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Probability distribution of residence times of grains in sand-pile models

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Abstract. We show that the probability distribution of the residence times of sand grains in sand-pile models, in the scaling limit, can be expressed in terms of the probability of survival of a single diffusing particle in a medium with absorbing boundaries and space-dependent jump rates. The scaling function for the probability distribution of residence times is non-universal, and depends on the probability distribution according to which grains are added at different sites. We determine this function exactly for the one-dimensional sand-pile when grains are added randomly only at the ends. For sand-piles with grains added everywhere with equal probability, in any dimension, and of arbitrary shape, we prove that, in the scaling limit, the probability that the residence time is greater than t is $\exp(-t/\bar{M})$, where $\bar{M}$ is the average mass of the pile in the steady state. We also study finite size corrections to this function.

Keywords: driven diffusive systems (theory), exact results, self-organized criticality (theory), granular matter

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In 1987, Bak, Tang, and Wiesenfeld (BTW) proposed a sand-pile model as a paradigm for self-organized critical systems in nature [1]. Since then, many different kinds of sand-pile models have been studied. These include models with discrete or continuous variables [2], different toppling rules [3], deterministic or stochastic particle transfer under toppling [4], with or without local particle conservation [5], driving mechanisms [6, 7] etc. By now, a fair amount of understanding of the critical steady state and critical exponents of avalanches has been achieved, by study of exactly solved models, and by means of numerical simulations. For a review of known results, see [8]–[11].

However, time-dependent properties of self-organized critical systems have not been studied as much theoretically so far, in spite of the fact that an explanation of $1/f$ noise was one of the main motivations for the initial proposal of self-organized criticality. While a power spectrum of mass fluctuations of $1/f$ type has been found in some one-dimensional models [12, 13], it appears that in higher dimensional sand-pile models, the behaviour is as $1/f^2$ [14]. Piles of long-grain rice have provided a very good experimental realization of the basic ideas of self-organized criticality [15, 16]. The probability distribution of residence times (DRT) of grains was studied experimentally in the Oslo rice-pile experiment, and by means of simulations [15, 17].

In this paper, we will study the DRT in sand-pile models. We argue that the problem of determining the DRT can be reduced to that of finding the probability distribution of survival times for a single particle diffusing to the absorbing boundary, in a medium with site-dependent jump rates. In the scaling limit of large system sizes, the DRT becomes a function of a single scaling variable t/L^b, where t is the residence time, L is the linear size of the system, and b is some exponent. This function is non-universal, and is a complicated function of the spatial distribution of added grains used to drive the pile to its steady state. We determine this function explicitly for a one-dimensional sand-pile when grains are added randomly only at the ends. When grains are added with equal probability everywhere, we prove that the exact scaling function of the DRT is a simple exponential. This result is independent of the dimension, and of the shape of the pile.

Let us consider the problem in the simplest setting first: the BTW model [1] on a line of L sites, labelled by integers 1 to L. At each site i we have a non-negative integer variable z_i called the height of the pile at that site. The site is stable if $z_i \leq 1$. If $z_i \geq 2$, the site is said to be unstable, and relaxes by toppling. In this process, z_i decreases by 2, and z_{i-1} and z_{i+1} increase by 1. Toppling at a boundary site causes the loss of one sand-grain from the pile. The pile is driven by adding grains at the right end, at a constant rate of one grain every P time steps. We assume that P is larger than the duration of the longest avalanche in the system, so that all avalanches have died before a new grain is added to the system.

This model has an Abelian property, and its properties are well understood [18]. The long-time behaviour under deterministic evolution is as follows: after an initial transient period, it falls into a cycle of period $L + 1$. The stable configurations of the pile that belong to the cycle are L configurations having all except one site with height 1, and one configuration with all $z_i = 1$. If we start with the state with all $z_i = 1$, adding a particle at $i = L$ gives a stable configuration with $z_1 = 0$. Adding a particle again, we get the recurrent configuration in which $z_2 = 0$. For each new added grain, the position of the zero shifts one step to the right, until after L steps it is at $i = L$. Then adding another grain, the zero disappears. We choose to say that in this case the zero is at $i = 0$. The

number of topplings to get the next stable configuration is also periodic with the same period: $L \to L-1 \to (L-2)\cdots 1 \to 0 \to L$.

