $\ell(ww') = \ell(w) + \ell(w')$. But given any $w \in W^T$, let $w' = w^{-1}w_0$. Then $ww' = w_0$, and $\ell(w) + \ell(w') = \ell(w_0)$, which implies that $H = \bigcap_{w \in W^T} wBw^{-1} = B \cap w_0 B w_0^{-1}$. Therefore, $H = B \cap w_0 B w_0^{-1} = P \cap w_0 P w_0^{-1}$. Now we note that $\mathbf{P} \cap w_0 \mathbf{P} w_0^{-1}$ is a smooth connected algebraic subgroup which contains the centralizer $Z_{\mathbf{G}}(\mathbf{S})$ of a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ (Propositions 3.5.12 and C.2.7 of [6]). Hence, $Z_{\mathbf{G}}(\mathbf{S})(k) \subseteq P \cap w_0 P w_0^{-1} = H$.

5.5. We will next prove that $\mathbf{P}$ is a minimal pseudo-parabolic k-subgroup of $\mathbf{G}$. Let us fix a minimal pseudo-parabolic k-subgroup $\mathbf{P}_0$ of $\mathbf{G}$ contained in $\mathbf{P}$ and containing $Z_{\mathbf{G}}(\mathbf{S})$. Let $P_0 = \mathbf{P}_0(k)$. Let Φ be the set of k-roots of $\mathbf{G}$ with respect to $\mathbf{S}$, Φ^+ the positive system of roots and $\Delta\, (\subseteq \Phi^+)$ the set of simple roots given by $\mathbf{P}_0$. For a nondivisible root $b \in \Phi$, let $\mathbf{U}_b$ be the root group corresponding to it (see [7], C.2.16). $\mathbf{U}_b$ is the largest smooth connected unipotent subgroup of $\mathbf{G}$ normalized by $\mathbf{S}$ on whose Lie algebra the only weights of $\mathbf{S}$ under the adjoint action are positive integral multiples of b. Then there is a subset Δ' of Δ such that $\mathbf{P}$ is generated by $\mathbf{P}_0$ and the root groups $\mathbf{U}_{-a}$, $a \in \Delta'$. We will now prove that $\mathbf{P} = \mathbf{P}_0$, or, equivalently, that Δ' is empty. The condition that $B\,(= P = \mathbf{P}(k))$ does not contain the group of k-rational points of any k-isotropic k-pseudo-simple normal subgroup of $\mathbf{G}$ is equivalent to the condition that Δ' does not contain any connected components of Δ.

Let $W = N_{\mathbf{G}}(\mathbf{S})(k)/Z_{\mathbf{G}}(\mathbf{S})(k)$ be the k-Weyl group and $R = \{r_a \,|\, a \in \Delta\}$, where for $a \in \Delta$, $r_a \in W$ is the reflection in a. The reflections in R generate W; for $w \in W$, we will denote the length of w, with respect to this generating set, by $\ell(w)$. The rank of W, i.e., the number of elements in Δ, is equal to the k-rank of $\mathscr{D}(\mathbf{G})$. For a subset X of R, we will denote by W_X the subgroup of W generated by the elements in X, and let $G^X = P_0 W_X P_0$. Any subgroup of G which contains P_0 equals G^X for a unique $X \subseteq R$. Now let $Y \subset R$ be such that $B = P = G^Y$. Then $Y = \{r_a \mid a \in \Delta'\}$. For $s \in S$, as $G_s = B \cup BsB$ is a subgroup containing $B = P$, there is a subset Y_s of R containing Y such that $G_s = G^{Y_s}$. As there is no subgroup lying properly between B and G_s, $Y_s = Y \cup \{r_{a_s}\}$, for some $a_s \in \Delta - \Delta'$, and then $BsB = Pr_{a_s}P$. For distinct elements s, s' of S, since $G_s \cap G_{s'} = B = P$, and the subgroups G_s, for $s \in S$, generate G, we conclude that if $s \neq s'$, then $a_s \neq a_{s'}$, and moreover, $\{a_s \,|\, s \in S\} = \Delta - \Delta'$.

