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On The Deconfinement Transition in SU(4) Lattice Gauge Theory

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ABSTRACT

The deconfinement transition in $SU(4)$ lattice gauge theory is investigated on $N_s^3 \times N_t$ lattices for $N_s = 8$–16 and $N_t = 4$–8 using a modified Wilson action which is expected to be free of any bulk transitions. The susceptibility $\chi_{|L|}^{\max}$, where L is the order parameter for deconfinement, is found to increase *linearly* with spatial volume for $N_t = 4$, 5, and 6, indicating a first order deconfinement phase transition. The latent heat of the transition is estimated to be $\approx \frac{2}{3}$ of the corresponding ideal gas energy density at T_c.

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1. INTRODUCTION

It is widely expected that the ongoing experiments at the Relativistic Heavy Ion Collider, BNL, New York, and those at the proposed Large Hadron Collider, CERN, Geneva, will provide us a strong evidence for Quark-Gluon Plasma and may even yield crucial details about this new phase and the nature of the phase transition(s) leading to it. The physics driving this phase transition is naturally of great interest. While the real world has two light (u,d) flavours of quarks and one somewhat heavier (s) flavour, both analytical and numerical methods in lattice Quantum Chromodynamics (QCD) begin from the limiting cases of either massless or infinitely massive quarks. One talks of the chiral symmetry restoring phase transition and the deconfinement phase transition in these two cases respectively and has suitable order parameters to investigate them. For quarks with N colours and N_f flavours, these transitions are related to spontaneous breaking of a global $Z(N)$ and $SU(N_f) \times SU(N_f)$ chiral symmetry, which QCD has in the infinite quark mass and zero quark mass limit respectively. Which of them is more relevant in the real world, is *a priori* not clear, since these symmetries are broken explicitly to various extents. The low masses of the light flavours suggest chiral symmetry to be the dominant one, leading one to expect a chiral transition in QCD at finite temperature. Indeed, the behaviour of the corresponding order parameter, the chiral condensate $\langle \bar{\psi}\psi \rangle$, confirms this expectation in numerical simulations. However, it is also seen in the same numerical simulations that even the order parameter for the deconfinement transition,

$$L(\vec{x}) = \frac{1}{N} \mathrm{tr} \prod_{t=1}^{N_t} U_4(\vec{x}, t) \quad , \tag{1}$$

acquires nonzero values at this chiral transition; in principle, it could have done so already at temperatures a lot below the chiral transition since no symmetry prevents it from doing so. The energy density also shows a large change at the transition which would be unexpected from a naive count of degrees of freedom at a chiral transition but is in fact consistent with that expected of a deconfinement transition. These apparently mysterious observations for QCD with light dynamical quarks can be explained [1] using arguments based on the large N limit, *if* the deconfinement transition for

$N \geq 4$ is of *second order*. $SU(4)$ is clearly the simplest case to check whether this is so.

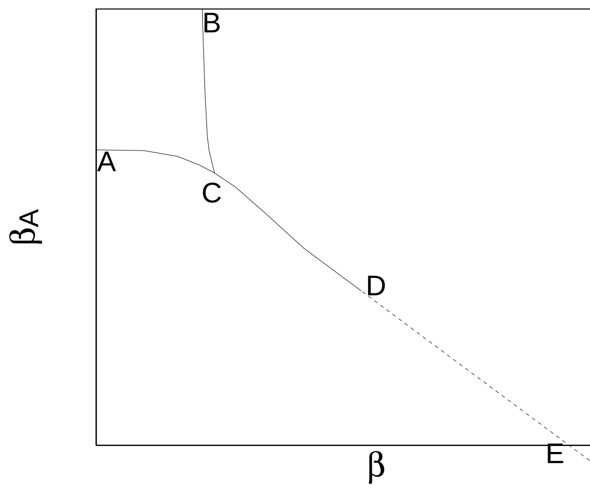

Figure 1: A schematic phase diagram in (β, β_A)-plane for the mixed action of eq. (2).