If we want to study the DRT in this model, we have to mark the grains. However, with marked gains, *the model is no longer Abelian*. This is because topplings at two adjacent unstable sites in a different order no longer give the same result. For a full specification of the rules governing the motion of grains in the model, we have to define precisely in which order the unstable sites are toppled, and how the grains are transferred under toppling. We choose the parallel update scheme: make a list of all sites which are unstable at a time t, choose at random two grains from each of these sites (if there are only two grains, both are selected), and randomly assign one of them to go to the left neighbour, and the other to the right. All these grains which are to be moved are then added to their destined sites, at the same time. This constitutes a single micro-step of evolution. Then we construct the new list of unstable sites for the next time micro-step, and repeat.

The constant time elapsed between two successive additions of grains (P micro-steps) will be called a time meso-step. We measure the residence time in units of meso-steps. We mark all grains with the time meso-step at which they were added to the pile. Then, if the grain numbered $T_{\rm in}$ (added at meso-step $T_{\rm in}$) gets out of the system at meso-step $T_{\rm out}$, we will say that its residence time is $T_{\rm out} - T_{\rm in}$.

It is easily seen that the first moment of the DRT is the average mass of the pile. Define a variable $\eta(i,j)$ as the function indicating that the ith grain is in the pile at the end of time meso-step j. Then, summation of $\eta(i,j)$ over j gives the residence time of particle i, and averaging over i gives the mean residence time. Conversely, summing over i, we get the mass of the pile at the end of time step j, and averaging over j gives the mean mass of the pile [19].

From our definition, it follows that the probabilities of different paths taken by a grain are exactly those of an unbiased random walker on the line. This is because when a grain moves under toppling, it is equally likely to take a step to the right or to the left. So, for example, the average number of steps a grain takes before it leaves the pile is equal to the average number of steps a random walker would take from that starting point. However, the time between two jumps of the grain is random, and has very non-trivial correlations with times of previous jumps, and also with jump times of other particles. This is what makes this problem non-trivial.

To calculate the DRT for the linear chain of L sites, we consider adding a marked grain at meso-time T_0. All other grains are unmarked, and indistinguishable. Then, stable configurations of the pile are L^2 in number. The configuration in which the site a has height 0 and the marked grain is at site b will be denoted by $\mathcal{C}_{a,b}$. All sites other than a and b are occupied by unmarked grains. For each value of a, $1 \le a \le L$, there are then $L-1$ possible configurations corresponding to different values of b. For the recurrent configuration with all $z_i = 1$, we define $a = 0$, and in this case there are L possible positions of b. Thus there are in total L^2 possible stable configurations of the pile.

Consider a particular configuration $\mathcal{C}_{a,b}$. Add another (unmarked) grain at $i = L$. If $b < a$, then it is easily seen that the wave of toppling [20] does not reach the marked grain, and the final configuration is $\mathcal{C}_{a+1,b}$.

When $b > a$, the wave of toppling, started at the right end, reaches the site b and the site will topple. The marked grain will move one step to the left or right, with equal probabilities. If the marked grain moves to the left, it will move again due to toppling,

