An Exactly Solvable Anisotropic Directed Percolation Model in Three Dimensions

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We solve exactly a special case of the anisotropic directed-bond percolation problem in three dimensions, in which the occupation probability is 1 along two spatial directions, by mapping it to a five-vertex model. We determine the asymptotic shape of the infinite cluster and hence the direction dependent critical probability. The exponents characterizing the fluctuations of the boundary of the wetted cluster in d dimensions are related to those of the $(d-2)$-dimensional KPZ equation. [S0031-9007(98)06887-2]

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It is well known that the critical behavior of a very large class of reaction-diffusion systems is describable in terms of the critical exponents of directed percolation (DP) [1]. Examples include heterogeneous catalysis [2], contact processes and epidemic models, self-organized criticality [3], and the driven depinning transition [4]. Although rather precise estimates of critical parameters are known from various numerical techniques such as series expansions and Monte Carlo simulations [5], an exact solution has not been possible so far, even in the simple case of $1+1$ dimensions. The only analytically tractable case known is the anisotropic bond percolation on a square lattice, solved by Domany and Kinzel (DK). They addressed the problem in which horizontal and vertical bonds are present with different concentrations and solved exactly the special case when one of these is set equal to 1 [6]. In this case, the wetted cluster has no holes, and this fact helps in reducing the problem to that of a simple random walk in one dimension [7].

In this paper, we solve the three-dimensional generalization of the DK model. We consider the DP on a three-dimensional simple cubic lattice with bond concentration in the three directions being p_x, p_y, and p_z, and solve the case $p_x = p_y = 1$, p_z arbitrary. We study the properties of the infinite cluster and obtain the macroscopic shape of the cluster and the scaling form for the fluctuations of its boundary. By universality, the behavior of fluctuations of the outer boundary of the infinite cluster at large length scales in this special soluble case is expected to be the same as in the more general case above the percolation threshold with p_x, p_y, and p_z arbitrary.

In addition to the usual critical exponents of the DP problem, which are defined in terms of the power-law behavior of various quantities *near* the critical point, there are universal power-law prefactors to the exponential decay of correlation functions away from the critical point. For example, for $p < p_c$, the probability of a finite cluster having s sites varies as $s^{-\theta}\exp(-As)$ as $s \to \infty$. For $p > p_c$, the probability varies as $s^{-\theta'}\exp(-Bs^{(d-1)/d})$ for large s. The exponents θ and θ' are examples of *off-critical* exponents [8]. We find that the fluctuations of the boundary of the infinite cluster in d dimensions are in the same universality class as those of a $(d-2)$-dimensional interface moving in a $(d-1)$-dimensional space. The latter is described by the well-known Karder-Parisi-Zhang (KPZ) equation [9]. Thus we identify the critical exponents of the KPZ equation as belonging to the class of off-critical exponents of DP.

Our solution falls in the general class of disorder solutions of statistical mechanical models [10]. These often show the phenomena of dimensional reduction. Thus a three-dimensional problem at a disorder point can be thought of as a two-dimensional system evolving in time, and correlation functions in its steady state would show large distance tails characteristic of two-dimensional systems. In our case, the choice $p_x = p_y = 1$ allows a further reduction of dimension by 1, and we show that the system is equivalent to describing the Markovian evolution of a system of hard core particles on a one-dimensional ring, where the particles are able to move in only one direction [11]. Our solution is also of interest as an exact solution of a nontrivial three-dimensional statistical mechanics problem with positive weights, of which not many are known [12].

Consider the general DP problem on a simple cubic lattice with nearest neighbor bonds, all directed in the direction of increasing coordinates. The concentration of bonds is p_x, p_y, and p_z along the x, y, and z axes. We imagine a source of fluid at the origin, which can spread along the occupied directed bonds. The sites wetted by this fluid form a cluster which we shall call the wetted cluster for brevity. One would like to calculate the probability, $P(\vec{R})$, that a site $\vec{R}$ belongs to the wetted cluster. When p_x, p_y, and p_z are small, the wetted cluster is finite. As the probabilities are increased, an infinite connected path appears. We shall call the special direction along which an infinite path first appears as the preferred direction. As the bond concentrations are further increased, percolation occurs in a narrow cone centered around the preferred direction. Following DK, we look at the angular dependence of $P(\vec{R})$. Its exponential decay for large $|\vec{R}|$ defines an

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angle dependent correlation length, ξ_Ω, in the direction Ω.