Numerical simulations of $SU(4)$ theory at finite temperatures have been done in the past [2] and recently [3] as well. All of them reported a first order deconfinement transition but they all used the Wilson action, or the more general mixed action:

$$S = \sum_P \left[\beta \left(1 - \frac{\text{Re tr } U_P}{N} \right) + \beta_A \left(1 - \frac{\text{tr}_\text{A}\, U_P}{N} \right) \right] . \tag{2}$$

Here U_P denotes the directed product around an elementary plaquette P of the link (gauge) variables, $U_\mu(x)$, located at site x in the direction μ =1–4, tr and tr_A denote the traces of U_P in the fundamental and adjoint representations, the β's are the corresponding couplings, and N is the number of colours. The sum over P is over all independent plaquettes. A well known problem in the simulations with these actions, especially for large N, is the presence of bulk transitions which are lattice artifacts. The generic phase diagram of the mixed action (2) in its coupling plane is depicted schematically in Fig. 1. The solid lines in it show first order bulk transition lines. The dotted line after the end point D is drawn to suggest the impact D may have on the Wilson axis ($\beta_A = 0$). For $N \geq 4$, D is expected to be where E is shown, causing a first order bulk transition for the usual Wilson action. In order to avoid it, simulations were made at negative β_A [2] and/or for larger [3] $N_t = 6$ and obtained a first order deconfinement phase transition for $SU(4)$.

Extensive studies [4] of the deconfinement phase transition for the action above for the $N = 2$ case have, however, shown that the presence of bulk transitions affect the order and location of the deconfinement transition in subtle and somewhat inexplicable ways. For example, recall that one expects the coupling at which the deconfinement transition takes place to move logarithmically with N_t, the temporal extension of the lattice whereas a shift in the bulk transition point is expected to be inversely proportional to the 4-volume of the lattice, $N_s^3 N_t$, where N_s is the spatial lattice size. This was the rationale behind the choice of simulation parameters in the earlier [2, 3] studies of the $SU(4)$ theory. However, we [4] found that either this expectation is incorrect or there are apparent qualitative violations of universality for the $SU(2)$ theory since its deconfinement transition was of second order for small β_A but was of first order for large enough β_A for an entire range of N_t from 2 up to 8, with hardly any shift with N_t. Indeed, universality was restored [5] in that case only after eliminating the bulk transitions associated with the $Z(2)$ vortices and $Z(2)$ monopoles by adding large chemical potentials for them. It seems natural to expect that the bulk transitions for $N = 4$ may also have affected the earlier results on the order of its deconfinement transition for the action above for $N_t = 4$ or 6. A cleaner determination of the order of the deconfinement transition for $SU(4)$ may be obtained by wiping off the corresponding bulk transitions by suppressing the relevant $Z(N)$ objects. We pursue this idea here to investigate the deconfinement phase transition in the $SU(4)$ theory.

2. SIMULATIONS AND RESULTS

Generalizing the idea of positive plaquette models [6] in the literature for the $SU(2)$ lattice gauge theory to the case of $SU(N)$ theories, we propose to simulate the action

$$S = \beta \sum_P \left(1 - \frac{\text{Re tr } U_P}{N}\right) \cdot \theta\left(\frac{\pi}{N} - |\alpha|\right) , \tag{3}$$

for $N = 4$, where $-\pi < \alpha \leq \pi$ is the phase of tr U_P. By adding the adjoint term of eq. (2) to the action (3), one sees that the phase diagram of the resultant mixed action should not have the bulk lines AC or BC and hence the endpoint D or E. This is because the action (3) restricts tr U_P to lie only in one $Z(N)$-sector for all, even small, β.

We have simulated the above action on $N_s^3 \times N_t$ lattices for N_s =8, 10, 12, 15, 16 and N_t = 4, 5, 6, 8 using a 15-hit Metropolis et al. algorithm. The calculations were done on a cluster of pentiums. Typically short runs to look for points of rapid variations in $\langle |L| \rangle$ were followed by long runs (a few million sweeps) to determine the susceptibility $\chi_{|L|}$,

$$\chi_{|L|} \ = N_s^3 \ (\langle |L|^2 \rangle - \langle |L| \rangle^2) \ , \tag{4}$$

as a function of β using the histogramming technique [7]. In order to monitor the presence of any bulk transition, we also studied the plaquette susceptibility given by,

$$\chi_P \ = 6 N_s^3 N_t \ (\langle P^2 \rangle - \langle P \rangle^2) \ , \tag{5}$$

where P denotes the average of (Re tr U_P)/N over the entire lattice. Usual finite size scaling techniques were used to determine the order of the transition and its exponents. According to the finite size scaling theory [8], the peak of the $|L|$ (or plaquette) susceptibility at the location of the deconfinement (or bulk) transition should grow on $N_s^3 \times N_t$ lattices like

