Eulerian Walkers as a model of Self-Organised Criticality

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We propose a new model of self-organized criticality. A particle is dropped at random on a lattice and moves along directions specified by arrows at each site. As it moves, it changes the direction of the arrows according to fixed rules. On closed graphs these walks generate Euler circuits. On open graphs, the particle eventually leaves the system, and a new particle is then added. The operators corresponding to particle addition generate an abelian group, same as the group for the Abelian Sandpile model on the graph. We determine the critical steady state and some critical exponents exactly, using this equivalence.

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In recent years, there has been much interest in the study of systems showing self-organized criticality (SOC) [1] and different models have been proposed for many systems such as sandpiles [1], earthquakes [2], forest-fires [3] and biological evolution [4]. All these models involve a slowly driven system, in which the externally-introduced disturbance propagates in a random medium using deterministic or stochastic rules. In the process it modifies the medium so that after many such disturbances, the medium develops long-range spatial correlations [5].

The most analytically tractable of all these models has been the Abelian sandpile model (ASM) [6,7]. In this Letter we introduce a new model of SOC called the Eulerian walkers model (EWM). This model is quite different from the ASM in some ways, but shares with it the abelian group property. This allows a determination of the steady state and exact calculation of some critical exponents. In fact we define a general abelian model of which both the EWM and ASM are special cases.

In the EWM self-organization occurs due to activity of a walker which moves deterministically in a medium while also modifying it. We show that on closed graphs, the walker finally settles into a limit cycle which is an Euler circuit visiting each directed bond exactly once in one cycle. On open graphs, the particle eventually leaves the system. We then add at a randomly chosen site another particle, which then moves, and so on. We define particle addition operators which act on the set of recurrent configurations of the system. These operators generate an abelian group, and satisfy same closure relations between themselves as in the ASM on the same graph. We show that recurrent configurations of the system are in one to one correspondence with spanning trees on the lattice.

The model is defined for a general graph as follows: consider a connected oriented graph G consisting of N points $i = 1, 2...N$. A point j has τ_j outgoing bonds, and an equal number of incoming bonds, connecting it to other points. The outgoing bonds at j are labelled by integers from 1 to τ_j. We associate with each point, an arrow which can point along one of the outgoing bonds (Fig. 1). Let n_j ($1 \leq n_j \leq \tau_j$) denote the current direction of the arrow that is the label of the bond along which the arrow points. The set $\{n_j\}$ specifies the arrow directions at all points and provides a complete description of the arrow configuration of the medium .

We now put a walker at some point on the graph. At each time step:
(i) the walker after arriving at a site j changes the arrow direction from n_j to $n_j + 1(\mathrm{mod}\tau_j)$,
(ii) the walker moves one step from j along the new arrow direction at j.

Thus the motion of the walker is deterministic, is affected by the medium and in turn affects the medium . We can interpret the rules (i),(ii) as an intention of the walker to maximize intervals between successive visits of the same bond each time the walker leaves a given site. The current position of the walker along with the value of the variable n_j at every site j specifies completely the state of the system [8].

In the absence of sinks the walker continues to walk forever. Since the system (walker+medium) has a finite number of possible states, it eventually settles into a limit cycle. In general, one would expect the size of these cycles to be of the order of Poincare recurrence times for the system, and grow exponentially with N. The surprising fact is that all the cycles are very short and, in fact, are of the same length $\sum_{j=1}^{N} \tau_j$. In each such cycle, all bonds are visited exactly once (Fig. 1). Such walks are known as Euler circuits [9] and their study has been an important problem in lattice statistics (If the circuit visits all *sites* exactly once, it is called a *Hamilton* circuit). There is a one to one correspondence between Euler circuits and spanning trees on the same graph. Clearly for any Euler walk ending at a site j, last exit bonds from all sites other than j form a spanning tree rooted at j. Kasteleyn also showed that each rooted spanning tree corresponds to a unique Euler circuit [10]. The number of all possible trees is known to be given in terms of the determinant of the adjacency matrix by the well known matrix tree theorem [10].