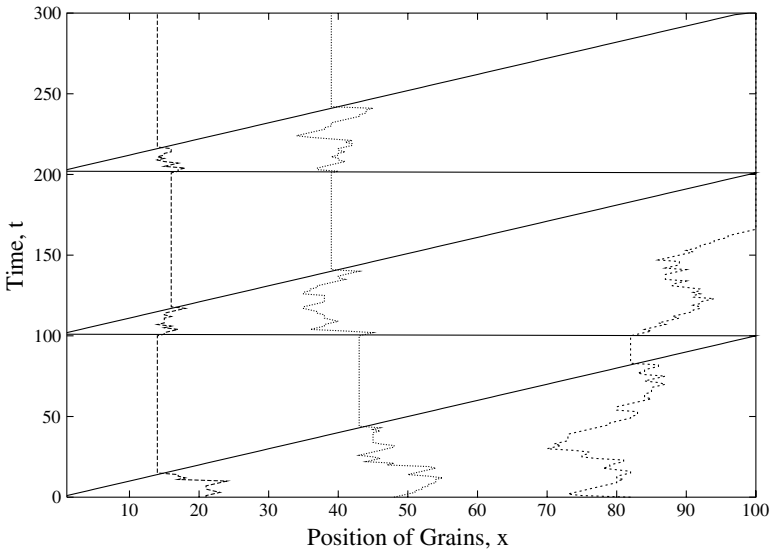

Figure 1. The motion of three grains starting from $x = 20, 50, 80$ on a one-dimensional sand-pile of length $L = 100$ where sand grains are added only at the right end.

unless that site has no grains. In this way, the marked grain can take zero, one, or more consecutive steps to the left in one meso-step. It stops diffusing as soon as it takes a right step or if the marked grain falls on a. We thus see that on adding more grains, if $b > a$, the final configuration is $\mathcal{C}_{a+1,b+\Delta b}$, with Δb taking values $1, 0, -1, \ldots, a+1-b, a-b$ with probabilities $2^{-1}, 2^{-2}, 2^{-3}, \ldots, 2^{a-b+1}, 2^{a-b}$. For $b-a$ large, the mean square displacement $\langle(\Delta b)^2\rangle$ tends to 2.

It is straightforward to construct the $L^2 \times L^2$ matrix $\mathcal{W}$ giving the probabilities of transitions between different configurations. Also, knowing the probabilities of different configurations in the steady state, we can write down the probabilities of different stable configurations just after the marked grain has been added. Then $\mathcal{W}^t P(t=0)$ gives the probabilities of different configurations at time t, and, summing over all configurations, we get the survival probabilities $S(t)$ for the marked grain remaining inside the system up to time t. In fact, using the fact that the position a of the site with height zero changes by 1 deterministically in time, one can rewrite this problem in terms of $L+1$ matrices $\mathcal{T}_j$, $j = 0, 1, 2, \ldots, L$, where $\mathcal{T}_j$ is an $(L-1)\times(L-1)$ matrix giving probabilities of transitions from the $L-1$ configurations with $a = j$ to the $L-1$ configurations with $a = j+1$. For $a = L+1$, there are L stable configurations with a marked grain, but the configuration with $b = L$ is transient, and hence one can work with the $L-1$ remaining configurations.

In an earlier paper, one of us used these to determine the DRT exactly numerically, for L up to 150 [19]. In the limit of large L, it was argued that $\mathrm{Prob}(t|L)$ tends to the scaling form

$$\mathrm{Prob}(t|L) \sim \frac{1}{L^3} f(t/L^2). \tag{1}$$

Here the function $f(x)$ varies as $x^{-3/2}$ for $x \ll 1$, and as $\exp(-Kx)$ for $x \gg 1$. We now obtain the exact functional form of the scaling function f.

Consider a marked grain at b, at some time T, with $b = \alpha L, 0 < \alpha < 1$, and L large. We consider the change in its position Δb after one cycle ($L + 1$ meso-steps). Figure 1 shows the motion of grains in a cycle in one realization. The grain diffuses for a while, and is stuck when the zero is to the right of the marked grain. The average fraction of time for which it moves is α. A grain at b is hit by b waves of toppling [20] in this interval. The net displacement Δb is the sum of the displacements due to these waves. Each wave causes a displacement with mean zero and variance 2. Then by the central limit theorem, the net displacement will be distributed normally with the variance given by $2b$. Thus Δb is of order $\sqrt{2\alpha L}$, and is much smaller than L when L is very large. Then, for times $t \gg L$, we can average over the motion in a cycle, and say that if the marked grain is at i, it moves to the left or right neighbour with a rate i/L per unit time. If $P(i,t)$ is the probability that the marked grain is at i at time t, the evolution equation for $P(i,t)$ for times $t \gg 1$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}P(i,t) = \frac{i+1}{L}P(i+1,t) + \frac{i-1}{L}P(i-1,t) - \frac{2i}{L}P(i,t). \tag{2}$$

