Persistence in a Stationary Time-series

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We study the persistence in a class of continuous stochastic processes that are stationary only under integer shifts of time. We show that under certain conditions, the persistence of such a continuous process reduces to the persistence of a corresponding discrete sequence obtained from the measurement of the process only at integer times. We then construct a specific sequence for which the persistence can be computed even though the sequence is non-Markovian. We show that this may be considered as a limiting case of persistence in the diffusion process on a hierarchical lattice.

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I. INTRODUCTION

In recent years, there has been a lot of interest in the study of persistence of fluctuations in different physical systems [1,2]. Persistence $P(t)$ is simply the probability that the deviation of the value of a fluctuating field from its mean value does not change sign up to time t. Persistence has been studied in many nonequilibrium systems [1] and also in diverse fields ranging from ecology [3] to seismology [4]. Theoretical studies include various models of phase ordering kinetics [5], diffusion equation [6,7], reaction diffusion systems in pure [8] as well as disordered environments [9], fluctuating interfaces [10], and various theoretical models [11]. Persistence or first passage properties find simple applications in various chemical [12], biological [13] and granular systems [14]. In laboratory experiments, persistence has been measured in various experimental systems including breath figures [15], liquid crystals [16], soap bubbles [17] and laser-polarized Xe gas using NMR techniques [18].

In many of the nonequilibrium systems discussed above, the underlying stochastic process $\psi(t)$ is nonstationary. For example, the two time correlation function $C(t_1,t_2) = \langle\psi(t_1)\psi(t_2)\rangle$ for the diffusion equation depends on the ratio of the two times t_1 and t_2, and not on their difference [6]. Persistence in such systems typically decays as a power law $P(t) \sim t^{-\theta}$ at late times t. The exponent θ, called the persistence exponent, is believed to be a new exponent and is apparently unrelated to the usual dynamical exponents that characterize the decay of n-point correlation functions with finite n. Persistence has also been studied for stationary processes [19,20] such as a stationary Gaussian process characterized by its two time correlation $C(t_1,t_2)$ which is invariant under *arbitrary* time translation, i.e., $C(t_1+t_0,t_2+t_0) = C(t_1,t_2)$ for all t_0. In the stationary case, persistence between times t_1 and t_2 typically decays exponentially, $P(t_1,t_2) \sim \exp[-\theta_s|t_2-t_1|]$ for large time difference [20]. For some processes such as the diffusion equation the nonstationary problem can be mapped onto a corresponding stationary one [6] and the exponent θ of the nonstationary process becomes identical to the inverse decay rate θ_s of the corresponding stationary process [1]. Despite many theoretical studies of either θ or θ_s, exact results are known only in relatively few cases [21]. The basic difficulty in computing either of them can be traced back to the fact that the underlying stochastic processes in both cases are usually non-Markovian [1].

In this paper we study the persistence in stochastic processes that are stationary under translations in time only by an *integer multiple of a basic period* (without loss of generality, this period may be chosen to be 1). Throughout this paper we will refer to such processes as SIS (stationary under integer shifts). For example, a Gaussian stochastic process will have the SIS property if its two-time correlation function $C(t_1,t_2)$ satisfies $C(t_1+n,t_2+n) = C(t_1,t_2)$ for all integer n. Such processes appear in many physical situations. For example, in weather records, there is an underlying non-random periodic forcing (the motion of earth round the sun), which makes the stochastic process not truly stationary in time. In nonlinear systems, even if one can filter out the periodic component, the properties of the filtered signal (say variance) would still be expected to show a periodic variation with time. It seems worthwhile to study in more detail persistence in such SIS processes.

When one wants to study the persistence of SIS processes, the following question arises naturally: Is the probability $P(t)$ that the process remains positive over the interval $[0,t]$ is same as the probability P_n that the process is positive only at all the n intermediate integer times between 0 and t? In other words, is the persistence of a 'continuous' SIS process is the same as the persistence of the corresponding 'discrete' sequence obtained by measuring the process only at integer times?

The question regarding the difference between 'continuous-time' and 'discrete-time' persistence was first raised in Ref. [22] for strictly stationary Gaussian processes, motivated from the observation that in experiment [18] as well as numerical simulation [23] of persistence in diffusion equation, the measurements (whether the process is positive) are done only at discrete points (separated by a fixed time window of size Δ) even though the actual process is continuous. In Ref. [22] it was shown that for general stationary Gaussian processes, the continuous persistence decays as $P(t) \sim \exp[-\theta_c t]$ for large t, where as the corresponding 'discrete-time' persistence (obtained from measuring the data only at the intermediate time points separated by a fixed Δ) decays as $P_n \sim \exp[-\theta_d n]$ where $T = n\Delta$. In general, one would expect that the exponent θ_c is strictly greater than θ_d for such a process since the process can change sign between two successive integer points. The exponent θ_d was computed analytically in Ref. [22] for a Gaussian stationary Markov process and was shown to depend continuously on the window size Δ.

