A K Mallik
Department of Mechanical Engineering
Indian Institute of Technology
Kanpur 208 016, India.
Email:akmallik@iitk.ac.in

Perfect Medians, Euler, and Ramanujan

Brian Hayes [1] defines a *perfect median* as follows. For the set of consecutive natural numbers 1, 2, 3, ..., m, ..., n to have m as its perfect median we must have

$$1 + 2 + 3 + ... + (m-1)$$

$$= (m+1) + (m+2) + (m+3) + ... + n,$$

or

$$\frac{m(m-1)}{2} = \frac{(n-m)(m+n+1)}{2},$$

$$\text{or} \quad m^2 = \frac{n(n+1)}{2}. \tag{1}$$

Following the classification of numbers practiced by the Pythagoreans, who used the shapes produced by the arrangement of dots or pebbles, $m^2$and $n(n+1)/2$ are called square and triangular numbers, respectively [2]. The determination of the values of m and n satisfying equation (1) was carried out by Hayes [1] using a small computer program, and he attributed this puzzle to David Gale who was interested in determining the underlying pattern in these numbers.

In 1733, Leonhard Euler at the end of his paper entitled "On the solution of problems of Diophantus about integer numbers" asked for what values of n, the triangular numbers are also square numbers. He also provided the answer [3]: "Triangular numbers with $n = 1$, 8, 49, 288, 1681, 9800, etc., are square numbers corresponding to $m = 1$, 6, 35, 204, 1189, 6930, etc., respectively". These answers can be obtained as follows.

Keywords
Perfect median, Pythagorean, triangular number, Pell's equation, Euler's equation, Brahmagupta's equation, Ramanujan, Mahanalobis, continued fraction.

Multiplying both sides of (1) by 8 and adding 1, we get

$$(2n+1)^2 - 8m^2 = 1. \tag{2}$$

Substituting

$$p = 2n + 1 \text{ and } q = 2m, \tag{3}$$

equation (2) can be written in the form of the so-called Pell's equation (see *Box* 1),

$$p^2 - 2q^2 = 1, \ or \left(p + \sqrt{2}q\right)\left(p - \sqrt{2}q\right) = 1. \tag{4}$$

In 1733, Euler asked about the triangular numbers which are also square numbers.

This has infinitely many solutions [4]. It is easy to see (by inspection) that the smallest possible solution to (4) is $p_1 = 3$, $q_1 = 2$. Hence, from (3) we get $n_1 = 1$, $m_1 = 1$. To get the next solution, first we write equation (4) with the smallest values of p and q as

$$\left(3 + 2\sqrt{2}\right)\left(3 - 2\sqrt{2}\right) = 1. \tag{5}$$

Box 1.

Following Euler, equations of the form $p^2 - Dq^2 = 1$ where p and q are integers, and D is a non-square positive integer, are historically (but wrongly) known as Pell's equation. 'Fermat's equation' would have been a more accurate name [3]. But even more accurately, this equation may be called 'Brahmagupta's equation'.

In about 650 AD Brahmagupta wrote, in his highly influential text *Brahma Sphuta Siddhantha*, "A person who can within a year solve the equation $x^2 - 92y^2 = 1$ is a mathematician" (see page 252 of the book by Beiler [6]). Since the smallest solution is $x = 1151$, $y = 120$, one does need to be a great mathematician to solve this equation without the aid of a computer!

If D is a perfect square, then Brahmagupta's equation implies that two consecutive integers are perfect squares, leading to the trivial solution $p = 1$, $q = 0$.

If D is a non-square positive integer, then Brahmagupta's equation has infinitely many solutions. The smallest solutions of p and q vary wildly with the value of D [4]. For example, the smallest solutions are $p = 31$, $q = 4$ with $D = 60$; $p = 63$, $q = 8$ with $D = 62$; and $p = 1766{,}319{,}049$, $q = 226{,}153{,}980$ with $D = 61$.