At time $t = 0$, we can assume that the marked particle is at $i = L$, so $P(i, t = 0) = \delta_{i,L}$. Integrating this equation, we determine the survival probability $S(t) = \sum_i P(i,t)$, and then the DRT is given by

$$\mathrm{Prob}(t|L) = S(t) - S(t+1). \tag{3}$$

We introduce the reduced coordinate $\xi = i/L$, and $\tau = t/L^2$, and consider equation (10) when L is large. In terms of these reduced variables, the evolution equation for the probability density $P(\xi,\tau)$ becomes, in the continuum limit,

$$\frac{\partial}{\partial\tau}P(\xi,\tau) = \frac{\partial^2}{\partial\xi^2}[\xi P(\xi,\tau)]. \tag{4}$$

We can integrate this equation numerically using the initial condition $P(\xi, t = 0) = \delta(\xi - 1 + 1/L)$. The scaling function $f(x)$ is given by

$$f(x) = \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1 P(\xi,\tau)\,\mathrm{d}\xi\right]_{\tau=x}. \tag{5}$$

Let $\varphi_j(\xi)$ be a solution to the eigenvalue equation corresponding to eigenvalue λ_j:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}[\xi\varphi_j(\xi)] = -\lambda_j\varphi_j(\xi), \tag{6}$$

where $\varphi_j(\xi = 1) = 0$, corresponding to an absorber being present at $i = L + 1$. At $\xi = 0$, we do not need to assume any special condition, as the absorber at $i = 0$ is automatically taken care of by the fact that the rate of jumps out of i is $2i/L$, which becomes zero at $i = 0$.

We look for a solution $\varphi_j(\xi)$ that does not diverge at $\xi = 0$. Expanding $\varphi_j(\xi)$ in a power series, and matching coefficients, we get

$$\varphi_j(\xi) = \sum_{n=0}^{\infty}\frac{(-\lambda_j\xi)^n}{n!(n+1)!} = I_1(2i\sqrt{\lambda_j\xi})/(i\sqrt{\lambda_j\xi}),$$

where $I_1(x)$ is the modified Bessel function of order 1 [21]. The eigenvalues λ_j are obtained by imposing the condition $\varphi(\xi = 1) = 0$. Thus if the jth zero $I_1(z)$ occurs

at $\pm 2ik_j$, then $\lambda_j = k_j^2$. At large times t, $S(t)$ varies as $\exp(-Ct/L^2)$, where we get $C = k_1^2 = 3.6705$. This value is in good agreement with the value obtained by extrapolation of estimates obtained by measuring the coefficients of the exponential determined by exact diagonalization of the master equation for finite L [19].

The generalization of these results to d dimensions is straightforward. We consider a d-dimensional sand-pile model on a lattice with number of sites V. We assume that when a new grain is added, the site $\vec{x}$ is chosen with probability $r(\vec{x})$. Clearly, the sum of $r(\vec{x})$ over all sites is 1. In the steady state, $n(\vec{x})$, the average number of topplings at $\vec{x}$ per added grain satisfies the equation (using conservation of sand grains)

$$\nabla^2 n(\vec{x}) = -r(\vec{x}), \tag{7}$$

with $n(\vec{x}) = 0$ at the boundary. The solution of this equation is

$$n(\vec{x}) = \sum_{\vec{x'}} G(\vec{x}, \vec{x'}) r(\vec{x'}), \tag{8}$$

where $G(\vec{x}, \vec{x'})$ is the average number of topplings at $\vec{x}$ due to addition of a grain at $\vec{x'}$, and is equal to the inverse of the toppling matrix Δ [18].

The important point to realize is that while avalanches in the sand-pile can spread quite far, the typical distance travelled by one marked grain in an avalanche is much smaller than L. In fact, in many cases, we expect it to be of order 1. During its motion to the boundary, the marked grain would be involved in a large number of avalanches. At timescales much larger than a meso-step, the motion is diffusive, with the rate of jumps out of different sites being space dependent because on average some parts of the lattice have more avalanche activity than others.