In this paper we study the 'continuous-time' versus 'discrete-time' persistence for SIS processes. We restrict ourselves to the study of discrete-time persistence only when the measurement points in time are integers. This is a natural choice since the process is stationary only under integer shifts in time. We show that unlike strictly stationary processes, in this case, the two exponents θ_c and θ_d can be equal under certain conditions. This discrete-time persistence P_n of a sequence is also rather interesting from a purely mathematical point of view, especially when the underlying process $\psi(t)$ is Gaussian. In that case, calculating P_n becomes the problem of calculating the probability that a set of n Gaussian random variables with a specified joint probability distribution are all positive. This 'one-sided barrier' problem has remained popular in the applied mathematics literature for many decades [19,20,24]. But the number of cases where this probability can be explicitly calculated for large n remains rather small [24].

In this paper, we study a specially simple case of a continuous-time stochastic process $\psi(t)$ which is obtained as a local smearing of a sequence of independent identically distributed random variables via a smearing function $f(t)$. This process becomes, by construction, a stochastic process whose probability distribution is invariant under time translations by integers, i.e. a SIS process. We construct examples where $\theta_d < \theta_c$, and also construct a family of smearing functions $f(t)$ for which $\theta_d = \theta_c$. We provide a physical example namely the diffusion equation on a hierarchical lattice where the diffusion field is a Gaussian stochastic process with the SIS property and we compute the corresponding smearing function exactly. We then determine exactly the exponent θ_d for a specific case when the correlations in the discrete sequence are nonzero only for consecutive values. We find that in this case, the exponent θ_d depends continuously on the value of the correlation.

The paper is organized as follows. In section II, we give some examples of continuous SIS processes where θ_c is strictly greater than θ_d, and also construct a class of processes for which $\theta_c = \theta_d$. In section III, we provide a physical example namely diffusion equation on a hierarchical lattice where the diffusion process shows log-periodic oscillations. After rescaling, and a change of variables from time t to $log(t)$, we get a stochastic process that has the SIS property. In section-III, we introduce a special sequence for which the persistence exponent θ_d can be computed exactly. Section IV contains a summary of our results.

II. CONTINUOUS-TIME VERSUS DISCRETE-TIME PERSISTENCE

In this section we discuss the conditions under which the continuous-time persistence of a SIS process is the same as the discrete-time persistence of a corresponding sequence obtained by measuring the process only at the integer points. As noted earlier, in general, we expect that $\theta_c > \theta_d$. Consider a stochastic process $\psi(t)$, which is known to be positive at the integer points $t = 1, 2, \ldots N$. Now consider the conditional distribution of $\psi(t_1)$ at some non-integer point t_1 lying within the interval $[1, N]$. This conditional distribution is a gaussian whose width is independent of the values of $\psi(t)$ at the known integer points. If this variance is finite, even in the limit when N is large, clearly, there will be finite probability of $\psi(t_1)$ becoming negative. Thus one would expect to get $\theta_c = \theta_d$ only if the conditional variance of $\psi(t_1)$ tends to zero for large N. In such a process, one should be able to determine $\psi(t)$ for all real t, if one knows its value at all integer points.

This suggests the following construction of $\psi(t)$: we consider a sequence of independent random variables $\{\phi(n)\}$ having zero mean, where $n \in \{-\infty, +\infty\}$, and define a stochastic process $\psi(t)$ by the convolution

$$\psi(t) = \sum_{n=-\infty}^{+\infty} f(t-n)\phi(n). \tag{1}$$

Knowing $\psi(t)$ at all integer points, one can expect to determine uniquely the constants $\{\phi(n)\}$ by solving coupled linear equations, which then determines $\psi(t)$ for all real values of t.

The behavior of this process depends only on the smearing function $f(t)$. In the following, we shall assume that $f(t)$ has some good properties, i.e. is a non-negative unimodal function of t, which decreases sufficiently fast for large $|t|$. By a shift of the origin of time t, and rescaling $\psi(t)$, we can assume that the maximum of $f(t)$ occurs at $t = 0$ and $f(0) = 1$.

What is the class of functions $f(t)$ such that θ_c equals θ_d? This class is not easy to characterize directly. A simple example illustrates this point clearly. Consider the simple case of triangular function

$$f(t) = 1 - |t|/a, \quad \text{for} \quad |t| < a \tag{2}$$
$$= 0, \text{otherwise}. \tag{3}$$

In this case, $\psi(t)$ is a piece-wise linear function of t. If $a < 1/2$, we have intervals in which $\psi(t)$ is identically zero. If, however, we define persistence probability as the probability that the function does not *change sign* upto time t, it is clear that for all $a < 1$, we have $\theta_c = \theta_d = log2$.