We now show that every limit cycle is an Euler circuit. We start from some arbitrary initial state of the medium with the walker at some point i. The walker leaves the point i along some bond b_1. We evolve it till after time T it returns to b_1 for the first time. Let the bond traversed at the jth step of path be b_j, so that the path is $b_1 b_2 ... b_T$ with $b_{T+1} = b_1$. We can show that no other bond in this path is visited twice. Proof: Assume the contrary and suppose that during the T steps the bond c, originating from the point j, is the first bond that is visited twice. Each successive exit from j is along a different direction so there will be $\tau_j + 1$ exits. But the number of visits to j equals number of exits. Hence there must exist some bond going into j which is also passed more than once. This contradicts the fact that c was taken to be the first bond to be passed twice. Thus all b_i, $i = 1$ to T are distinct and hence $T \leq \sum_{i=1}^{N} \tau_i$. If every bond in G is visited we have an Euler circuit with $T = \sum_{i=1}^{N} \tau_i$. If not, consider the path $b_2 b_3 ... b_{T+1} b_{T+2}$. If $b_{T+2} = b_2$ we have another circuit of length T. We keep shifting the path thus till we reach a t such that $b_{T+t} \neq b_t$. Such a $t < T$ exists so long as there are points j on the path which have not been visited τ_j number of times. Let T' be the first time when $b_{T'+t} = b_t$. Clearly $T < T'$. Now define the new circuit formed by the T' steps starting with the tth step. Iterating this we get circuits of increasing lengths $T < T' < T''...$ where each is $\leq \sum_{i=1}^{N} \tau_i$ and so finally we will get an Euler circuit when $T = \sum_{i=1}^{N} \tau_i$. All the configurations which the system goes through before it enters the cycle are transients.

To illustrate the process of self-organization, consider the motion of a walker on an infinite line starting with a random initial configuration of the medium. This walk has a simple structure (Fig. 2). The walker turns the arrow at the origin and starts the motion along the new direction of this arrow reversing the arrows at all sites it passes through. It moves on till it encounters a site with an arrow pointing in the direction of motion. The walker now reverses its direction and retraces its path entirely, passing over all the sites traversed since the last reversal of its direction. Then it continues to move ahead till it again encounters an arrow pointing in the direction of motion and so on. Thus the arrows in the region already visited get organized into an almost Eulerian circuit so that, if at time t the number of sites visited is $S(t)$, then in the previous $2S(t)$ time steps, most of these sites have been visited exactly 2 times. In addition, the boundary of the cluster advances by a finite amount Δ, as some new sites are visited. For compact clusters $S(t) \sim R(t)$, the average distance of the of the walker from the origin, at time t. Thus we get

$$\frac{dR(t)}{dt} \sim \frac{\Delta}{R} \ . \tag{1}$$

which implies that $R \sim t^{\frac{1}{2}}$ for large t. The average number of sites visited till time t, $S(t)$, goes as $t^{\frac{1}{2}}$.

In higher dimensions, the motion of the EW is not so simple.In Fig. 3 we show the results of a simulation of the model on a square lattice with random initial configuration of arrows. Before each step the arrow is turned clockwise by 90^o. The sites visited at least once by the walker form a cluster with few holes, whose radius $R(t)$ increases with time t. In the region visited by the walker, all arrows are not aligned parallel, but are organized into an almost Euler circuit so that in the time between T and $T - 4S(T)$, only a very small fraction of sites is *not* visited *exactly* 4 times (here $\tau_j = 4$ for all sites j). Arguing as in the one dimensional case we get $R \sim t^{\frac{1}{3}}$ for large t. We have carried out Monte-carlo simulations and verified this to very good accuracy.

However, for $d > 2$, a random walker does not return to previously visited sites often enough, and we expect the motion of a walker in an initially random medium to be diffusive ($R^2 \sim t$). Our numerical simulations show that this is indeed the case for $d = 3$.

Now consider an open graph for which all the external perimeter sites are identified with a single sink site, i_0, at which the walker gets absorbed. We place a walker at some point i, with probability p_i ($\sum p_i = 1$), and let it evolve according to the rules specified before, until it leaves the system [The walker will not get into a cycle as every cycle would contain all points of the graph including i_0]. Now the system is specified only by the values of n_i, $i = 1, N$. We define operators a_i acting on the space of recurrent configurations of the EWM as follows: for any recurrent configuration C, $a_i C = C'$, where C' is the resulting configuration of the medium obtained by adding a particle at site i on the configuration C, and evolving it until it leaves the system.