$$\chi^{\max}_{|L| \text{ or } P} \propto N_s^{\omega} \ , \tag{6}$$

for fixed N_t. For a second order transition, $\omega \equiv \gamma / \nu$, where γ and ν characterize the growth of the $|L|$ (plaquette)-susceptibility and the correlation length near the critical temperature (coupling) on an infinite spatial lattice. If the phase transition were to be of first order instead, then one expects [9] the exponent $\omega = 3$, corresponding to the dimensionality of the space. In addition, of course, the $\langle |L| \rangle$ or $\langle P \rangle$ is expected to exhibit a sharp, or even discontinuous, jump. The corresponding probability distribution should then show a double peak structure in stead of a usual gaussian distribution.

In simulations on $N_s^3 \times 4$ lattices, N_s = 8, 10, 12, we found the hot (all link variables random) and cold (all links equal to identity) starts to converge quickly to a unique value of $|L|$ at couplings a little away from the transition point on its both sides but a clear co-existence of states was visible for all

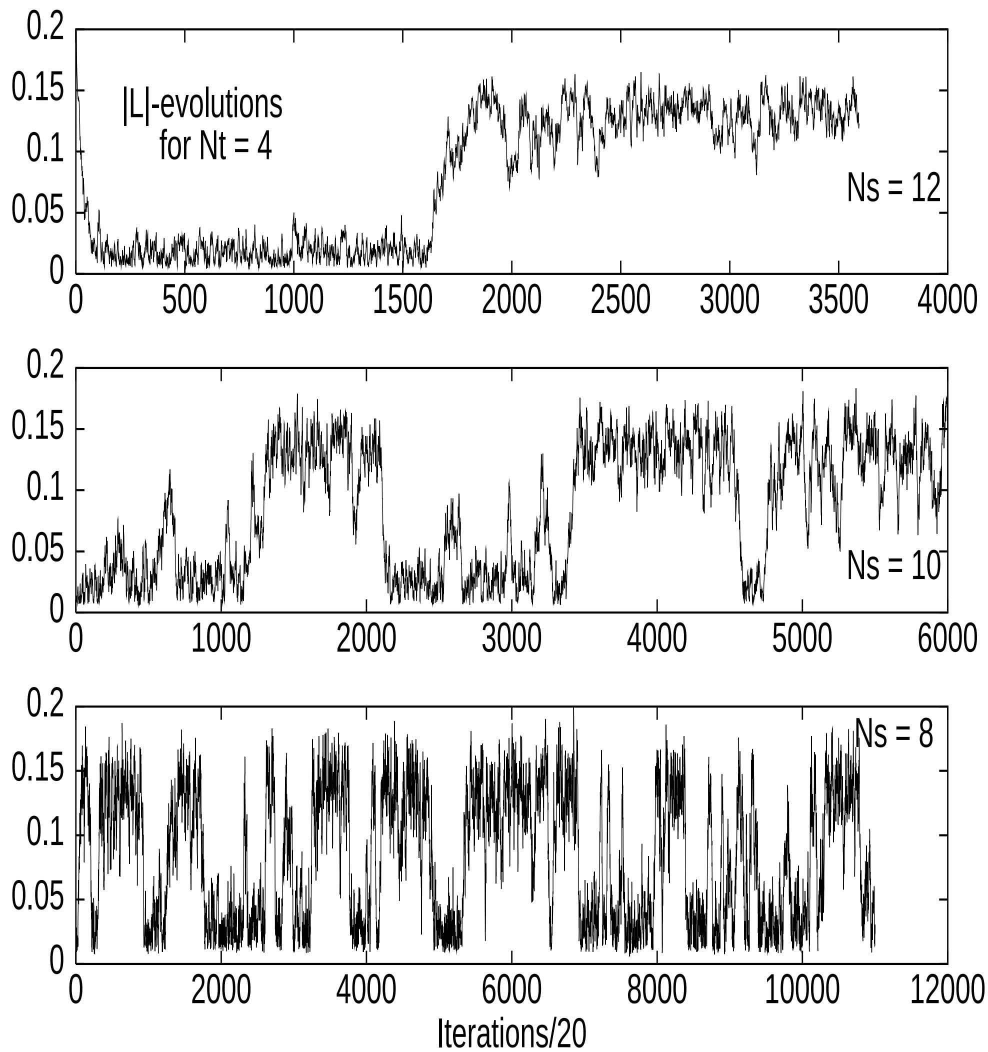

Figure 2: The evolution of $|L|$ as a function of Monte Carlo time for $N_s^3 \times 4$ lattices close to the transition point.