We now show that $\theta_c \neq \theta_d$ if $a > 1$. For this purpose, it is sufficient to show that there are sequences $\{\phi_n\}$ such that the corresponding ψ-process is positive at all integer points, but takes negative value for non-integer t. As such events would occur with non-zero probability along the sequence, $\theta_c > \theta_d$.

Let n be the integer just below a. We consider a periodic sequence of ϕ_n with

$$\begin{aligned} \phi_i &= 1, \text{ if } i = 0 \bmod (2n+1); \\ &= -c, \text{ if } i = n \text{ or } n+1 \bmod (2n+1); \\ &= 0 \text{ otherwise.} \end{aligned} \tag{4}$$

Then it is easy to see that if we choose c such that $(a - n) > (2a - 1)c > (a - n - 1/2)$, then $\psi(t = i)$ is positive for all integers i, but $\psi(t = n + 1/2)$ is negative. Clearly, these signs are not changed if all ϕ's deviate from these values by sufficiently small amount. Then such sequences (of finite length) will occur with non-zero frequency, and hence for any $a > 1$, θ_c is strictly greater than θ_d.

However, there are functions $f(t)$ for which $\theta_c = \theta_d$. The simplest example of this class is $f(t) = \exp(-|t|/a)$. In this case, it is easy to see from Eq. (1) that $\psi(t)$ at any non-integer point t can be expressed as a positive linear combination of its value at the two nearest integer points, so that for all $t = n + \delta t$ with $0 \leq \delta t \leq 1$ we have

$$\psi(n + \delta t) = [\sinh \frac{\delta t}{a} \psi(n+1) \tag{5}$$
$$+ \sinh \frac{(1 - \delta t)}{a} \psi(n)] / [\sinh(1/a)] \tag{6}$$

Thus, if $\psi(n)$ and $\psi(n+1)$ are positive, Eq. (5) implies that $\psi(t)$ is positive for all $n \leq t \leq n+1$. Hence one gets, $\theta_c = \theta_d$.

The example above can be generalized. For example, one can introduce a two parameter family of functions, $f(t) = \exp(\kappa_1 t)$ for $t < 0$, and $f(t) = \exp(-\kappa_2 t)$ for $t > 0$ with $\kappa_1 > 0$ and $\kappa_2 > 0$ and are not equal in general. In fact, one can even introduce two arbitrary periodic functions $g_1(t)$ and $g_2(t)$ (with period 1), and take

$$f(t) = \exp(\kappa_1 t - g_1(t)), \quad \text{for} \quad t < 0, \tag{7}$$
$$= \exp(-\kappa_2 t - g_2(t)), \quad \text{for} \quad t > 0 \tag{8}$$

without destroying the equality of θ_c and θ_d. One only has to impose some conditions on $g_1(t)$ and $g_2(t)$ to ensure that $f(t)$ is unimodal. Effectively, we can take any unimodal function $f(t)$ defined in the interval $-1 \leq t \leq 1$, and extend it to all real t using the conditions $f(t-1) = e^{-\kappa_1} f(t)$, for $t < 0$, and $f(t+1) = e^{-\kappa_2} f(t)$ for $t > 0$, to get a function $f(t)$ for which θ_d and θ_c are equal.

III. PERSISTENCE IN DIFFUSION EQUATION ON A HIERARCHICAL LATTICE

A simple example of a physics problem where functions of the type given by Eq.(1) show up is the persistence of a diffusion field on a hierarchical lattice. The lattice may be thought of as a line having $N = 2^n$ sites labelled by an n-bit binary integer $i, 0 \leq i \leq N-1$ [25,26]. We define the ultrametric distance between two sites i and j as d, if the binary integers denoting i and j differ at the $n - d + 1$ bit counting from the left. Thus we have $d = 1$ between sites 2 and 3, but $d = 4$ between sites 7 and 8. , At each site i, we have a real variable $\psi(i)$. At time $t = 0$, the fields at different sites are assumed to be independent identically distributed random variables (say gaussians of mean zero, and variance 1). The fields $\psi(i)$ are assumed to evolve in time by the deterministic equation

$$\frac{d}{dt}\psi(i) = \sum_{j=0}^{N-1} K_{i,j}[\psi(j) - \psi(i)] \tag{9}$$

Here the spring constants $K_{i,j}$ are assumed to be functions of the distance $d_{i,j}$ between the two points. In the following, we shall assume that $K_{i,j} = a^{-d_{i,j}}$, where a is a constant > 1.

