Eulerian Walkers as a model of Self-Organised Criticality

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We propose a new model of self-organized criticality. A particle is dropped at random on a lattice and moves along directions specified by arrows at each site. As it moves, it changes the direction of the arrows according to fixed rules. On closed graphs these walks generate Euler circuits. On open graphs, the particle eventually leaves the system, and a new particle is then added. The operators corresponding to particle addition generate an abelian group, same as the group for the Abelian Sandpile model on the graph. We determine the critical steady state and some critical exponents exactly, using this equivalence.

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In recent years, there has been much interest in the study of systems showing self-organized criticality (SOC)\textsuperscript{1} and different models have been proposed for many systems such as sandpiles\textsuperscript{2}, earthquakes\textsuperscript{3}, forest-fires\textsuperscript{4} and biological evolution\textsuperscript{5}. All these models involve a slowly driven system, in which the externally-introduced disturbance propagates in a random medium using deterministic or stochastic rules. In the process it modifies the medium so that after many such disturbances, the medium develops long-range spatial correlations\textsuperscript{6}.

The most analytically tractable of all these models has been the Abelian sandpile model (ASM)\textsuperscript{6,7}. In this Letter we introduce a new model of SOC called the Eulerian walkers model (EWM). This model is quite different from the ASM in some ways, but shares with it the abelian group property. This allows a determination of the steady state and exact calculation of some critical exponents. In fact we define a general abelian model of which both the EWM and ASM are special cases.

In the EWM self-organization occurs due to activity of a walker which moves deterministically in a medium while also modifying it. We show that on closed graphs, the walker finally settles into a limit cycle which is an Eulerian walk ending at a site\textsuperscript{j} on the graph. On open graphs, the particle eventually leaves the system. We then add at a randomly chosen site another particle, which then moves, and so on. We define particle addition operators which act on the set of recurrent configurations of the system. These operators generate an abelian group, and satisfy same closure relations between themselves as in the ASM on the same graph. We show that recurrent configurations of the system are in one to one correspondence with spanning trees on the lattice.

The model is defined for a general graph as follows: consider a connected oriented graph \( G \) consisting of \( N \) points \( i = 1,2,...,N \). A point \( j \) has \( \tau_j \) outgoing bonds, and an equal number of incoming bonds, connecting it to other points. The outgoing bonds at \( j \) are labelled by integers from 1 to \( \tau_j \). We associate with each point, an arrow which can point along one of the outgoing bonds (Fig. 1). Let \( n_j \) (\( 1 \leq n_j \leq \tau_j \)) denote the current direction of the arrow that is the label of the bond along which the arrow points. The set \( \{ n_j \} \) specifies the arrow directions at all points and provides a complete description of the arrow configuration of the medium.

We now put a walker at some point on the graph. At each time step:

(i) the walker after arriving at a site \( j \) changes the arrow direction from \( n_j \) to \( n_j + 1 \text{mod} \tau_j \),

(ii) the walker moves one step from \( j \) along the new arrow direction at \( j \).

Thus the motion of the walker is deterministic, is affected by the medium and in turn affects the medium. We can interpret the rules (i),(ii) as an intention of the walker to maximize intervals between successive visits of the same bond each time the walker leaves a given site. The current position of the walker along with the value of the variable \( n_j \) at every site \( j \) specifies completely the state of the system\textsuperscript{8}.

In the absence of sinks the walker continues to walk forever. Since the system (walker+medium) has a finite number of possible states, it eventually settles into a limit cycle. In general, one would expect the size of these cycles to be of the order of Poincare recurrence times for the system, and grow exponentially with \( N \). The surprising fact is that all the cycles are very short and, in fact, are of the same length \( \sum_{j=1}^{N} \tau_j \). In each such cycle, all bonds are visited exactly once (Fig. 1). Such walks are known as Euler circuits\textsuperscript{9} and their study has been an important problem in lattice statistics (If the circuit visits all sites exactly once, it is called a Hamilton circuit). There is a one to one correspondence between Euler circuits and spanning trees on the graph. Clearly for any Euler walk ending at a site \( j \), last exit bonds from all sites other than \( j \) form a spanning tree rooted at \( j \). Kasteleyn also showed that each rooted spanning tree corresponds to a unique Euler circuit.\textsuperscript{10} The number of all possible
trees is known to be given in terms of the determinant of the adjacency matrix by the well known matrix tree theorem \[10\].

We now show that every limit cycle is an Euler circuit. We start from some arbitrary initial state of the medium with the walker at some point \(i\). The walker leaves the point \(i\) along some bond \(b_1\). We evolve it till after time \(T\) it returns to \(b_1\) for the first time. Let the bond traversed at the \(j\)th step of path be \(b_j\), so that the path is \(b_1b_2\ldots b_T\) with \(b_{T+1} = b_1\). We can show that no other bond in this path is visited twice. Proof: Assume the contrary and suppose that during the \(T\) steps the bond \(c\), originating from the point \(j\), is the first bond that is visited twice. Each successive exit from \(j\) is along a different direction so there will be \(\tau_j + 1\) exits. But the number of visits to \(j\) equals number of exits. Hence there must exist some bond going into \(j\) which is also passed more than once. This contradicts the fact that \(c\) was taken to be the first bond to be passed twice. Thus we get \(\sum_{i=1}^N \tau_i \leq N\).