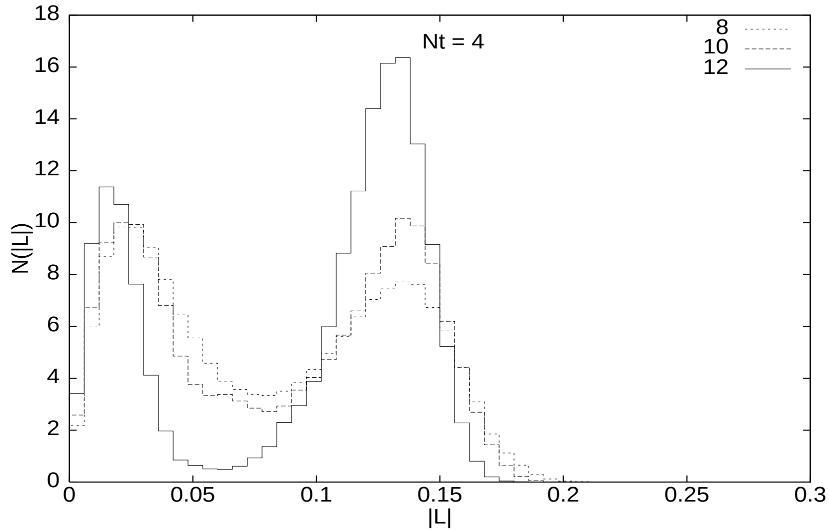

Figure 3: The histograms of $|L|$ on $N_s^3 \times 4$ lattices for long runs close to the transition point.

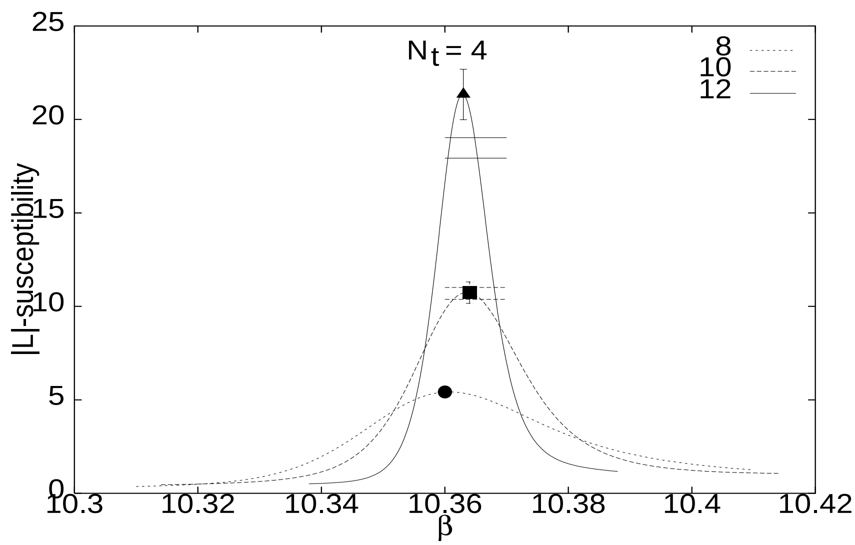

Figure 4: The susceptibility $\chi_{|L|}$ as a function of β for $N_s^3 \times 4$ lattices.