The integration of the equations of evolution is made particularly simple by the hierarchical nature of the spring couplings. It is easily verified that we have 2^{n-r} independent eigenmodes of relaxation rate $(a-1)^{-1}a^{-r+1}$ $(r = 1, 2, \ldots N-1)$ satisfying

$$\frac{d}{dt} S_j^{(r)} = (a-1)^{-1} a^{-r+1} S_j^{(r)} \tag{10}$$

where

$$S_j^{(r)} = \sum_{k=0}^{2^{r-1}-1} \phi(j2^r - 2^{r-1} - k) - \phi(j2^r - k) \tag{11}$$

where $j = 1$ to 2^{n-r}.

Expanding any particular $\psi(i)$, say for $i = 1$, in terms of these eigenvectors, and we get,

$$\psi_1(t) = \sum_{r=1}^{n-1} 2^{-r/2} \exp\left[-(a-1)^{-1} a^{-r+1} t\right] \phi(r), \tag{12}$$

where $\phi(r)$'s are i.i.d. Gaussian variables of zero mean and unit variance that characterize the initial condition.

This formula for the hierarchical model may be compared with the corresponding formula one writes in the Euclidean space in $d-$dimensions

$$\psi(\vec{r}=0) = \int_0^\infty dk \exp(-k^2 t)\eta(k) \tag{13}$$

where $\eta(k)$ are white-noise process with variance

$$< \eta(k)\eta(k') >= \delta_{k,k'} k^{d-1} \tag{14}$$

We eliminate the time variable t in terms a logarithmic time variable τ using the identification $a^\tau = t$, and change $\psi(t)$ by a change of scale, $\psi(\tau) = [a^{\tau/2}]\psi(t = (a-1)a^\tau)$. Then we have

$$\psi(\tau) = \sum_{r=0}^{\infty} \xi(r) \exp\left[-a^{\tau-r}\right] a^{(\tau-r)/2}. \tag{15}$$

For large τ, the summation over r can be extended from $-\infty$ to $+\infty$, and the process $\psi(\tau)$ then becomes a Gaussian process with the SIS property, i.e., is stationary only under integer shifts in time and is obtained by local smearing of the discrete white noises $\phi(r)$'s,

$$\psi(\tau) = \sum_{r=-\infty}^{+\infty} f(\tau - r)\phi(r), \tag{16}$$

where the convolution function $f(r)$ clearly goes to zero when r tends to $\pm\infty$. Thus, the problem of calculating the persistence exponent reduces that of calculating the exponents θ_c and θ_d for a process defined by given convolution function $f(t) = \exp(-a^t)a^{t/2}$. The origin of the SIS property here comes from the discrete scale invariance of the original model, which gives rise to log-periodic oscillations in the relaxation processes [27].

We have not been able to determine whether this function $f(t)$, the exponents θ_c and θ_d coincide, or are different. However, in a simple Monte Carlo realization of a sequence of 10^5 Gaussian variables $\{\phi_i\}$'s of zero mean and unit variance, we did not find any instance where the function $\psi(t)$ changed sign twice between two consecutive integers. This indicates that these exponents, if not equal, are likely to be quite close to each other.

IV. EXACT RESULTS FOR A SPECIAL CASE

For a smearing function $f(t)$ for which $\theta_c = \theta_d$, the computation of the persistence exponent simplifies considerably, and reduces to its determination for a discrete sequence rather than a continuous process. But even then, the exponent θ_d is quite nontrivial for an arbitrary smearing function $f(t)$. For calculating θ_d, only the values of $f(t)$ at integer points are relevant. In the following, we shall consider in detail the calculation of θ_d when only $f(0)$ and $f(-1)$ are non-zero. This can be thought of a crude approximation to the smearing function $f(t) = \exp(-a^t)a^{t/2}$, as in the diffusion equation on a hierarchical lattice, which decreases superexponentially for $t > 0$ and only exponentially for $t < 0$ for $a > 1$. We will show below that the exact computation of θ_d is nontrivial even for this toy smearing function since the resulting sequence is non-Markovian.

In this special case, Eq. (16) becomes

$$\psi_i = \phi_i + \epsilon\, \phi_{i+1}, \quad i = 1,2,\ldots,n \tag{17}$$

where we shall assume that $\{\phi_i\}$ are independent identically distributed random variables, not necessarily Gaussian, each drawn from the same distribution $\rho(\phi)$. Here ϵ is a mixing parameter. For convenience, we relabel the ϕ's without any loss of generality to consider the following sequence,

$$\psi_i = \phi_i + \epsilon\, \phi_{i-1}, \quad i = 1,2,\ldots,n. \tag{18}$$

For simplicity, we will assume that $\rho(\phi)$ is symmetric about the origin. The mean value of ϕ is then zero. We now ask: what is the probability $P_n(\epsilon)$ that $\psi_1, \psi_2, \ldots \psi_n$ are all positive for a given ϵ?