Consider a grain at site $\vec{x}$ at time t. Let its position be $\vec{x} + \Delta\vec{x}$ after Δt new grains have been added, where $L^d \gg \Delta t \gg 1$. As the path of the grain is an unbiased random walk, we have $\langle(\vec{\Delta x})^2\rangle = s$, where s is the average number of jumps that the grain makes in this interval. Assuming that $|\vec{\Delta x}| \ll L$, and that $n(\vec{x})$ is a slowly varying function of $\vec{x}$, we see that s has to be proportional to $n(\vec{x})\Delta t$, the total number of toppling waves during time interval Δt. Let us say $s = Kn(\vec{x})\Delta t$, where K is some constant. Writing $\langle(\vec{\Delta x})^2\rangle = \Gamma(\vec{x})\Delta t$, we get

$$\Gamma(\vec{x}) = Kn(\vec{x}), \tag{9}$$

where the constant K depends on the details of the model. For large times t, the probability density $P(\vec{x}, t)$ satisfies the equation

$$\frac{\partial}{\partial t} P(\vec{x}, t) = \frac{1}{2} K \nabla^2 [n(\vec{x}) P(\vec{x}, t)] \tag{10}$$

with the initial condition given by

$$P(\vec{x}, t = 0) = r(\vec{x}). \tag{11}$$

It may be noted that equation (10) is not the diffusion equation with the space-dependent diffusion constant $D(\vec{x})$, where the right-hand side would have been of the form $\nabla(D(\vec{x})\nabla P(\vec{x}))$. The net current between two sites depends on the difference between the products nP at the two sites, and can be non-zero even if ∇P is zero.

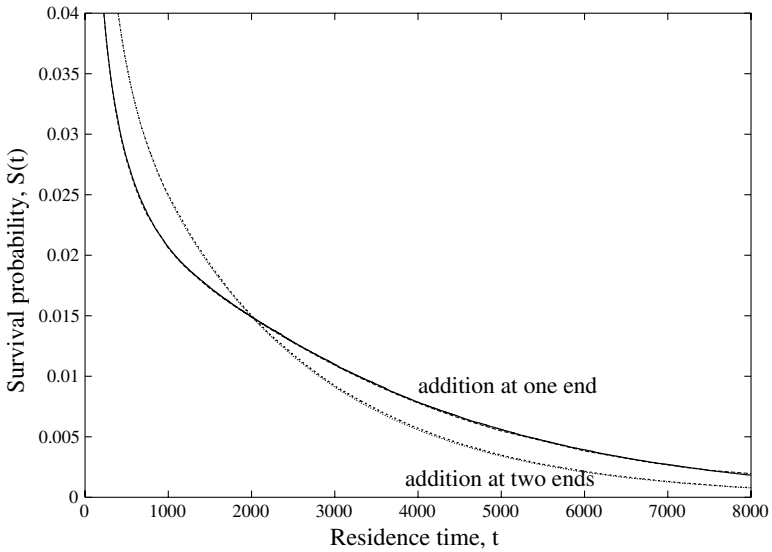

Figure 2. Survival probability versus residence time t for the one-dimensional BTW model in two cases with grains added only at one or both ends. The theoretical result (full curve) and the simulation result (dotted curve) match perfectly. $L = 100$ for both the cases.

Solving this differential equation, with the condition that $P(\vec{x}, t)$ is zero at the boundary corresponding to the absorbing boundaries, we can determine $P(\vec{x}, t)$ at any time t. Integrating over $\vec{x}$ determines the probability that the marked particle remains in the system at time t, and the DRT is obtained from the survival probability using equation (3).