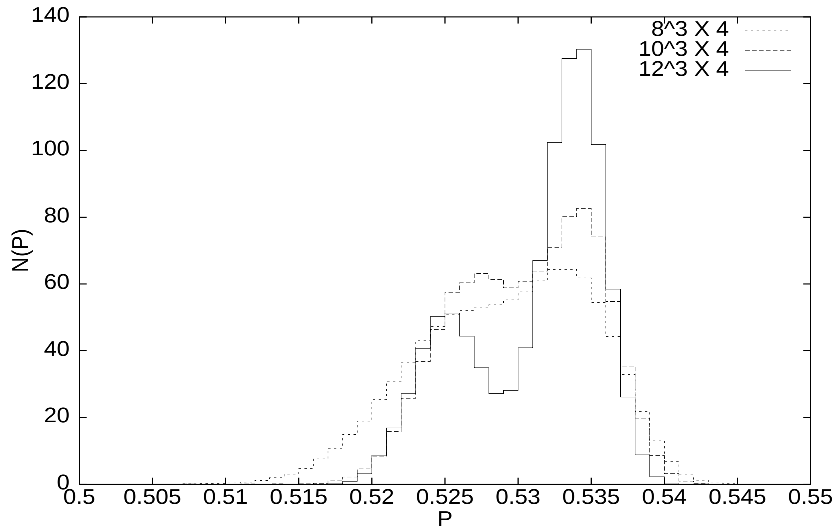

Figure 5: The histograms of the plaquette P on $N_s^3 \times 4$ lattices for the same runs as in Fig. 3.

lattices at the transition point, as shown in Fig. 2. It is clearly seen that the tunnelling frequency goes down with spatial volume. The histograms of $|L|$, shown in Fig. 3, display peaks which become narrower with increasing volume while the gap between them remains unchanged. These classic signs of a first order phase transition are confirmed by a quantitative analysis of the growth of $\chi_{|L|}$ with volume, as seen in Fig. 4. The horizontal lines in each case are predictions obtained by scaling the $N_s = 8$ results linearly with volume, as expected from eq. (6) and $\omega = 3$ for a first order transition.

Although the above first order transition is clearly a deconfinement transition, as the behaviour of the order parameter $|L|$ certifies, it could, of course be due to a coincident bulk transition. One can check for this possibility by analogous studies of the average plaquette P. Fig. 5 displays the histograms of P for the same runs as in Figs. 2 and 3. These histograms do exhibit curious non-gaussian structures, with a suggestion to develop a discontinuity as the spatial volume grows. The corresponding susceptibilities turn out to be too noisy to be conclusive, especially when compared to the behaviour of $|L|$-susceptibility in Fig. 4. However, their growth with volume is not inconsistent with being linear. We therefore made simulations on larger symmetric lattices up to the sizes 12^4 at these couplings and in their immediate neighbourhood and found only usual gaussian distributions for the plaquette. For a genuine bulk transition, the behaviour in Fig. 5 should have been accentuated with the three-fold increase in the 4-volume. Recall that the presence of latent heat at a first order deconfinement phase transition can also be a source of the behaviour in Fig. 5. This should then lead to specific predictions as the temporal lattice size is increased. We postpone discussing them in the next section, turning in stead to investigations of larger N_t.

For larger N_t, we used many longer runs in the region of strong variation of $\langle|L|\rangle$ to obtain the susceptibility directly and used the histogramming technique only for the finer determinations of the critical coupling. Our results for $\langle|L|\rangle$ as a function of β clearly show the expected shift for a deconfinement phase transition for $N_t = 4$ and 6, as can be seen in Fig. 6. As is usual for the $SU(2)$ and $SU(3)$ theories, the rise of the order parameter

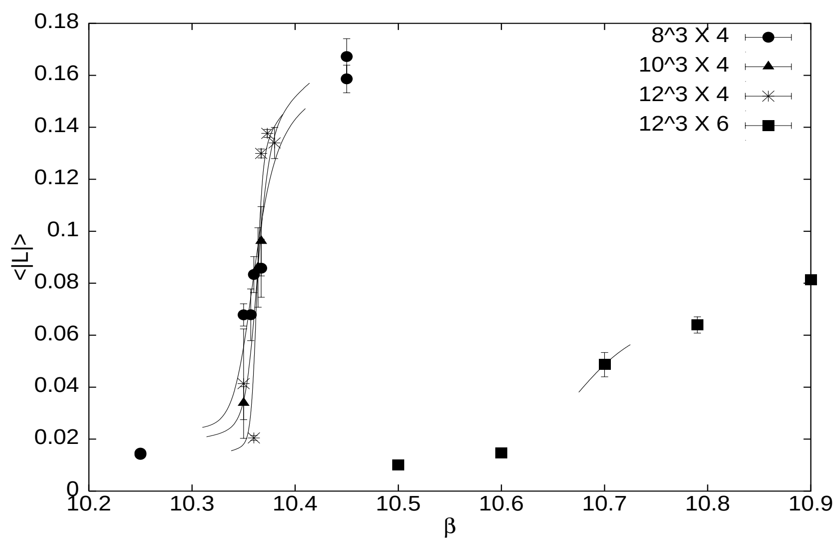

Figure 6: $|L|$ as a function of β for lattices with N_t 4 and 6.