The correlation length ξ_Ω diverges as $[p_c(\Omega) - p]^{-\nu}$, where $p_c(\Omega)$ is the Ω dependent critical probability. The value of the exponent ν depends on the direction Ω. We show below that in most directions ν takes the same value $3/2$. ν is different if Ω is the preferred direction (when $\nu = \nu_\parallel^{3d}$, the DP exponent), or when it lies in one of the coordinate planes (xy, yz, and zx). For percolation in these planes, clearly all perpendicular bonds can be ignored, and the problem reduces to the known two-dimensional case. Consider, for simplicity, the isotropic case ($p_x = p_y = p_z$). Then the easy direction is (111), and along this direction $\nu_\parallel^{3d} \approx 1.29$. Along (110), (011), and (101) directions $\nu_\parallel^{2d} \approx 1.733$. Other directions in the xy, yz, and zx planes have $\nu = 2$. The x, y, and z axes have $\nu = 1$.

Consider a coordinate system in which x and y axes are in the plane of the paper with the z axis pointing out of it (see Fig. 1). Since $p_x = p_y = 1$, all of the bonds aligned along the x axis and y axis are present. Hence, if a point (x, y, z) is wetted, then so are all of the points (x', y', z) with $x' \geq x, y' \geq y$. It is easy to see that, if (x, y, z) is wet, so are all the points directly below it in the z direction. Therefore, we can define an integer height function $h(x, y)$ such that all points (x, y, z) with $z > h(x, y)$ are dry while the points with $z \leq h(x, y)$ are wet. The wetted cluster has no holes and can be specified completely by its bounding surface, $h(x, y)$.

We first determine $\overline{h}(x, y)$, the mean value of $h(x, y)$. We do so by mapping the problem to a five-vertex model [13]. Consider the orthogonal projection of the wetted cluster onto the xy plane [see Fig. 2(a)]. It consists of paths running from the y axis to the x axis via steps in the right and down directions obeying the constraint that paths do not cross each other. The kth path separates the sites with $h(x, y) < k$ from those sites with $h(x, y) \geq k$. We now slide each point in the kth path by (k, k) [see Fig. 2(b)], thereby mapping the point (x, y) to the point (ζ, η) given by

$$\begin{aligned} \zeta &= x + h(x, y), \\ \eta &= y + h(x, y). \end{aligned} \tag{1}$$

This ensures that two paths will not have common edges. Now, every site has either zero or two bonds connecting it to its neighbors. Thus a given configuration of shifted paths will be made up of five kinds of vertices. It is easy to check that the correct weights of each of these vertices are shown in Fig. 3, where $q = 1 - p$. Under this mapping, it is an elementary exercise to show that the gradients are related through

$$\begin{aligned} \overline{h}_\zeta &= \overline{h}_x/(1 + \overline{h}_x + \overline{h}_y), \\ \overline{h}_\eta &= \overline{h}_y/(1 + \overline{h}_x + \overline{h}_y), \end{aligned} \tag{2}$$

where $\overline{h}_\zeta$ stands for $\frac{\partial \overline{h}}{\partial \zeta}$, keeping the second coordinate η fixed, and similarly for other partial derivatives.

Consider a point (ζ, η) with $\zeta, \eta \gg 1$. In the five-vertex problem, $\overline{h}_\zeta$ and $\overline{h}_\eta$ are the mean densities of lines in the horizontal and vertical directions, respectively. They are, however, not independent and knowledge of one determines the other. The relation between $\overline{h}_\zeta$ and $\overline{h}_\eta$ is a local relation and is most easily determined using a different set of boundary conditions when $\overline{h}_\zeta$ and $\overline{h}_\eta$ are uniform everywhere. Thus we consider the five-vertex model on an open cylinder of length N in the y direction and infinite in the x direction.

The five-vertex model can be solved via the transfer matrix technique. We transfer from column to column along the x axis. Consider the sector in which there are n lines in a column. The ice rule plus periodic boundary conditions ensure that there are n lines in each column [14]. The transfer matrix, $T(\{y'\};\{y\})$, which is the weight of going from configuration $\{y\}$ to $\{y'\}$ (see Fig. 4, where the labeling is self-explanatory) is the product of the weights of each vertex and can be written as

$$T(\{y'\};\{y\}) = \prod_{i=1}^{n} (p^{1-\delta_{y_i', y_i}} q^{y_i' - y_{i-1} - 1}). \tag{3}$$

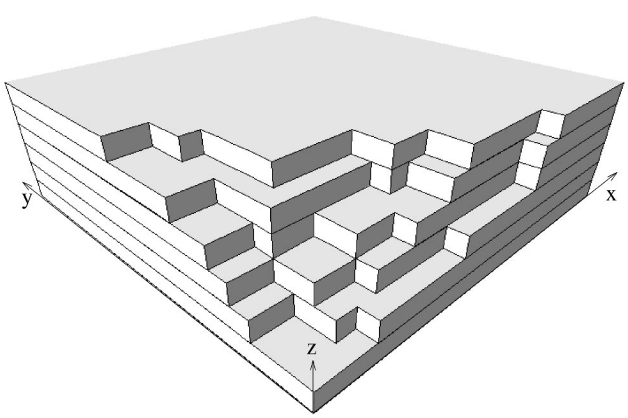

FIG. 1. A typical wetted cluster shown up to $z = 6$ (for $p = 0.3$). The cluster extends infinitely in the x and y directions.