We keep shifting the path thus till we reach a \(t\) such that \(b_{T+t} \neq b_t\). Clearly \(T < T'\). Now define the new circuit formed by the \(T'\) steps starting with the \(t\)th step. Iterating this we get circuits of increasing lengths \(T < T' < T''\ldots\) where each is \(\leq \sum_{i=1}^N \tau_i\) and so finally we will get an Euler circuit when \(T = \sum_{i=1}^N \tau_i\). All the configurations which the system goes through before it enters the cycle are transients.

To illustrate the process of self-organization, consider the motion of a walker on an infinite line starting with a random initial configuration of the medium. This walk has a simple structure (Fig. 2). The walker turns the arrow at the origin and starts the motion along the new direction of this arrow reversing the arrows at all sites it passes through. It moves on till it encounters a site with an arrow pointing in the direction of motion. The walker now reverses its direction and retraces its path entirely, passing over all the sites traversed since the last reversal of its direction. Then it continues to move ahead till it again encounters an arrow pointing in the direction of motion and so on. Thus arrows in the region already visited get organized into an almost Eulerian circuit so that, if at time \(t\) the number of sites visited is \(S(t)\), then in the previous \(2S(t)\) time steps, most of these sites have been visited exactly 2 times. In addition, the boundary of the cluster advances by a finite amount \(\Delta\), as some new sites are visited. For compact clusters \(S(t) \sim R(t)\), the average distance of the of the walker from the origin, at time \(t\). Thus we get

\[
\frac{dR(t)}{dt} \sim \frac{\Delta}{R}.
\] (1)

which implies that \(R \sim t^\frac{1}{2}\) for large \(t\). The average number of sites visited till time \(t\), \(S(t)\), goes as \(t^\frac{1}{2}\).

In higher dimensions, the motion of the EW is not so simple. In Fig. 3 we show the results of a simulation of the model on a square lattice with random initial configuration of arrows. Before each step the arrow is turned clockwise by \(90^\circ\). The sites visited at least once by the walker form a cluster with few holes, whose radius \(R(t)\) increases with time \(t\). In the region visited by the walker, all arrows are not aligned parallel, but are organized into an almost Euler circuit so that in the time between \(T\) and \(T - 4S(T)\), only a very small fraction of sites is not visited exactly 4 times (here \(\tau_j = 4\) for all sites \(j\)). Arguing as in the one dimensional case we get \(R \sim t^\frac{1}{4}\) for large \(t\). We have carried out Monte-carlo simulations and verified this to very good accuracy.

However, for \(d > 2\), a random walker does not return to previously visited sites often enough, and we expect the motion of a walker in an initially random medium to be diffusive \((R^2 \sim t)\). Our numerical simulations show that this is indeed the case for \(d = 3\).

Now consider an open graph for which all the external perimeter sites are identified with a single sink site, \(i_0\), at which the walker gets absorbed. We place a walker at some point \(i\), with probability \(p_i\) (\(\sum p_i = 1\)), and let it evolve according to the rules specified before, until it leaves the system [The walker will not get into a cycle as every cycle would contain all points of the graph including \(i_0\)]. Now the system is specified only by the values of \(n_i\), \(i = 1, N\). We define operators \(a_i\) acting on the space of recurrent configurations of the EWM as follows: for any recurrent configuration \(C\), \(a_iC = C'\), where \(C'\) is the resulting configuration of the medium obtained by adding a particle at site \(i\) on the configuration \(C\), and evolving it until it leaves the system.

It is easy to see that the operators at different sites commute. Treat the motion of each walker as a sequence of elementary steps. Then if two particles (walkers) are added to the lattice at sites \(j\) and \(j'\), the elementary moves on two sites \(j \neq j'\) commute. If \(j = j'\), they also commute due to identity of particles. Therefore

\[
[a_i, a_j] = 0 .
\] (2)

Within the space of recurrent configurations the operators \(a_i\) will have unique inverses. If we define the \(N \times N\) matrix, \(\Delta\), such that \(\Delta_{ij}\) gives the number of outgoing bonds from \(i\) and \(-\Delta_{ij}\) gives the number of bonds from \(i\) to \(j\) then

\[
\prod_j a_i^ {\Delta_{ij}} = I , \text{ for all } i
\] (3)

which simply reflects the fact that \(\tau_j\) particles added at \(i\) produce the same effect as 1 particle added at each
mediate times where the motion of the EW may lead to
rows form a spanning tree. This is not true for inter-

system, the medium is in a recurrent state, and the ar-
hence exhibits self-organized criticality.

steady state of the model has long range correlations and
$R$ varies as $\phi$ varies as $R$.

of spanning trees with these bonds occupied to the num-

Average number of steps $s$ taken by the walker till it
leaves the system is given by $< n > = z < s >_{ASM}$, where $s$ is the number of topplings in ASM avalanches,
for regular graphs with coordination number $z$. Hence
$n \sim L^2$, where $L$ is the length of the system.