Consider, as an example, the case of a linear chain with L sites, when we add particles at each step at either of the two ends with probability 1/2. In this case, we get $n(x) = \frac{1}{2}$, independent of x, and $K = 2$. One can then solve equation (10) analytically, and a straightforward calculation gives $S(\tau) = \theta_3(0, \tau) - \theta_3(\pi/2, \tau)$, where $\tau = \pi^2 t/2L^2$, and $\theta_3(z, \tau)$ is the Jacobi theta function [22] defined by

$$\theta_3(z, \tau) = 1 + 2 \sum_{n=1}^{\infty} \exp(-n^2 \tau) \cos(2nz).$$

In figure 2, we have plotted the analytically computed survival probability $S(t)$ versus t/L^2 for this case and compared with the results of simulation for $L = 100$ using 10^6 grains. We have also shown the result of the numerical integration of equation (2) to determine the scaling function in the case where the grains are added only at one end, and compared it with the simulation data obtained for $L = 100$ using 10^6 grains. Clearly, the agreement is excellent. This supports our arguments used to obtain equation (10).

The equation (10) is very easy to solve in the special case when sand grains are added randomly at any sites in the system. Clearly here $r(\vec{x}) = 1/V$ where V is the number of sites in the lattice. Then $n(\vec{x})$ is a solution to the equation

$$\nabla^2 n(\vec{x}) = -1/V. \tag{12}$$

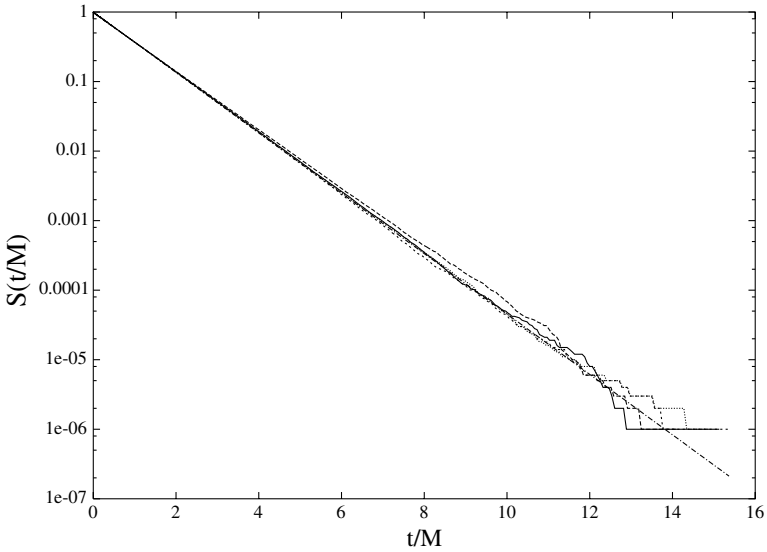

Figure 3. A semilog plot of the probability of survival of the marked grain as a function of the scaled residence time t/M for four different cases: (1) the one-dimensional BTW model, (2) the one-dimensional Manna model, (3) the two-dimensional BTW model, (4) the two-dimensional Manna model. We have chosen $L = 100$ in one dimension and the 70×70 cylindrical square lattice in two dimensions.

The function $P(x,t) = T(t)$, for all x, satisfies equation (10) in any dimension if

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -\frac{K}{2V}T(t).$$

With the initial condition $P(x, t = 0) = 1/V$, it is easy to see that the full solution is given by

$$P(\vec{x},t) = \frac{1}{V}\exp(-Kt/2V). \tag{13}$$

The probability of survival up to time t is $VP(\vec{x},t)$, and we see that it decays in time as a simple exponential. Using the fact that the mean residence time is the average mass $\bar{M}$ in the pile, we see that $K/2V = 1/\bar{M}$. This then implies that

$$S(t) = \exp(-t/\bar{M}). \tag{14}$$

We note that our derivation depends only on equations (7) and (9). These two equations are valid under the conditions of local conservation of sand grains, transfer of fixed number of grains at each toppling, and isotropy. Thus, the results would be equally applicable to models in which toppling conditions are different, or the transfer of particles is stochastic, as in the Manna model.

