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Ion acoustic solitary wave in an inhomogeneous plasma with non-uniform temperature

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Abstract. A modified *K-dV* equation, which describes the propagation of an ion acoustic solitary wave in an inhomogeneous plasma with both density and temperature gradients, is derived. It is shown that, the velocity of a solitary wave increases as it propagates towards regions of increasing temperature.

Keywords. Inhomogeneous plasma; *K-dV* equation; soliton.

1. Introduction

It is well known (Washimi and Taniuti 1966; Davidson, 1972) that in a homogeneous plasma the *K-dV* equation (Korteweg and de Vries, 1895) gives a weakly nonlinear description of one-dimensional ion-sound wave disturbances travelling near the ion-sound speed. The *K-dV* equation possesses stationary, solitary wave solutions (Karpman 1973, Davidson 1972, Washimi and Taniuti 1966) which for the ion-acoustic waves result due to a balance between the steepening effects due to nonlinearity and the dispersion effects. In reality, however, the plasma is far from being homogeneous. Recently Nishikawa and Kaw (1975) have derived an equation describing the propagation of a weakly nonlinear ion-acoustic wave in a plasma with a density gradient. These authors have shown that the amplitude of a solitary wave (normalized to local density) decreases as it propagates towards the region of increasing density. This is because of the fact that the strength of the dispersion, which is proportional to $\lambda_D{}^2 = T_e/4\pi n_0 e^2$, where T_e is the electron temperature and n_0 is the equilibrium density, decreases as one move towards regions of increasing density. We note that, the strength of the dispersion gets further modified if on the top of a density inhomogeneity, a temperature inhomogeneity is also present. Based on this physical ground, in this paper, we investigate the propagation of a ion acoustic solitary wave in the presence of both density and temperature inhomogeneities.

We assume that the temperature gradient is produced by the presence of a finite thermal conductivity. Since the coefficient of thermal conductivity goes as $T^{5/2}$, where T is the temperature, thermal conduction is generally large for high temperature plasmas. Hence an appreciable temperature gradient is not to be expected in a high temperature plasma. Nevertheless, for not too high temperature plasmas in the presence of a magnetic field or in presence of collisions temperature gradient can be sustained. Because of this reason we shall assume that the temperature gradient scale-length is larger than the density gradient scale-length. In section 2, we assume that the equilibrium density gradients are produced by zero-order electric field E_0 and an effective gravity field. In the presence of a temperature gradient the electric field needed to sustain the density gradients is smaller than the one needed in the absence of temperature gradients. Due to the presence of the collisions which give rise to the finite thermal conductivity, one needs a smaller field to maintain the gradients. The absence of collisions simply shorts out the electric field and hence a higher field is required to maintain the gradients.

In section 2, using reductive perturbation method for a inhomogeneous plasma (Asano 1974) we have derived a modified *K-dV* equation governing the propagation of an ion acoustic wave in the presence of both density and temperature inhomogeneities. In section 3, we show that a solitary wave of a given amplitude propagating towards increasing temperature and decreasing density has a higher velocity compared to the case when temperature gradients are absent.

2. Derivation of the modified *K-dV* equation

The basic equations governing the system are the electron and ion momentum transfer equations, ion continuity equation and Poisson's equation, namely

$$en_e \frac{\partial \phi}{\partial x} - \frac{\partial}{\partial x}[n_e T_e(x)] = 0, \tag{1}$$

$$m_i \left[\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x}\right] + e\frac{\partial \phi}{\partial x} = m_i g, \tag{2}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}[n_i v_i] = 0 \tag{3}$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e [n_e - n_i]. \tag{4}$$

In eqs (1)–(4), ϕ is the potential, $n_{e,i}$ are the electron and ion densities respectively, $T_e(x)$ electron temperature, v_i is ion velocity, $-e$ is the electronic charge and g is acceleration due to the gravity field. Assuming that the scale-length for the temperature gradients is L_T, we write $T_e(x) = T_e(o)[1 + x/L_T]$. Thus, eq. (1) shows that the equilibrium density gradients are balanced by the zero order electric field $-e(\partial\phi_0/\partial x)$ and partly by the temperature gradients. If the scale-length for equilibrium density gradients be L_N one can write $N_e(x) = N_i(x) \equiv N(x) = N(o)[1 - x/L_N]$. Equation (2) shows that, for ions in equilibrium the gravitational force is balanced by the zero-order electric field.