We note that the variables ψ_i's are now correlated. The two point correlation function, $C_{i,j} = \langle \psi_i \psi_j \rangle$ can be easily computed from Eq. (18),

$$C_{i,j} = \sigma^2 \left[(1+\epsilon^2)\delta_{i,j} + \epsilon(\delta_{i-1,j} + \delta_{i,j-1})\right], \tag{19}$$

where $\delta_{i,j}$ is the Kronecker delta function and $\sigma^2 = \int_{-\infty}^{\infty} \phi^2 \rho(\phi) d\phi$. Thus the parameter ϵ serves as a measure of the correlation and it is this correlation that makes the calculation of $P_n(\epsilon)$ nontrivial for nonzero ϵ.

The sequence $\{\psi_n\}$ defined by Eq. (18) is non-Markovian in the sense that if only $\{\psi_n\}$ are observed, and not the ϕ_n's, ψ_n depends not just on the previous member of the sequence ψ_{n-1}, but rather on the whole history of the sequence. For example, from Eq. (18) one can express ψ_n as,

$$\psi_n = \sum_{k=0}^{n-1} (-1)^{k-1} \epsilon^k \psi_{n-k} + \phi_n - \epsilon^n \phi_0 \tag{20}$$

which demonstrates explicitly the history dependence of the sequence. For non-Markovian sequences, it is generally hard to compute the persistence exponent. Fortunately progress can be made for this special sequence even though it is non-Markovian.

In order to calculate $P_n(\epsilon)$, it is first useful to define the following probabilities,

$$\begin{aligned} Q_1(x) &= \int_x^\infty d\phi_0 \rho(\phi_0), \\ Q_n(x) &= \int_x^\infty d\phi_0 \rho(\phi_0) \int_{-\epsilon\phi_0}^\infty d\phi_1 \rho(\phi_1) \int_{-\epsilon\phi_1}^\infty d\phi_2 \rho(\phi_2) \ldots \\ &\quad \ldots \int_{-\epsilon\phi_{n-2}}^\infty d\phi_{n-1} \rho(\phi_{n-1}), \quad n \geq 2. \end{aligned} \tag{21}$$

Using the definitions in Eq. (18), it is then easy to see that the persistence $P_n(\epsilon) = Q_{n+1}(-\infty)$. This is due to the fact that for all the ψ_i's in Eq. (18) to be positive, while ϕ_0 is free to take any value, ϕ_1 must be bigger than $-\epsilon\phi_0$, ϕ_2 must be bigger than $-\epsilon\phi_1$ and so on. Differentiating Eq. (21) with respect to x, we get the recursion relation

$$\frac{dQ_n(x)}{dx} = -\rho(x) Q_{n-1}(-\epsilon x), \quad n \geq 1, \tag{22}$$

with $Q_0(x) = 1$ and the boundary condition, $Q_n(\infty) = 0$ for all $n \geq 1$. Let us define the generating function

$$F(x,z) = \sum_{n=1}^{\infty} Q_n(x) z^n. \tag{23}$$

¿From Eq. (22), it follows that $F(x,z)$ satisfies a first order *non-local* differential equation,

$$\frac{\partial F(x,z)}{\partial x} = -\rho(x) z \left[1 + F(-\epsilon x, z)\right], \tag{24}$$

with the boundary condition, $F(\infty, z) = 0$ for any z. Once we know the function $F(x,z)$, $P_n(\epsilon)$ can be obtained by evaluating the Cauchy integral,

$$P_n(\epsilon) = Q_{n+1}(-\infty) = \frac{1}{2\pi i} \int_{C_0} \frac{F(-\infty, z)}{z^{n+2}} dz, \tag{25}$$

over a contour C_0 encircling the origin in the complex z plane.

Before proceeding to solve Eq. (24), we make the simple observation that,

$$P_n(\epsilon) = P_n\left(\frac{1}{\epsilon}\right), \tag{26}$$

true for any ϵ. To see this, we first rescale the ψ_i variables, ${\psi_i}' = \psi_i/\epsilon$. Clearly the persistence of ${\psi_i}'$'s is the same as that of the ψ_i's. Dividing Eq. (18) by ϵ, we see that in order for the ${\psi_i}'$'s to be positive, we need to satisfy the conditions: $\phi_0 > -\phi_1/\epsilon$, $\phi_1 > -\phi_2/\epsilon$, ..., $\phi_{n-1} > -\phi_n/\epsilon$ where ϕ_n can be arbitrary. Eq. (26) then follows once we relabel $\phi_i \to \phi_{n-i}$ for all $0 \leq i \leq n$. Thus it is sufficient to compute $P_n(\epsilon)$ for ϵ only in the range, $-1 \leq \epsilon \leq 1$. Once we know this, $P_n(\epsilon)$ for $|\epsilon| > 1$ can be obtained from Eq. (26).