at β_c is seen to be slower in Fig. 6 as N_t is increased due to the well-known ultra-violet effects. The β_c values, determined from the $\chi_{|L|}^{\max}$ and listed in Table 1 for $N_t = 4$, 5 and 6 for different spatial volumes, also evidently

Table 1: The values of β at which long simulations were performed on $N_s^3 \times N_t$ lattices, β_c and the height of the $|L|$-susceptibility peak, $\chi_{|L|}^{\max}$.

$N_s^3 \times N_t$	β	β_{c,N_s}	$\chi_{\|L\|}^{\max}$	$\chi_{\text{predicted}}^{\max}$
$8^3 \times 4$	10.360	10.360(2)	5.48(16)	–
$10^3 \times 4$	10.364	10.364(5)	11.16(35)	10.71(32)
$12^3 \times 4$	10.363	10.363(5)	22.41(1.11)	18.5(6)
$10^3 \times 5$	10.520	10.515(5)	4.63(40)	–
$15^3 \times 5$	10.520	10.525(5)	14.02(1.50)	15.6(1.3)
$12^3 \times 6$	10.675	10.686(5)	4.36(35)	–
$16^3 \times 6$	10.675	10.676(5)	10.43(95)	10.3(8)

suggest strong shifts in β_c with N_t. Comparing our results in Table 1 with those of Ref. [3] for $N_t = 6$ for the usual Wilson action, i.e, for eq.(2) with $\beta_A = 0$, one finds a shift by ≈ 0.1 towards smaller coupling. Using in each case the peak height for the smaller spatial volume, the $\chi^{\max}$ on the bigger lattice can be predicted, assuming a first order deconfinement phase transition. The predictions listed in Table 1 can be seen to be in very good agreement with the direct Monte Carlo determinations. Along with the shifts in β, these confirm that the same *physical* first order deconfinement phase transition is being simulated on these lattices as the continuum limit of $a \to 0$ is approached in a progressive manner by increasing the temporal lattice size N_t.

3. SCALING AND LATENT HEAT

A quantitative test of the fact that the first order transition reported in the previous section is indeed a physical deconfinement transition (and not a bulk transition) consists of translating all the β_c in Table 1 to the corresponding transition temperature in physical units and thus checking whether it is constant while the β_c's shift with N_t as expected from the renormalization group equation,

$$aT_c = \frac{1}{N_\tau} = \frac{T_c}{\Lambda_L} \left(\frac{4b_0}{\beta}\right)^{-b_1/b_0^2} \exp\left(-\frac{\beta}{8b_0}\right) \quad , \tag{7}$$

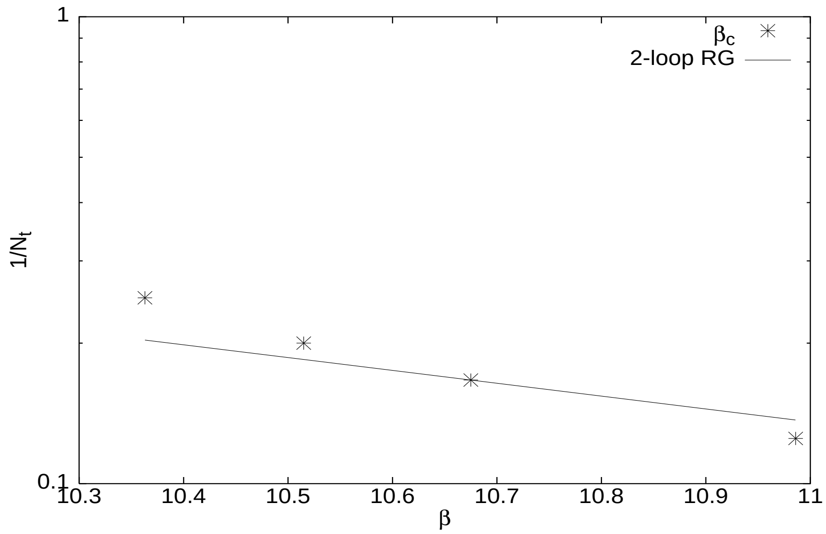

Figure 7: N_t^{-1} as a function of β along with the RG-curve mentioned in the text.