We write the densities $n_{e,i} = N(x) + \triangle n_{e,i}(x, t) = N(x)[1 + \tilde{n}_{e,i}(x, t)]$, where $\tilde{n}_{e,i}(x, t)$ are the perturbed electron and ion densities respectively, normalized to the local equilibrium values. It is convenient for subsequent calculations, if we write eqs (1)–(4) in dimensionless form. For this reason we normalize ϕ to $T_e(o)/e$, lengths to Debye length $\lambda_D(o)$ $[\lambda_D^2(o) = T_e(o)/4\pi N(o)e^2]$ at $x = 0$, time to ion plasma period $\omega_{pi}^{-1}(o)\,[\omega_{pi}^2(o) = 4\pi N(o)\, e^2/m_i]$ and velocity to ion-sound speed $C_s(o)\,[C_s^2(o) = T_e(o)/m_i]$ at $x = 0$. In terms of these normalized quantities the equations governing the perturbed quantities can be written from eqs (1)-(4) as

$$\frac{\partial \Phi}{\partial x} + \tilde{n}_e \frac{\partial \Phi}{\partial x} - T_e(x) \frac{\partial \tilde{n}_e}{\partial x} = 0, \tag{5}$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{\partial \Phi}{\partial x} = 0, \tag{6}$$

$$\frac{\partial \tilde{n}_i}{\partial t} + v_i \frac{\partial \tilde{n}_i}{\partial x} + (1 + \tilde{n}_i) \frac{\partial v_i}{\partial x} - \bar{\kappa}(1 + \tilde{n}_i)\, v_i = 0 \tag{7}$$

and

$$\frac{N(o)}{N(x)} \frac{\partial^2 \Phi}{\partial x^2} = \tilde{n}_e - \tilde{n}_i, \tag{8}$$

where $\bar{\kappa} = -[\lambda_D(o)/N(x)]\,[dN(x)/dx]$ and $T_e(x) = 1 + \alpha\bar{\kappa}x$ with $\alpha = L_N/L_T$. We now introduce stretched variables (Asano, 1974) $\xi = \epsilon^{1/2}(x - t)$ and $\eta = \epsilon^{3/2} x$. The smallness parameter ϵ is defined in terms of the scale-length of the density gradients such that $\bar{\kappa} = \lambda_D(o)/L_N = \epsilon^{3/2}\kappa$. We shall also assume that $\alpha = O(\epsilon)$. The perturbed quantities can now be expanded as

$$\tilde{n}_i = \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \ldots$$

$$\tilde{n}_e = \epsilon n_e^{(1)} + \epsilon^2 n_e^{(2)} + \ldots$$

$$\Phi = \epsilon \Phi^{(1)} + \epsilon^2 \phi^2 + \ldots$$

and

$$v_i = \epsilon v_i^{(1)} + \epsilon^2 v_i^{(2)} + \ldots .$$

To the lowest order eqs (5)-(8) give, $n_i^{(1)} = n_i^{(1)} = \Phi^{(1)} = v_i^{(1)}$. To the next higher order, eqs (5)-(8) can be written as

$$\frac{\partial \Phi^{(2)}}{\partial \xi} - \frac{\partial n_e^{(2)}}{\partial \xi} + n_e^{(1)} \frac{\partial \Phi^{(1)}}{\partial \xi} - \alpha\eta \frac{\partial n_e^{(1)}}{\partial \xi} = 0, \tag{9}$$

$$-\frac{\partial v_i^{(2)}}{\partial \xi} + n_i^{(1)} \frac{\partial n_i^{(1)}}{\partial \xi} + \frac{\partial n_i^{(1)}}{\partial \eta} + \frac{\partial \phi^{(2)}}{\partial \xi} = 0, \tag{10}$$

$$-\frac{\partial n_i^{(2)}}{\partial \xi} + \frac{\partial v_i^{(2)}}{\partial \xi} + \frac{\partial n_i^{(1)}}{\partial \eta} + 2n_i^{(1)} \frac{\partial n_i^{(1)}}{\partial \xi} - \kappa n_i^{(1)} = 0 \tag{11}$$

and

$$\frac{N(o)}{N(x)} \frac{\partial^2 \Phi^{(1)}}{\partial \xi^2} = n_e^{(2)} - n_i^{(2)}. \tag{12}$$