Let us summarize our main results. We show that for $-1 < \epsilon \leq 1$, $P_n(\epsilon) \sim \exp(-\theta(\epsilon) n)$ for large n, where $\theta(\epsilon)$ depends continuously on ϵ and also depends on the distribution $\rho(\phi)$. In contrast, at $\epsilon = 1$, the exponent $\theta(1) = \log(\frac{\pi}{2})$ is independent of the distribution $\rho(\phi)$. The exponent $\theta(\epsilon)$ diverges as $\epsilon \to -1$, indicating a faster than exponential decay of P_n for large n. We show that $P_n(-1) = 1/(n+1)!$ exactly for all $n \geq 1$, again independent of the distribution $\rho(\phi)$.

A. The case when $\epsilon = -1$

Let us first consider the case $\epsilon = -1$. In this case, the Eq. (24) becomes local and can be easily solved by integration. For symmetric $\rho(\phi)$ with zero mean, the exact solution is given by

$$F(x,z) = -1 + \exp\left[z\left(\frac{1}{2} - \int_0^x \rho(x')dx'\right)\right], \tag{27}$$

which satisfies the boundary condition $F(\infty, z) = 0$ for all z. Expanding the exponential in Eq. (27) in powers of z and using the definition in Eq. (23), we find, $Q_n(x) = (\frac{1}{2} - \int_0^x \rho(x')dx')^n/n!$. Using the relation $P_n = Q_{n+1}(-\infty)$ and the normalization condition $\int_{-\infty}^{\infty} \rho(x')dx' = 1$, we get

$$P_n(-1) = \frac{1}{(n+1)!}, \tag{28}$$

for all $n \geq 1$. Remarkably $P_n(-1)$ is independent of the distribution $\rho(\phi)$ for *all* $n \geq 0$.

B. The case when $\epsilon = 1$

Next we consider the case $\epsilon = 1$. We first make a change of variable, $u(x) = \int_0^x \rho(\phi)d\phi$. Let $F(x,z) = \tilde{F}(u,z)$. Since $\rho(\phi)$ is symmetric about zero, $u(-x) = -u(x)$ and hence $F(-x,z) = \tilde{F}(-u,z)$. Using this in Eq. (24) with $\epsilon = 1$, we find

$$\frac{\partial \tilde{F}(u,z)}{\partial u} = -z\left[1 + \tilde{F}(-u,z)\right], \tag{29}$$

where u varies from $-1/2$ to $1/2$ and the boundary condition is, $\tilde{F}(1/2, z) = 0$ for all z. Differentiating Eq. (29) with respect to u we get a local second order differential equation

$$\frac{\partial^2 \tilde{F}(u,z)}{\partial u^2} = -z^2\left[1 + \tilde{F}(u,z)\right], \tag{30}$$

whose general solution is given by

$$\tilde{F}(u,z) = -1 + \left[A(z)\cos(zu) + B(z)\sin(zu)\right]. \tag{31}$$

If this solution also has to satisfy Eq. (29), we have additionally $B(z) = -A(z)$. The boundary condition $\tilde{F}(1/2, z) = 0$ determines $A(z)$ and we finally get

$$F(x,z) = -1 + \frac{\cos(u(x)z) - \sin(u(x)z)}{\cos(z/2) - \sin(z/2)}. \tag{32}$$

Thus, $F(-\infty, z) = 2/[\cot(z/2) - 1]$. This function has poles at $z = \pi/2 + 2m\pi$, where m is an integer. One can then easily evaluate the contour integration in Eq. (25) and we get the exact expression,

$$P_n(1) = 2\sum_{-\infty}^{\infty} \frac{1}{\left(\frac{\pi}{2}+2m\pi\right)^{n+2}}, \tag{33}$$

valid for any $n \geq 0$. For example, by summing the series in Eq. (33) we find, $P_0(1) = 1$, $P_1(1) = 1/2$, $P_2(1) = 1/3$, $P_3(1) = 5/24$, etc. which can also be verified by performing the direct integration in Eq. (21). The remarkable fact is $P_n(1)$ is universal for *all* $n \geq 0$ in the sense that it is independent of the distribution $\rho(\phi)$, as in the $\epsilon = -1$ case. Clearly the leading asymptotic behavior is governed by the $m = 0$ term in Eq. (33) and we get, $P_n(1) \sim \exp(-\theta n)$ for large n, with $\theta(1) = \log(\pi/2)$. Clearly the exponent $\theta(1)$ is also universal.