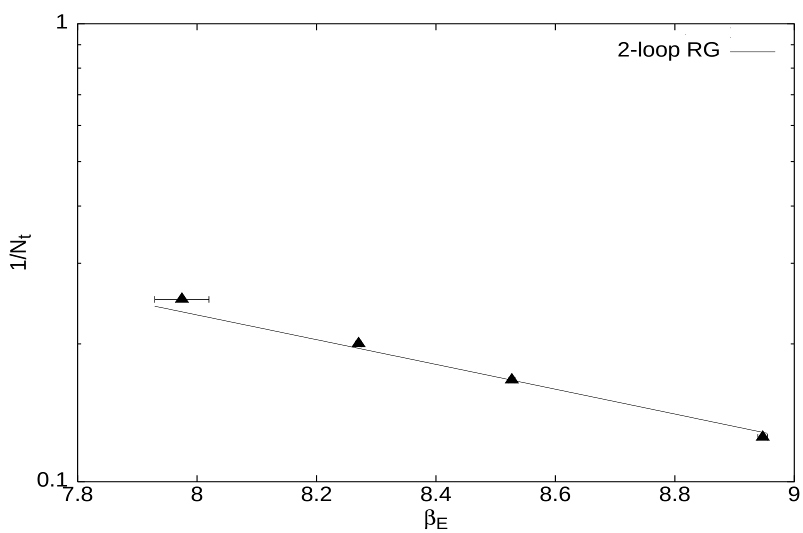

Figure 8: N_t^{-1} as a function of β_E along with the corresponding RG-curve.

where

$$b_0 = \frac{11}{12\pi^2}, \quad \text{and} \quad b_1 = \frac{17}{24\pi^4} \quad , \tag{8}$$

are the first two coefficients of the perturbative β-function for the $SU(4)$ Yang-Mills theory. Fig. 7 exhibits a comparison of our set of β_c for N_t =4, 5, 6 and 8 with eq.(7). The normalization was chosen by demanding the line to pass through the point at $N_t = 6$. While the shifts are in accord with the expectations, one also sees strong quantitative deviations. These are, however, not unusual: similar deviations have been seen in the studies for N = 2 and 3 [10] as well. As pointed earlier [11], and tested successfully [10] for $SU(2)$ and $SU(3)$, one possible cure for removing these deviations is to employ a better choice of the coupling in eq.(7) which consists of replacing β there by

$$\beta_E = \frac{0.25(N^2-1)}{1-\langle P\rangle} \ . \tag{9}$$

Fig. 8 displays our data using this new variable along with the corresponding RG-curve of eq.(7) normalized the same way as before. One sees an excellent agreement, providing a concrete quantitative evidence in favour of the physical nature of the transition. Furthermore, it leads to an estimate of T_c $= 18.5 \pm 0.5 \Lambda_L$, where the error is estimated by requiring all the data points to lie within the band generated by it.

Apart from the characteristic (logarithmic) shift of the transition point with N_t, seen above, the latent heat of a first order deconfinement phase transition provides yet another quantitative test of the continuum limit since it should also remain constant as $N_t \to \infty$. Requiring the pressure to be continuous at the deconfinement phase transition, the latent heat can be obtained from two different observables $\Delta_1 \equiv \Delta(\epsilon - 3p)/T_c^4$, and $\Delta_2 \equiv \Delta(\epsilon + p)/T_c^4$, where Δ denotes discontinuities across the transition in the respective variables,

$$\begin{aligned} \Delta_1 &= -48 N_t^4 a \frac{\partial g^{-2}}{\partial a} \Delta P \ , \\ \Delta_2 &= 32 N_t^4 \frac{C(g^2)}{g^2} (\Delta P_t - \Delta P_s) \ , \end{aligned} \tag{10}$$

and $C(g^2) = (1 - 0.2366 g^2 + O(g^4))$ for $SU(4)$ [3]. We will employ the perturbative β-function in Δ_1 with its coefficients given by eq. (8). In order