By eliminating $\Phi^{(2)}$, $n_i^{(2)}$, $n_e^{(2)}$ and $v_i^{(2)}$ from eqs (9)-(12) one obtains:

$$\frac{\partial n_i^{(1)}}{\partial \eta} + (n_i^{(1)} + \tfrac{1}{2}\alpha\eta) \frac{\partial n_i^{(1)}}{\partial \xi} + \tfrac{1}{2} f(\eta) \frac{\partial^3 n_i^{(1)}}{\partial \xi^3} - \tfrac{1}{2}\kappa n_i^{(1)} = 0, \tag{13}$$

where $f(\eta) = N(o)/N(x)$. In writing eq. (13) use has been made of the relation $n_i^{(1)} = n_e^{(1)} = \Phi^{(1)} = v_i^{(1)}$. Equation (13) is a modified *K-dV* equation that governs the propagation of weakly nonlinear ion-acoustic waves in an inhomogeneous plasma with both density and temperature inhomogeneities. We notice that the term $\frac{1}{2}\alpha\eta\,(\partial n_i^{(1)}/\partial\xi)$ appears entirely due to temperature gradients. We also notice that, in terms of our stretched variables the coefficient of the dispersion term becomes a function of η only.

3. Steady state solution of equation (13)

By making a co-ordinate transformation of the type $\tau = \xi - \alpha\eta^2/4$ and $\theta = \eta$, eq. (13) can be written as

$$\frac{\partial n_i^{(1)}}{\partial\theta} + n_i^{(1)}\frac{\partial n_i^{(1)}}{\partial\tau} + \frac{1}{2}f(\theta)\frac{\partial^3 n_i^{(1)}}{\partial\tau^3} - \frac{1}{2}\kappa n_i^{(1)} = 0. \tag{14}$$

Now we introduce a change of variable $n = [f(\theta)]^{-\frac{1}{2}} n_i^{(1)}$ and a transformation,

$$\mu = [f(\theta)]^{\beta}\,\tau$$

and

$$v = \int^{\theta} [f(\theta')]^{\gamma}\, d\theta'.$$

For $\gamma = 1/4$ and $\beta = -1/4$, this transformation reduces eq. (14) to

$$\frac{\partial n}{\partial v} + n\frac{\partial n}{\partial\mu} + \frac{1}{2}\frac{\partial^3 n}{\partial\mu^3} = 0. \tag{15}$$

In writing eq. (15) terms of order $\beta\kappa\mu\,\partial/\partial\mu$ are neglected because of smallness of $\kappa\mu$ (Nishikawa and Kaw 1975). Equation (15) is a *K-dV* equation and if one assumes that n depends on μ and v only through $\Psi = (\mu - uv)$, one obtains a stationary 'soliton' solution (Davidson 1972) which is given by,

$$n = 3u\,\mathrm{sech}^2\left[\left(\frac{u}{2}\right)^{\frac{1}{2}}(\mu - uv)\right], \tag{16}$$

where u is the velocity of the solution in $(\mu - v)$ plane. Since

$f(\theta) = \dfrac{N(o)}{N(x)} \approx (1 + \kappa\theta) \approx \exp(\kappa\theta)$, eq. (16) can be written as

$$n_i^{(1)} = 3u\exp\left(\frac{\kappa\eta}{2}\right)\mathrm{sech}^2\left[\left(\frac{u}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{\kappa\eta}{4}\right)\left\{\xi - \left(\frac{\alpha\eta^2}{4} + \frac{4u}{\kappa}e^{\kappa\eta/2}\right)\right\}\right] \tag{17}$$

Therefore, the absolute density perturbation is given as

$$\triangle n_i = 3uN(x)\exp\left(\frac{\kappa\eta}{2}\right)\mathrm{sech}^2\left[\left(\frac{u}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{\kappa\eta}{4}\right)\right.$$
$$\left.\times\left\{\xi - \left(\frac{\alpha\eta^2}{4} + \frac{4u}{\kappa}e^{\kappa\eta/2}\right)\right\}\right] \tag{18}$$