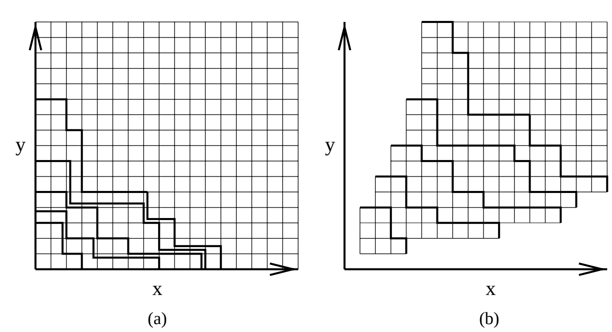

FIG. 2. (a) The projection of the wetted cluster onto the x-y plane. Overlapping edges have been shown separated for clarity. (b) The shifted paths.

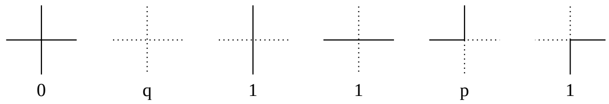

FIG. 3. The vertex weights.

It is easy to verify that $\sum_{\{y'\}} T(\{y'\};\{y\}) = 1$, where the summation is over all allowed configurations. Therefore $T(\{y'\};\{y\})$ is a properly normalized transition probability of $(\{y\} \rightarrow \{y'\})$.

One can visualize this process as a system of n hard core particles, each hopping only to the right on a ring of size N. If the particle on the right is at a distance m then the particle can hop up to $m - 1$ steps during one time step, with the probability for k steps being $P(k \mid m) = p^{1-\delta_{k0}} q^{m-k-1}$. For large times, the above, being a Markov process, will evolve into its steady state. The steady state will be one in which all states are equally likely. To see this, we note that $\sum_{\{y\}} T(\{y'\};\{y\}) = 1$. Therefore T^t, the transpose of T, is also a stochastic matrix. Hence it has the state $(1,\dots,1)$ as the left eigenvector with an eigenvalue of 1. Taking transpose, $(1,\dots,1)^t$, is a right eigenvector with an eigenvalue of 1.

Knowing the steady state, we can determine the average number of steps, $\overline{d}$, of a particle in the y direction for each transfer in the x direction. Simple algebra gives

$$\overline{d}_\rho = \frac{p(1-\rho)}{\rho(p + q\rho)}, \tag{4}$$

where $\rho = n/N$. Making the identification $\overline{d}_\rho = \overline{h}_\zeta / \overline{h}_\eta$ and $\rho = \overline{h}_\eta$ in Eq. (4), we get

$$p(1 - \overline{h}_\zeta - \overline{h}_\eta) = q\overline{h}_\zeta \overline{h}_\eta\,. \tag{5}$$

Equation (5) can be rewritten in x, y coordinates with the help of Eq. (2) to give

$$p(1 + \overline{h}_x + \overline{h}_y) = q\overline{h}_x \overline{h}_y\,. \tag{6}$$

For large x, y, Eq. (6) has the scaling solution

$$\overline{h}(x,y) = \frac{1}{\Lambda}\,\overline{h}(\Lambda x, \Lambda y)\,. \tag{7}$$

Putting the ansatz $\overline{h}(x,y) = A(x + y) + B\sqrt{xy}$, we find that it satisfies the equation with $A = p/q$ and $B = 2\sqrt{p}/q$. Thus the asymptotic shape of the surface is given by

$$\overline{h}(x,y,p) = \big(px + py + 2\sqrt{pxy}\big)/q, \quad \text{for } x, y \gg 1\,. \tag{8}$$

The expression is symmetric in x and y. If $y = 0$, then $\overline{h} = px/q$ as expected. For given p, and given the azimuthal angle, ϕ, there is a critical polar angle θ_c such that all points with $\theta > \theta_c$ are wetted. The critical angle is determined through the relation $\cot(\theta_c) = \overline{h}/\sqrt{x^2 + y^2}$. Equation (8) can then be inverted to derive the direction dependent critical probability, the smallest probability for which (θ, ϕ) is wetted, as

$$p_c(\theta,\phi) = \left(\frac{\sqrt{(\alpha + 1)(\beta + 1)} - \sqrt{\alpha\beta}}{\alpha + \beta + 1}\right)^2, \tag{9}$$

where $\alpha = \tan(\theta)\cos(\phi)$ and $\beta = \tan(\theta)\sin(\phi)$.