Interestingly, $P_n(1)$ is related to the fraction of metastable states in an Ising spin glass on a 1-dimensional lattice of n sites at zero temperature [28]. Consider the spin glass Hamiltonian on a chain, $H = -\sum_i J_{i,i+1}s_i s_{i+1}$ where $s_i = \pm 1$ are Ising variables and the bonds $J_{i,i+1}$'s are independent and identically distributed variables each drawn from the same symmetric distribution with zero mean. Out of the 2^n number of total configurations, how many are metastable with respect to single spin flip Glauber dynamics at zero temperature? A configuration is metastable at zero temperature if the energy change $\Delta E_i = 2s_i[J_{i-1,i}s_{i-1} + J_{i,i+1}s_{i+1}] \geq 0$ due to the flip of every spin s_i. Defining the new variable $\phi_i = 2J_{i,i+1}s_i s_{i+1}$, we see that the variables ϕ_i's are also independent and identically distributed and the probability that a configuration is metastable is precisely the probability that the variables, $\psi_i = \phi_i + \phi_{i-1}$ are positive for each i. This is precisely $P_n(1)$ as computed in the previous paragraph. We note that the average number of metastable configurations $\langle N_s\rangle$ for the 1-d spin glass was computed exactly by Derrida and Gardner [29] by a different method and they found $\langle N_s\rangle \sim (4/\pi)^n$ for large n. Thus the fraction of metastable configurations scales as $\langle N_s\rangle/2^n \sim (\pi/2)^{-n}$, in agreement with our exact result for $P_n(1)$.

C. The case $-1 < \epsilon < 1$

We now turn to the range, $-1 < \epsilon < 1$. In this range, we were unable to calculate $P_n(\epsilon)$ exactly for arbitrary distribution $\rho(\phi)$. However progress can be made for the uniform distribution,

$$\begin{aligned}\rho(\phi) &= \frac{1}{2}, \quad \text{for } -1 \leq \phi \leq 1 \\ &= 0, \quad \text{otherwise.}\end{aligned} \tag{34}$$

For this case, it follows from Eq. (24) that $F(x,z)$ is independent of x for $x < -1$ and hence, $F(-\infty,z) = F(-1,z)$. Similarly, $F(x,z) = 0$ for all $x \geq 1$. In the range, $-1 \leq x \leq 1$, we expand $F(x,z) = \sum_0^\infty b_m(z)x^m$ in a power series in x. Subsituting this series in Eq. (24), we get the recursion relation, $b_m = -b_{m-1}z(-\epsilon)^{m-1}/2m$ for all $m \geq 1$. Thus the function $F(x,z)$ can be expressed completely in terms of only $b_0(z)$ which is then determined from the boundary condition, $F(1,z) = 0$. This determines $F(x,z)$ completely in the range $-1 \leq x \leq 1$ and we find, $F(x,z) = -1 + \frac{f(xz)}{f(z)}$, where

$$f(z) = \sum_{m=0}^{\infty} \frac{(-1)^{m(m+1)/2}}{m!}\left(\frac{z}{2}\right)^m \epsilon^{m(m-1)/2}. \tag{35}$$

Using $F(-\infty,z) = F(-1,z)$, we finally get

$$F(-\infty,z) = -1 + \frac{f(-z)}{f(z)}, \tag{36}$$

where $f(z)$ is given by Eq. (35). We note that the series in Eq. (35) and hence in Eq. (36) is convergent for all z as long as $-1 < \epsilon \leq 1$. In fact, for $\epsilon = 1$, it is easy to see that Eq. (36) gives $F(-\infty,z) = 2/[\cot(z/2) - 1]$ as before.

Substituting Eq. (36) in the expression of $P_n(\epsilon)$ in Eq. (25), we find that the leading asymptotic decay of P_n for large n is governed by the pole of $F(-\infty,z)$ that is closest to the origin. From Eq. (36), the poles of $F(-\infty,z)$ in the z plane are precisely the zeroes of the function $f(z)$ in Eq. (35) in the z plane. In particular, $P_n(\epsilon) \sim {z_+}^{-n}$ for large n where z_+ is the zero of $f(z)$ in Eq. (35) closest to the origin. The persistence exponent is then simply, $\theta = \log(z_+)$. Let us first consider a few special cases. For $\epsilon = 1$, we find from Eq. (35), $f(z) = \cos(z/2) - \sin(z/2)$ indicating $z_+ = \pi/2$, a result we already obtained. For $\epsilon = 0$, we find from Eq. (35), $f(z) = 1 - z/2$, indicating $z_+ = 2$, as expected for the persistence of uncorrelated variables. As $\epsilon \to -1^+$, the function $f(z)$ is Eq. (35) approaches to, $f(z) \to \exp(-z)$ indicating $z_+ \to \infty$. Indeed putting $\epsilon = -1 + \delta$ in Eq. (35), it is easy to see that, $z_+ \approx \sqrt{8/\delta}$ as $\delta \to 0$. Thus the persistence exponent diverges as, $\theta \approx \log[\sqrt{8/(1+\epsilon)}]$ as $\epsilon \to -1$.