Assume, if possible, that Δ' is nonempty. Then since Δ' does not contain any connected components of Δ, we can find a root $a \in \Delta'$ such that there is a root $b \in \Delta - \Delta'$ connected to a. Let $s \in S$ be the element such that $b = a_s$. Let $\mathbf{P}'$ be the pseudo-parabolic k-subgroup generated by $\mathbf{P}$ and the root group $\mathbf{U}_{-b}$. Let $\overline{\mathbf{M}}' = \mathbf{P}'/\mathscr{R}_{us,k}(\mathbf{P}')$ be the maximal pseudo-reductive quotient of $\mathbf{P}'$ and $\overline{M}' := \overline{\mathbf{M}}'(k)$. Then $P' := \mathbf{P}'(k) = G_s = P \cup Pr_bP$. Since $\mathbf{P}' \supset \mathbf{P}$, $\mathscr{R}_{us,k}(\mathbf{P}') \subset \mathscr{R}_{us,k}(\mathbf{P})$, and hence, $\mathscr{R}_{us,k}(\mathbf{P}')(k) \subset P = B$. Since (P, N'), with $N' = N \cap P'$, is a Tits system of rank 1 in P', $(\overline{P}, \overline{N}')$, where $\overline{P}'$ and $\overline{N}'$ are respectively the images of P and N' in $\overline{M}'$, is a Tits system of rank 1 in $\overline{M}'$. Arguing as in 5.3 we see that this Tits system is weakly-split. Now since the k-pseudo-simple normal subgroup of $\overline{\mathbf{M}}'$, whose k-root system contains the roots $\pm a$ and $\pm b$, is of k-rank at least two, we conclude from

Proposition 2 that the group of k-rational points of this k-pseudo-simple subgroup must be contained in $\overline{P}$. As every connected component of $\Delta' \cup \{b\}$, except the one containing a and b, is also a connected component of Δ', we now see that the group of k-rational points of every k-isotropic k-pseudo-simple normal subgroup of $\overline{\mathbf{M}}'$ is contained in $\overline{P}$, and hence, $\overline{P} = \overline{M}'$ (since $P \supset Z_{\mathbf{G}}(\mathbf{S})(k)$) , which is a contradiction since $P(\subset P')$ cannot map onto $\overline{M}'$. Thus we have proved that Δ' is empty, and hence $\mathbf{P} = \mathbf{P}_0$ is a minimal pseudo-parabolic k-subgroup of $\mathbf{G}$. In particular, the map $s \mapsto r_{a_s}$ from S into R is bijective.

5.6. Having proved that $B = P = \mathbf{P}(k)$, and $\mathbf{P}$ is a minimal pseudo-parabolic k-subgroup of $\mathbf{G}$, we will now prove that $H = Z_{\mathbf{G}}(\mathbf{S})(k) =: Z$ and $N = N_{\mathbf{G}}(\mathbf{S})(k) = N_G(Z)$. We begin by recalling from [3], Ch. IV, §2, Theorem 2, that

($*$) *for* $w \in W^T$ *(resp.,* $w \in W$*) and* $s \in S$ *(resp.,* $r \in R$*)* $\ell(sw) > \ell(w)$ *(resp.,* $\ell(rw) > \ell(w)$*) if and only if* $BsB \cdot BwB = BswB$ *(resp.,* $PrP \cdot PwP = PrwP$*), and moreover, if* $\ell(sw) < \ell(w)$ *then* $BsB \cdot BwB$ *is the union of two distinct* $\{B, B\}$*-double cosets.*