Difficult puzzles, ultimately involving solutions to Brahmagupta's equation, have been devised with values of D that yield large values for even the smallest solution. $D = 1597$ and 9781 result in astronomically large values for the smallest solutions of p and q. The systematic procedure of obtaining the smallest possible solution is given in textbooks on number theory.

Squaring both sides of (5), we get

$$\left(17+12\sqrt{2}\right)\ \left(17-12\sqrt{2}\right)=1.$$

Thus, we obtain the second solution as $p_2 = 17,\ q_2 = 12$. Using (3), one gets $n_2 = 8,\ m_2 = 6$.

Taking the cube of both sides of (5), we get

$$\left(99+70\sqrt{2}\right)\ \left(99-70\sqrt{2}\right)=1,$$

yielding $p_3 = 99,\ q_3 = 70$. Thus, from equation (3) one obtains $n_3 = 49,\ m_3 = 35$. All the infinitely many solutions can be obtained in this way by taking successively higher powers of both sides of (5).

Kanigel [5] tells us the following story about the mathematical genius S Ramanujan. In December 1914, Ramanujan was asked by his friend P C Mahalanobis (founder of the Indian Statistical Institute) to solve a puzzle, "Puzzles at a Village Inn", that appeared in the popular English magazine *Strand*. The puzzle stated that n houses on one side of a street are numbered sequentially starting from 1. The sum of the house-numbers on the left of a particular house having the number m, equals that of the houses lying on the right of this particular house. It is given that n lies between 50 and 500 and one has to determine the values of m and n. Thus, the puzzle basically boiled down to finding the perfect median m for n lying between 50 and 500, which is 204 with $n = 288$. Ramanujan, while stirring vegetables over the gas fire, rattled out a continued fraction generating all the perfect medians. When his astounded friend asked how he had solved the problem, Ramanujan answered, "Immediately I heard the problem it was clear that the solution should obviously be a continued fraction; I then thought, which continued fraction? And the answer came to my mind".

In 650 AD, Brahmagupta wrote, "A person who can within a year solve the equation $x^2 - 92y^2 = 1$ is a mathematician". The smallest solution is $x = 1151$, $y = 120$.

Ramanujan got the first two values of m (namely, 1 and

6) mentally, and gave the following continued fraction:

$$\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6-\ldots}}}$$

The numerator and the denominator of every rational convergent of this continued fraction give two consecutive values of m. The first rational approximation is $1/6$, 1 and 6 being the first two perfect medians. The second rational approximation $\frac{1}{6-\frac{1}{6}}= 6/35$ again gives two consecutive values of m, viz., 6 and 35. The next one will give 35 and 204, and so on.

"... [When] I heard the problem, it was clear that the solution should be a continued fraction. I then thought, which continued fraction? And the answer came to my mind."

S Ramanujan

It may be mentioned that the continued fraction of Ramanujan,

$$x = \cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6-\ldots}}} = \frac{1}{6-x},$$

when solved yields $x = 3 \pm 2\sqrt{2}$. Since $x < 1, x = 3 - 2\sqrt{2}$ is the correct solution; the other root is more than unity. The reciprocal of this root is $\frac{1}{x} = 3 + 2\sqrt{2}$. So, we get back equation (5), which generates all the solutions.

Suggested Reading

[1] B Hayes, *American Scientist*, Vol.96, p. 362, 2008.

[2] M Kline, *Mathematics and the Physical World*, Dover Publications, Inc., New York, p.27–28, 1981.

[3] C E Sandifer, The Early Mathematics of Leonhard Euler, *The MAA Tercentenary Euler Celebration*, The Mathematical Association of America, Washington, D.C., p.105, 2007.

[4] I Stewart, *Math Hysteria*, Oxford University Press, Oxford, p.50, 2004.

[5] R Kanigel, *The Man Who Knew Infinity, A Life of the Genius Ramanujan*, Simon and Schuster Inc., New York, p. 214–215, 1991.

[6] Albert H Beiler, *Recreations in the theory of numbers*, 2nd ed. Dover Publications Inc., New York, 1964.