It is easy to see that the operators at different sites commute. Treat the motion of each walker as a sequence of elementary steps. Then if two particles (walkers) are added to the lattice at sites j and j', the elementary moves on two sites $j \neq j'$ commute. If $j = j'$, they also commute due to identity of particles. Therefore

$$[a_i, a_j] = 0 \ . \tag{2}$$

Within the space of recurrent configurations the operators a_i will have unique inverses. If we define the $N \times N$ matrix, Δ, such that Δ_{ii} gives the number of outgoing bonds from i and $-\Delta_{ij}$ gives the number of bonds from i to j then

$$\prod_j a_i^{\Delta_{ij}} = I \quad , \text{ for all i} \tag{3}$$

which simply reflects the fact that τ_i particles added at i produce the same effect as 1 particle added at each nearest neighbor of i. Thus, operators a_i satisfy the same algebra as the particle addition operators in the ASM. In fact one can define commuting operators $a_i(\epsilon)$, which have a phase factor proportional to the number of steps taken by the walker in going from C to $C^{'}$ as in the case of the ASM [11].

So, as in the ASM, the number of recurrent configurations $R = Det(\Delta)$ and they occur with equal probability. The first result also follows from the one to one correspondence between steady state configurations of the EWM and spanning trees which is obtained by drawing the last exit directions at all points. Since the a_i's commute, they can be diagonalized simultaneously. Then as for the ASM Eq.(3) determines all the eigenvalues of a's. Thus one can diagonalize the evolution operator $W = \sum p_i a_i$, where p_i is the probability that a new particle is added at site i. For a lattice of size L on a d-dimensional lattice, this implies that the largest relaxation time of the system varies as L^d.

Let G_{ij} be the expected number of full rotations of the arrow at the site j due to addition of a walker at the site i. During the walk, the expected number of steps leaving j is $G_{ij}\Delta_{jj}$ whereas $-\sum_{k\neq j} G_{ik}\Delta_{kj}$ is the average flux into j. Equating both fluxes one gets

$$\sum_k G_{ik}\Delta_{kj} = \delta_{ij} \qquad \text{or}$$

$$G_{ij} = [\Delta^{-1}]_{ij}. \tag{4}$$

Average number of steps n taken by the walker till it leaves the system is given by $< n >= z < s >_{ASM}$, where s is the number of topplings in ASM avalanches, for regular graphs with coordination number z. Hence $< n > \sim L^2$, where L is the length of the system.

It is quite straightforward to calculate the arrow-arrow correlation function in the steady state using the equivalence of the problem to spanning trees. For two given sites $\vec{R_1}$ and $\vec{R_2}$ the probabilities that arrows at the sites are in the directions $\vec{e_1}$ and $\vec{e_2}$ respectively is the ratio of spanning trees with these bonds occupied to the number of all spanning trees. This is easily calculated. For large R_{12} the leading term in the connected part of this probability is given by

$$C(\vec{R_{12}};\vec{e_1},\vec{e_2}) \sim (\vec{e_1}.\vec{\nabla}\phi(\vec{R_{12}}))(\vec{e_2}.\vec{\nabla}\phi(\vec{R_{12}})) \tag{5}$$

where $\phi(\vec{R_{12}}) = G_{\vec{R_1}\vec{R_2}} - G_{\vec{R_1}\vec{R_1}}$. In d dimensions $\phi(R)$ varies as R^{2-d}, hence the correlation function $C(R)$ varies as R^{2-2d} for large separations R [12]. Thus the steady state of the model has long range correlations and hence exhibits self-organized criticality.

As pointed out earlier, when the walker has left the system, the medium is in a recurrent state, and the arrows form a spanning tree. This is not true for intermediate times where the motion of the EW may lead to a cyclic configuration of arrows. Thus a typical evolution of medium has periods of cyclicity interspersed between 'normal' acyclic states. In the EWM, the durations of these intervals of cyclicity have a power-law distribution. In two dimensions, numerical simulations [13] show that the probability of intervals of cyclicity of duration τ varies approximately as $1/\tau^{1.75}$.