For $s \in S$, let a_s be as in 5.5. We will first prove that the map $S \to W$ which maps s onto r_{a_s} extends to an isomorphism of W^T onto W. For this purpose, let $s, s' \in S$ and m be the order of $s's$ in W^T. Let w (resp., $\widetilde{w}$) be the product of the first m-terms of the sequence $\{s', s, s', s, \ldots\}$ (resp., of the sequence $\{r_{a_{s'}}, r_{a_s}, r_{a_{s'}}, r_{a_s}, \ldots\}$). Then $\ell(w) = m$, and $\ell(sw) = m - 1$. Since $B = P$, $BsB = Pr_{a_s}P$, and $Bs'B = Pr_{a_{s'}}P$, by repeated application of ($*$) we easily see that $\ell(\widetilde{w}) = m$, and $\ell(r_{a_s}\widetilde{w}) = m - 1$. This observation implies that the order of the product $r_{a_{s'}} r_{a_s}$ (in the k-Weyl group W) is also m. Now since (W^T, S) and (W, R) are Coxeter groups, we conclude that the map $s \mapsto r_{a_s}$ of S into R extends uniquely to an isomorphism $W^T \to W$. In particular, W^T and W are of equal order, say n.

We have shown in 5.4 that $H \supseteq Z_{\mathbf{G}}(\mathbf{S})(k) = Z$. We will now prove that these two subgroups are equal. For this, we note that since $\mathbf{P}$ is a minimal pseudo-parabolic k-subgroup of $\mathbf{G}$, $B = P$ has precisely n distinct conjugates containing Z, these are $wPw^{-1}, w \in W$. The intersection of these conjugates equals Z. On the other hand, since the Weyl group W^T of the Tits system (B, N) is of order n, at least n distinct conjugates of $B = P$ contain H, and hence also Z. Therefore, there are exactly n distinct conjugates of $B = P$ which contain H, and these conjugates are $wBw^{-1}, w \in W^T$. Thus the collections $\{wPw^{-1} \mid w \in W\}$ and $\{wBw^{-1} \mid w \in W^T\}$ are the same, so $H = \bigcap_{w \in W^T} wBw^{-1} = \bigcap_{w \in W} wPw^{-1} = Z$.

The subgroup N normalizes $H = Z$, and hence it is contained in the normalizer $N_G(Z)$ of Z in G, and the order of $N/Z(= W^T)$ is n. But the index of Z in $N_G(Z)$ is n since $N_G(Z)/Z = W$, which implies that $N = N_G(Z) = N_{\mathbf{G}}(\mathbf{S})(k)$. Thus we have proved that the Tits system (B, N) is a standard Tits system (i.e., it is as in 1.2). This completes the proof of Theorem B.

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References

1. P. Abramenko, M. Zaremsky, *Some reductive anisotropic groups that admit no non-trivial split spherical BN-pairs*, a paper posted on arXiv.

2. A. Borel, *Linear Algebraic Groups*, 2nd edition, Springer-Verlag, New York (1991).

3. N. Bourbaki, *Lie algebras and Lie Groups*, Chapters IV, V, VI, Springer-Verlag, New York (2002).

4. J. Brundan, *Double coset density in exceptional algebraic groups*, J. London Math. Soc. **58** (1998), 63-83.

5. P-E. Caprace, T. Marquis, *Can an anisotropic reductive group admit a Tits system?*, Pure and Appl. Math. Q. **7**(2011), 539-557.

6. B. Conrad, O. Gabber, G. Prasad, *Pseudo-reductive groups*, first edition, New Mathematical Monographs #17, Cambridge U. Press (2010).

7. B. Conrad, O. Gabber, G. Prasad, *Pseudo-reductive groups*, 2nd edition, New Mathematical Monographs #17, Cambridge U. Press. (The new material to be included in the second edition is posted on the author's homepage. We refer to [7] in this paper only if the required result is not contained in [6].)

8. M. Liebeck, G. M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, AMS Monograph (2012).

9. T. De Medts, F. Haot, K. Tent, H. Van Maldeghem, *Split BN-pairs of rank at least* 2 *and the uniqueness of splittings*, J. Group Theory **8**(2005), 1-10.

10. G. Prasad, M. S. Raghunathan, *Topological central extensions of* $SL_1(D)$, Inventiones Math. **92**(1988), 645-689.

11. J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics No. 386, Springer-Verlag, New York (1974).

12. J. Tits, R. M. Weiss, *Moufang Polygons*, Springer Monographs in Mathematics, Springer-Verlag, New York (2002).

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