For other values of ϵ in the range $-1 < \epsilon < 1$, it is easy to evaluate z_+ to any arbitrary accuracy from Eq. (35) using Mathematica. The exponent $\theta = \log(z_+)$ for some representative values of ϵ in the range $-1 < \epsilon \leq 1$ are listed in Table 1. The exponent $\theta(\epsilon)$ increases monotonically as ϵ decreases from $+1$ to -1, diverging as $\epsilon \to -1$. For $|\epsilon| > 1$, the exponent is determined from the relation, $\theta(\epsilon) = \theta(1/\epsilon)$. Thus in the whole range, $-\infty < \epsilon < \infty$, the exponent $\theta(\epsilon)$ is a nonmonotonic function of ϵ. As ϵ varies from $-\infty$ to ∞, $\theta(\epsilon)$ increases monotonically in the range $[-\infty, -1]$, then decreases monotonically in $[-1, 1]$ followed by a further monotonic increase in the range $[1, \infty]$. The slowest decay of P_n occurs at $\epsilon = 1$, where $\theta(\epsilon)$ is minimum and given by the universal value, $\theta(1) = \log(\pi/2)$.

ϵ	θ
1.0	$\log(\pi/2) = 0.4515\ldots$
3/4	$0.4690\ldots$
1/2	$0.5155\ldots$
1/4	$0.5882\ldots$
0	$\log(2) = 0.6931\ldots$
-1/4	$0.8465\ldots$
-1/2	$1.0906\ldots$
-3/4	$1.5686\ldots$

Table 1. The exponent $\theta(\epsilon)$ for some representative values of ϵ in the range, $-1 < \epsilon \leq 1$ in the case of the uniform distribution in Eq. (34).

Except at $\epsilon = 0$, 1 and -1, the exponent $\theta(\epsilon)$ is nonuniversal in the sense that its value depends on the details of the distribution $\rho(\phi)$. To see this explicitly, we now compute $\theta(\epsilon)$ perturbatively for small ϵ. We expand the right hand side of Eq. (24) upto order ϵ and then solve the resulting local differential equation exactly to determine $F(x,z)$ upto $O(\epsilon)$. Taking $x \to -\infty$ limit in the expression of $F(x,z)$, we find

$$F(-\infty, z) = \frac{2z}{[2 - z - 2c\rho(0)\epsilon z^2]}\ , \tag{37}$$

where $c = \int_0^\infty \phi\rho(\phi)d\phi$. From Eq. (37), the pole closest to the origin is given by,

$$z_+ = 2\left[1 - 4c\rho(0)\epsilon + O(\epsilon^2)\right]. \tag{38}$$

From Eq. (25), it then follows that $P_n(\epsilon) \sim {z_+}^{-n}$ for large n. Hence $\theta(\epsilon) = \log(z_+) = \log(2) - 4c\rho(0)\epsilon + O(\epsilon^2)$ and is clearly nonuniversal, as seen from the nonuniversality of the $O(\epsilon)$ term in the previous equation. For example, for the uniform distribution in Eq. (34), we get $\theta(\epsilon) = \log(2) - \epsilon/2 + O(\epsilon^2)$. On the other hand for the Gaussian distribution, $\rho(\phi) = (2\pi)^{-1/2}\exp(-\phi^2/2)$, we get $\theta(\epsilon) = \log(2) - 2\epsilon/\pi + O(\epsilon^2)$.

V. CONCLUSION

In summary, we have discussed persistence in stochastic processes that are stationary only integer translations of time. Such a process can be explicitly constructed by smearing independent noises with a convolution function $f(t)$. A physical example of such a process is provided by the diffusion field on a hierarchical lattice for which we have computed the smearing function $f(t)$ exactly. However, we could not compute the persistence exponents θ_c or θ_d in this case. We showed that under certain conditions, the continuous-time persistence of such a process reduces to the persistence of a discrete sequence obtained by measuring the process only at integer times. We have constructed a specific non-Markovian sequence where the smearing function is nonzero only at two consecutive integer points leading to nonzero correlations only between consecutive values of the sequence and computed the persistence exponent θ_d exactly for this sequence. The exponent θ_d depends continuously on the strength ϵ of the correlation. Remarkably for $\epsilon = 1$ and $\epsilon = -1$, the persistence becomes universal. For $\epsilon = 1$, we have shown an interesting connection between the persistence of this sequence to the average fraction of metastable states in a one dimensional spin glass.

The class of functions $f(t)$ for which we could show that $\theta_c = \theta_d$ is perhaps not the most general. A precise characterization of this class seems like an interesting problem. Calculation of the persistence for SIS processes, or sequences, with correlations extending beyond nearest neighbors may be possible in some special cases, and would help understand the general question about the dependence of the persistence exponent on the correlations in the sequence.

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