In figure 3, we have shown the results of a MC simulation study of the DRT in four different models:

(a) the one-dimensional BTW model for $L = 100$,

(b) the two-dimensional BTW model defined on a cylinder of size 50×50,

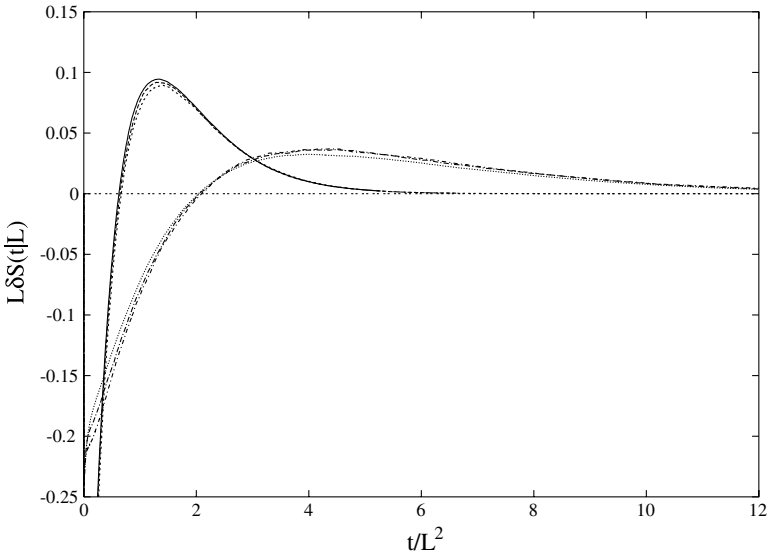

Figure 4. The scaled correction term $L\delta S(t|L)$ versus t/L^2 in the two-dimensional BTW and Manna model from simulations for the three different values $L = 13, 20, 30$. The curve with the higher peak is for the Manna model.

(c) the one-dimensional Manna model with $L = 100$ with the rule that if z_i exceeds 1, then two particles are transferred, each randomly to one of the neighbouring sites, and

(d) the two-dimensional Manna model on a 50×50 cylinder with two grains transferred at each toppling, each grain in a randomly chosen direction.

The number of particles used in each simulation was 10^6. We plotted the probability of survival of the marked grain as a function of time $t/\bar{M}$, where $\bar{M}$ was determined from the simulation directly. We see a very good collapse and agreement with the theoretical prediction that $\exp(-t/\bar{M})$ for $t \gg 1$.

For small t, the probability distribution is determined by the grains which are added very near the boundary. Boundary avalanches are not properly taken care of by our analysis. In particular, it is easy to see that the probability of an added grain coming out immediately is non-zero in d dimensions, and varies as $1/L$ for large L (this being the ratio of surface to volume). But equation (13) would give this as $\mathcal{O}(L^{-d})$. This comes from the fact that near the boundary the height distribution is modified, resulting in the effective K becoming different near the boundary. Equation (13) is valid for $t \gg 1$.

In figure 4, we have plotted the difference $\delta S(t|L)$ between the survival probability from MC simulation data and the scaling theory prediction (equation (14)), for a two-dimensional BTW and the Manna models on an $L \times L$ square lattice for different values of L. We find that the curves for different L collapse onto each other if $L\delta S(t|L)$ is plotted versus t/L^2, indicating that the correction $\delta S(t|L)$ to the equation (14) has the scaling form

$$\delta S(t|L) \sim \frac{1}{L} g(t/L^2), \tag{15}$$

where the correction-to-scaling functions $g(x)$ are different for the two models.