Though we can establish a one to one correspondence between the recurrent configurations of the EWM and ASM, the relaxation process in the two models is quite different. In the latter (in more than one dimensions) in almost all cases particle addition leads to a stable configuration after a finite number of topplings, and the fraction of avalanches which reach the boundary is very small. In contrast, in the present model, each walker must travel to the boundary before it leaves the system, and thus the fraction of events involving a finite number of steps of the walker is zero in the limit of large system sizes. This leads to the interesting conclusion that the statistics of avalanches is not completely determined by the operator algebra of ASM.

However, in one dimension, the probability distribution of number of steps can be computed exactly, and we find that apart from trivial numerical factors, it is exactly the same form as the limiting distribution found by Ruelle and Sen [14] for avalanches in the 1d ASM.

To bring out the relationship of the present model to the ASM more clearly, we observe that due to the abelian nature of the evolution rules, we can add and evolve two or more walkers in the system in arbitrary order without effecting the final state. Let us choose the following rules: each walker arriving at a site waits there until the number of particles waiting at that site is $\geq$ r. Then these r particles take 1 step each in the directions n_j+1,$n_j+2...n_j+r$, and the arrow is reset to $n_j + r(\text{mod}\tau_j)$. Clearly $r = 1$ corresponds to the EWM, and $r = \tau_j$ corresponds to the ASM. In the latter case the arrow configuration does not evolve at all, and may be omitted from discussion.In Fig. 4, we have shown the results of Monte Carlo simulation of this general model on a square lattice of size 200×200 for $r = 1$ to 4. We see that we get the same general behaviour of distribution of avalanches for all $r > 1$, but the case $r = 1$ is special. It belongs to a different universality class. For small s, the distribution $P(s)$ is dominated by boundary avalanches, therefore, it does not have a simple thermodynamic limit. However, the model has long range correlations, and hence is critical.

In brief, we have introduced a new analytically tractable model of SOC. It is hoped that further studies of the model will contribute to a better understanding of self-organizing systems in general.

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Figure Captions

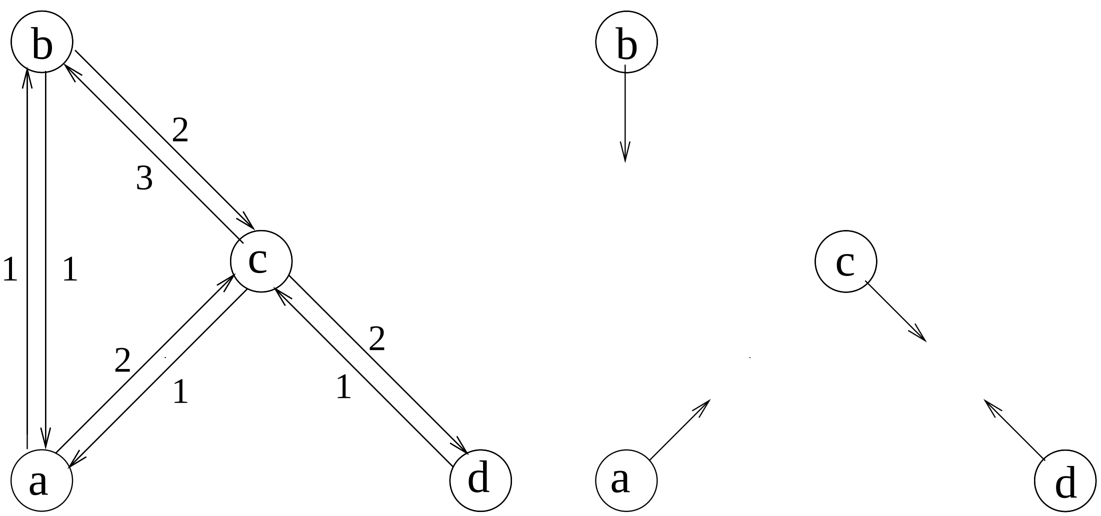
FIG. 1. (a) A directed graph. The outgoing bonds at each site are labelled by integers $1, 2....$ An initial state with a configuration of arrows as in (b) and a walker starting at the site a moves along the path $abcb...$ which eventually settles to the Euler circuit $abcdcbaca$.