The problem can also be viewed as a one-dimensional cellular automaton evolving in time. Consider the coordinate transformation $t = x + y, u = x - y$. Then from the definition of the model it is easy to check that for $t \geq 0$, $h(u,t)$ follows the rule,

$$h(u, t + 1) = \max[h(u - 1, t), h(u + 1, t)] + \eta(u,t)\,, \tag{10}$$

where $\eta(u,t)$'s are independent identically distributed random variables taking non-negative integer values with $Prob(\eta = k) = (1 - p)p^k$. The rules of evolution are local and we expect the fluctuations to be governed by the KPZ equation [9].

We now look at the correlations in the system. There are two correlation lengths of interest. $\xi_\parallel(\Omega)$ has been defined through the exponentially decaying probability of wetting. $\xi_\parallel(\Omega)$ diverges as $[p_c(\Omega) - p]^{-\nu}$. We denote by $\xi_\perp$ the correlation length along equal height contours but along the wetted surface. We can study the fluctuations by looking at the scaling properties of the five-vertex model. The transfer matrix can be diagonalized via the Bethe ansatz [15]. From this analysis it is known that the dynamic exponent $z = 3/2$, and the roughening exponent $\chi = 1/2$ [16]. These exponents are the well-known exponents of the KPZ equation.

The height fluctuations at a point are linearly related to the fluctuations in the trajectories in the five-vertex problem. In the KPZ problem, it is known that the magnitude of the fluctuations of height varies as $h^{1/3}$. Thus we get $\sqrt{\langle(\delta h)^2\rangle} \sim h^{1/3}$. $\xi_\parallel$ is the correlation length at an angle $\theta_c + \delta\theta$. Clearly $\xi_\parallel \delta\theta \sim h^{1/3} \sim \xi_\parallel^{1/3}$ or $\xi_\parallel \sim (\delta\theta)^{-3/2}$. From Eq. (8), we note that, for $p \neq 0$, $\delta\theta \sim \delta p$. Thus we get $\nu = 3/2$, to be compared with the value $\nu = 2$ for the two-dimensional DK. The behavior of $\xi_\perp$ can be obtained from the relation $\xi_\perp \sim \xi_\parallel^{1/z}$. Thus we get $\xi_\perp \sim x_\parallel^{2/3}$, where $x_\parallel$ is the distance of the height contour from the origin.

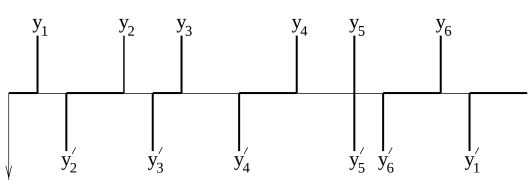

FIG. 4. Two typical arrangements of lines in the adjacent column (for $n = 6$). The position of lines $\{y_i\}$ in the upper column must interlace with the position $\{y'_i\}$ in the lower column.

Other three-dimensional lattices can be treated similarly. We consider lattices made up of two-dimensional layers stacked on top of each other with vertical bonds connecting a site to the one vertically above it. If each layer is a triangular lattice, then it may be described as a square lattice with one set of diagonal bonds. However, the diagonal bonds provide no additional connections if $p_x = p_y = 1$ and can be neglected. Thus the problem reduces to the square lattice. If each layer is a honeycomb lattice, there are two types of sites: those having only two or three outgoing bonds. If we integrate over the former, the problem again reduces to a simple cubic lattice, with a renormalized probability p_z.

In higher dimensions, $d > 3$, the exponents are related to the higher-dimensional KPZ equation. The d-dimensional DK model is equivalent to a $(d - 2)$-dimensional interface growing in time. As in the case of a depinning transition of a driven tilted interface [17], the height-height correlation function in d dimensions will have the scaling form

$$\langle \delta h(x) \delta h(x') \rangle = |x_\parallel - x'_\parallel|^{(2\chi_{d-2})/(z_{d-2})} F \times \left(\frac{|x_\perp - x'_\perp|^{z_{d-2}}}{|x_\parallel - x'_\parallel|} \right), \qquad (11)$$

where $x_\parallel$ is measured along the radial direction, $x_\perp$ is measured along the equal height contour, and z_d and χ_d are KPZ exponents of a growing d-dimensional interface.

In summary, we have been able to exactly solve the three-dimensional anisotropic directed percolation problem for the special case $p_x = p_y = 1$. We used the fact that in this case the wetted cluster has no holes within. This reduces the problem to studying the two-dimensional surface of the cluster. The weights of different surfaces were shown to be the same as those of different configurations of a five-vertex model. From the already known exact solution of the latter, we obtained an exact expression for the average height profile $h(x, y)$ and determined the asymptotic correlations of the surface fluctuations of the wetted cluster for large separations.

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