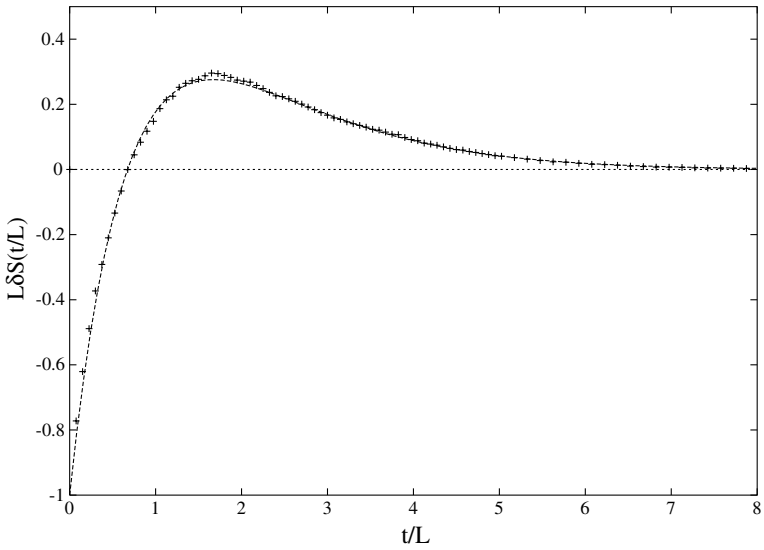

Figure 5. The finite size correction for the one-dimensional BTW model with grains added everywhere for $L = 40$. Plotted is the scaled deviation, $L\delta S(t/L)$, from the simple exponential scaling solution (equation (14)), for the simulation data and the theory (equation (16)).

For the one-dimensional BTW model, with grains added everywhere with equal probability, we can determine exactly the leading $\mathcal{O}(1/L)$ correction to the scaling solutions of equation (14). Thus, from the scaling solution of equation (13), we get $\mathrm{Prob}(T = 0|L) = 1/L + \mathcal{O}(L^{-2})$. But a straightforward calculation shows that actually $\mathrm{Prob}(T = 0|L) = 2/L + \mathcal{O}(L^{-2})$. However, $\mathrm{Prob}(t|L)$ for $t = 1, 2, \ldots$ is correctly given to the lowest order by $1/L$. Assuming that the remaining distribution is a simple exponential, we get

$$S_{1d\ \mathrm{BTW}}(t) = \frac{1}{L}\delta_{t,0} + \left(1 - \frac{1}{L}\right) \exp\left[-\frac{t}{L}\left(1 - \frac{a}{L}\right)\right], \tag{16}$$

where we have used the normalization condition $S(0) = 1$, and added a $\mathcal{O}(1/L)$ correction term to the coefficient in the exponential. Using the condition that the first moment of this distribution is $\bar{M} = L^2/(L + 1)$, we get $a = 1/2$. In figure 5, have shown the results of a Monte Carlo simulation of this case for $L = 40$ and number of grains equal to 10^6, and compared with the theoretical prediction (equation (16)). We see that the agreement is very good.

It is interesting to note that the DRT (equation (13)) has a simple universal form, and does not depend on the critical exponents for the distribution of avalanches, which differ for BTW and Manna models, and also depend on the dimension in a non-trivial way. In contrast, for the Oslo rice-pile model, Christensen *et al* [15] and Boguna and Corral [17] found that the DRT for the one-dimensional rice-pile of size L, with $L \gg 1$, does involve non-trivial exponents, and has the form

$$\mathrm{Prob}(T|L) = \frac{1}{L^{\nu}} f(T/L^{\nu}). \tag{17}$$

The exponent $\nu \simeq 1.3$ is related to the roughness of the rice-pile surface. The function $f(x)$ takes a constant value for x small, and varies as x^{-b} for large x, with $b \simeq 2.4$.

To study such models theoretically, we have to consider sand-pile models where there is a stack at each site into which grains are put, and there are specific rules about which grains leave the stack. For example, one could assume that incoming grains are put on the top of the stack, and outgoing ones leave from the top. This corresponds to the last-in–first-out rule. Alternatively, we can choose the first-in–first-out rule, or choose the outgoing grains probabilistically, with the probability of selection depending on the distance of the grain from the top of the stack. Clearly, the DRT will have different behaviour for different rules. We hope to address this question in a future publication.

For the Oslo rice-pile model, if we consider the grains that are not permanently stuck in the pile as constituting the 'active mass' of the rice-pile, all mass above the minimum slope of the pile is active. The configuration with the minimum slope is recurrent, and will recur infinitely often in the steady state as grains are added to the pile. Therefore, all the grains added after the pile has reached the minimum slope have a finite residence time in the pile. Then the argument given earlier in this paper implies that the mean active mass of the rice-pile should vary as L^2. But simulations in [15] and [17] estimated that $\bar{M} \simeq L^{1.3}$. The reason for the discrepancy in the estimate of the mean mass in [15, 17] is presumably the very long residence times of grains which happen to get deeply embedded, making the estimate of the first moment of the DRT from short simulations unreliable.

Acknowledgment

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