It is quite straightforward to calculate the arrow-arrow
correlation function in the steady state using the equiv-
ance of the problem to spanning trees. For two given
sites $R_i$ and $R_j$, the probabilities that arrows at the sites
are in the directions $e_1^i$ and $e_2^j$ respectively is the ratio
of spanning trees with these bonds occupied to the num-
ber of all spanning trees. This is easily calculated. For
large $R_{12}$ the leading term in the connected part of this
probability is given by

$$C(R_{12}; e_1^i, e_2^j) \sim (e_1^i, \vec{\nabla} \phi(R_{12})) (e_2^j, \vec{\nabla} \phi(R_{12}))$$

where $\phi(R_{12}) = G_{R_1 R_2} - G_{R_1 R_1}$. In $d$ dimensions $\phi(R)$
vary as $R^{2-d}$, hence the correlation function $C(R)$
vary as $R^{2-2d}$ for large separations $R$.

Thus the steady state of the model has long range correlations and
hence exhibits self-organized criticality.

As pointed out earlier, when the walker has left the
system, the medium is in a recurrent state, and the ar-
rows form a spanning tree. This is not true for inter-
mediate times where the motion of the EW may lead to

a cyclic configuration of arrows. Thus a typical evolu-
tion of medium has periods of cyclicity interspersed be-

between ‘normal’ acyclic states. In the EWM, the durations
of these intervals of cyclicity have a power-law distribution.

In two dimensions, numerical simulations [13] show
that the probability of intervals of cyclicity of duration $\tau$
varies approximately as $1/\tau^{1.75}$.

Though we can establish a one to one correspondence
between the recurrent configurations of the EWM and
ASM, the relaxation process in the two models is quite
different. In the latter (in more than one dimensions) in
almost all cases particle addition leads to a stable config-
uration after a finite number of topplings, and the fraction
of avalanches which reach the boundary is very small.

In contrast, in the present model, each walker must travel to
the boundary before it leaves the system, and thus the
fraction of events involving a finite number of steps of
the walker is zero in the limit of large system sizes. This

leads to the interesting conclusion that the statistics of
avalanches is not completely determined by the operator
algebra of ASM.

However, in one dimension, the probability distribu-
tion of number of steps can be computed exactly, and we
find that apart from trivial numerical factors, it is ex-
actly the same form as the limiting distribution found by
Ruelle and Sen [14] for avalanches in the 1d ASM.

To bring out the relationship of the present model to
the ASM more clearly, we observe that due to the abelian
nature of the evolution rules, we can add and evolve two
or more walkers in the system in arbitrary order without
ffecting the final state. Let us choose the following rules:
each walker arriving at a site waits there until the number
of particles waiting at that site is $\geq r$. Then these $r$
particles take 1 step each in the directions $n_j + 1, n_j + 2 ... n_j + r$,
and the arrow is reset to $n_j + r (mod r)$. Clearly $r = 1$
corresponds to the EWM, and $r = r_j$ corresponds to the
ASM. In the latter case the arrow configuration does not
evolve at all, and may be omitted from discussion.

In Fig. 4, we have shown the results of Monte Carlo simulation
of this general model on a square lattice of size $200 \times 200$
for $r = 1$ to $4$. We see that we get the same general be-
aviour of distribution of avalanches for all $r > 1$, but the
case $r = 1$ is special. It belongs to a different universality
class. For small $s$, the distribution $P(s)$ is dominated by
boundary avalanches, therefore, it does not have a simple
thermodynamic limit. However, the model has long
range correlations, and hence is critical.

In brief, we have introduced a new analytically
tractable model of SOC. It is hoped that further studies of
the model will contribute to a better understanding of
self-organizing systems in general.
[8] A model of self-organization using somewhat similar rules was earlier defined by Langton. The emphasis was, however, on evolution of complex structures by deterministic rules, starting from simple initial conditions. Langton, Physica D 22, 120 (1986).
[9] “Graph Theory” by F. Harary, Addison-Wesley, chapters 7 and 16.
[12] The decay of correlations can be faster in special directions if \( e_1 \) or \( e_2 \) are perpendicular to \( \mathbf{R}_{12} \) so that the coefficient of \( R^{d-2d} \) vanishes.

Figure Captions

FIG. 1. (a) A directed graph. The outgoing bonds at each site are labelled by integers 1, 2,... An initial state with a configuration of arrows as in (b) and a walker starting at the site \( a \) moves along the path \( abc... \) which eventually settles to the Euler circuit \( abedcabc \).

FIG. 2. A random initial state of a lattice in one dimension and the motion of a walker on this lattice. The medium is organized into a state in which all arrows point in the same direction.

FIG. 3. Simulation of the Euler walk on a square lattice with random initial conditions. The whole cluster consists of sites covered by the walker after \( 10^5 \) steps. The white region shows the cluster of approximately 12500 sites visited exactly four times in the last 50,000 steps. The grey sites at the boundary of the cluster are visited less than four times.

FIG. 4. Plot of probability \( P(s) \) of avalanche of size \( s \) vs. \( s \) for different values of \( r \).
figure 1
figure 3