FIG. 2. A random initial state of a lattice in one dimension and the motion of a walker on this lattice. The medium is organized into a state in which all arrows point in the same direction.

FIG. 3. Simulation of the Euler walk on a square lattice with random initial conditions. The whole cluster consists of sites covered by the walker after 10^5 steps. The white region shows the cluster of approximately 12500 sites visited exactly four times in the last $50,000$ steps. The grey sites at the boundary of the cluster are visited less than four times.

FIG. 4. Plot of probability $P(s)$ of avalanche of size s vs. s for different values of r.

figure 1



(a) (b)

figure 2

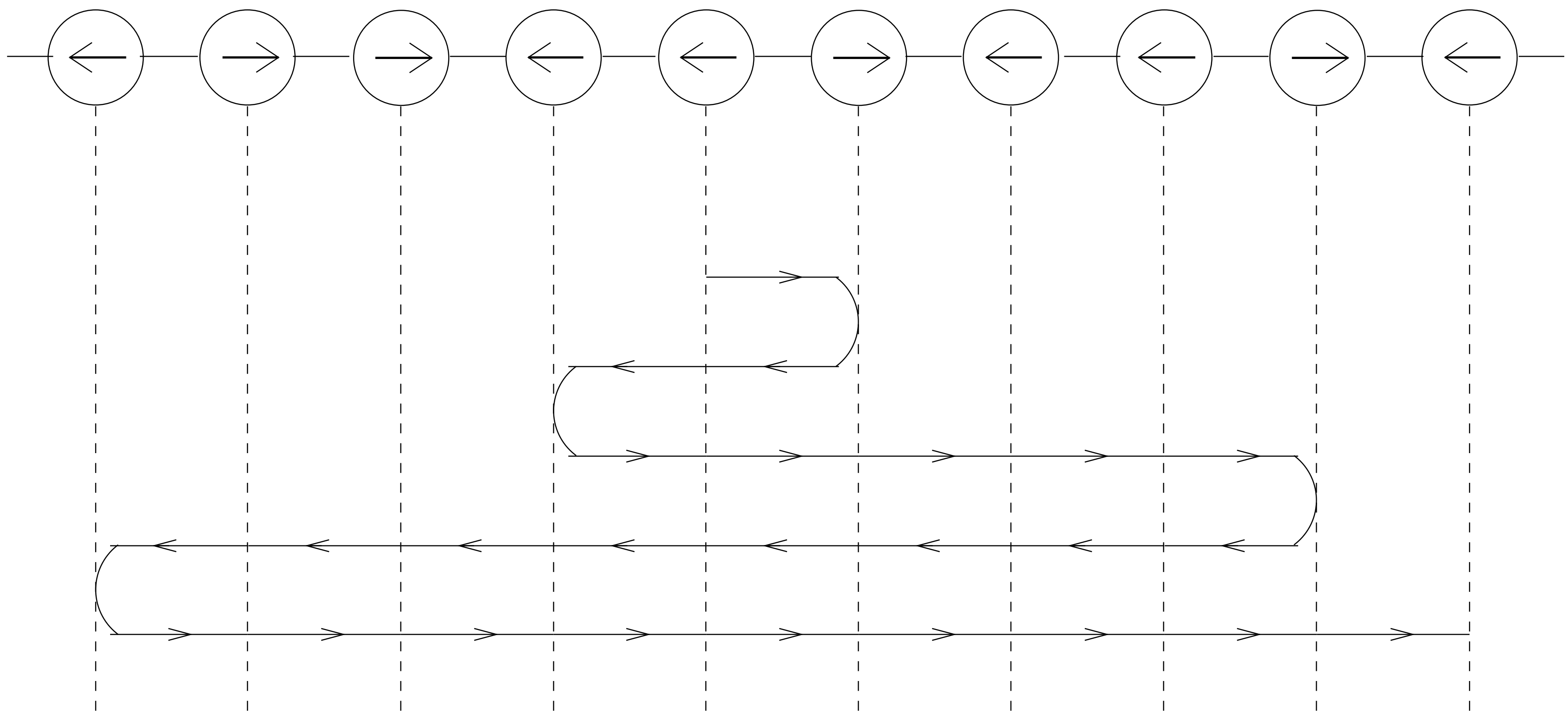


figure 3

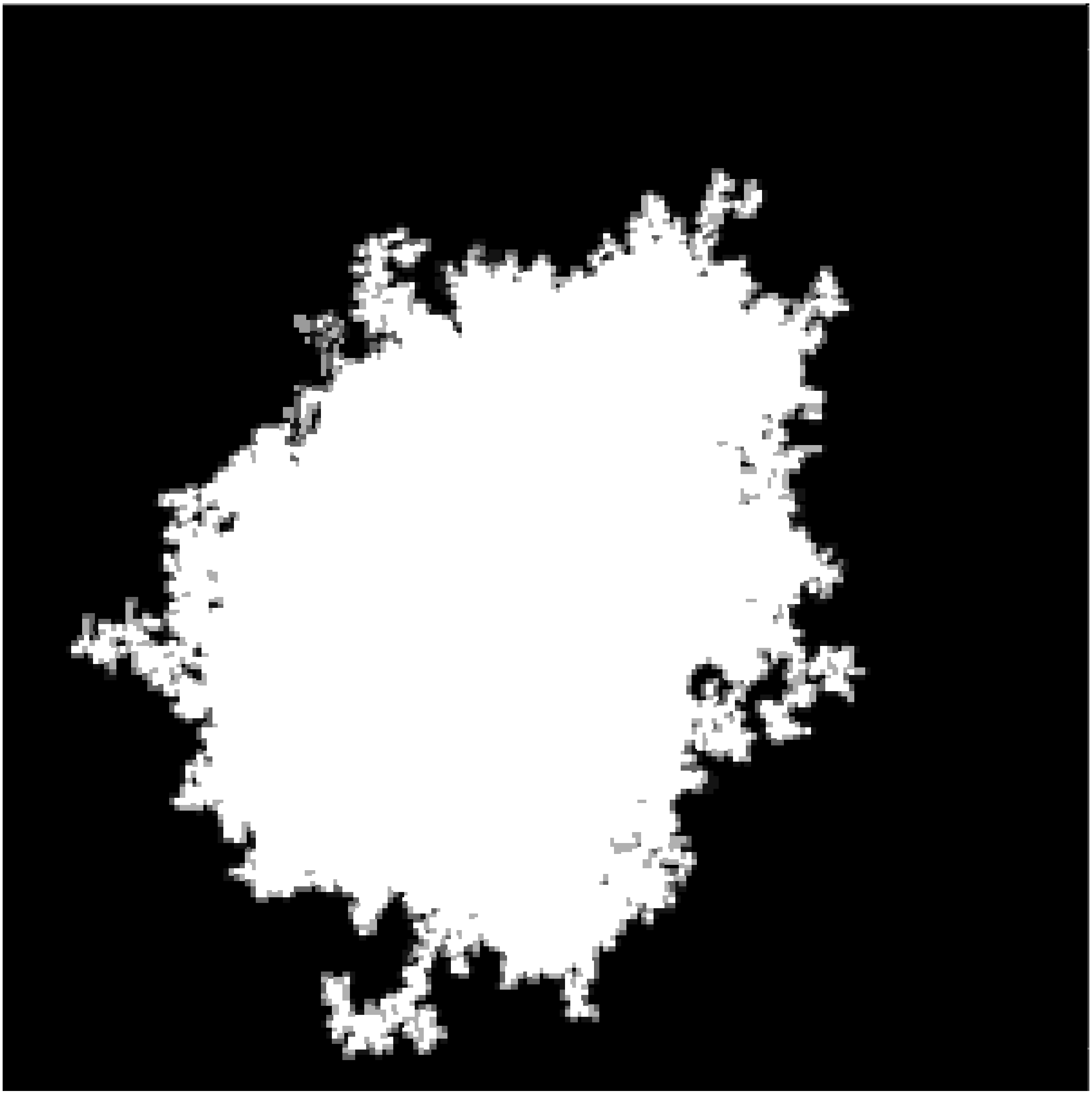


figure 4