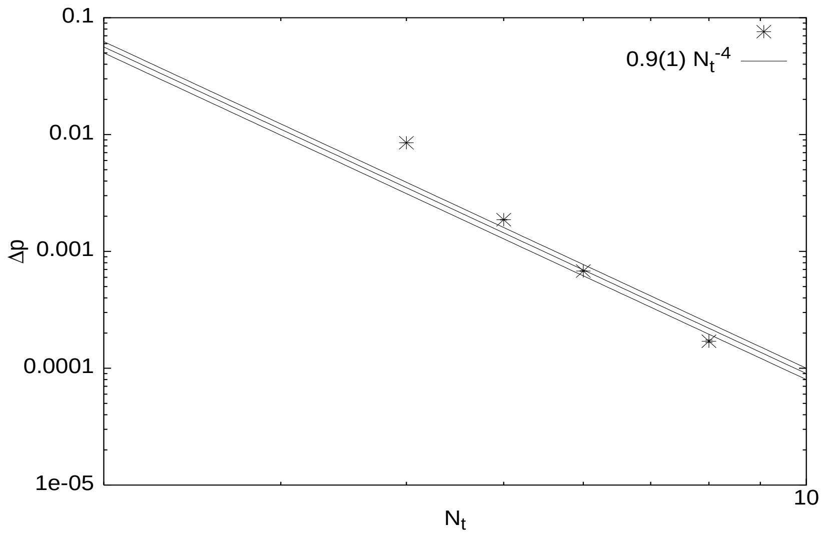

Figure 9: The plaquette discontinuity ΔP as a function of N_t.

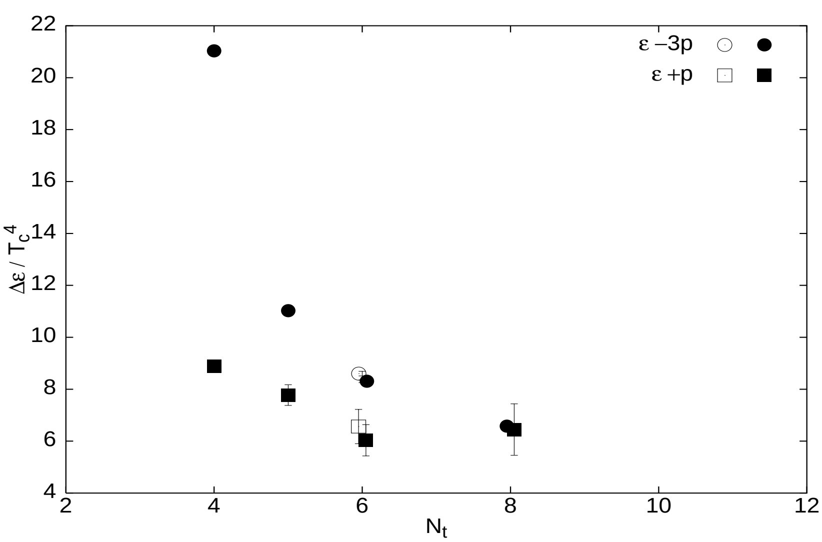

Figure 10: The two latent heat heat estimates of eq. (10) as a function of N_t.

Table 2: Both the latent heat estimates of eq.(10) as a function of N_t.

N_t	4	5	6	8
Δ_1	21.03(5)	11.02(6)	8.31(5)	6.57(16)
Δ_2	9.89(14)	7.77(40)	6.04(60)	6.45(99)

to obtain the ΔP, ΔP_s and ΔP_t, the minimum of the histogram $N(|L|)$ was used to separate the two phases in each case. The errors were estimated by varying it within the limits suggested by the corresponding histograms. From eq. (10), it is clear that the plaquette discontinuity $\Delta P \propto N_t^{-4}$ if one is to obtain the same latent heat in physical units, since the rest of the factors vary mildly with N_t. In case of a bulk phase transition, on the other hand, one would expect the plaquette discontinuity to remain constant or exhibit a mild increase. Indeed, as Fig. 9 displays, its decrease with N_t is seen to be consistent with these expectations for $N_t \geq 5$. Our estimates of latent heat from Δ_1 and Δ_2 are shown in Fig. 10 as a function of N_t (filled symbols) along with the corresponding results of Ref. [3] for $N_t = 6$ (open symbols) which can be seen to be in good agreement with ours. Both the estimates in eq.(10) must agree with each other as $N_t \to \infty$, since the neglected cut-off corrections become then insignificant. Table 2 and Fig. 10 verify this to be the case. One may therefore take the results for N_t =8 as a reasonable approximation to the continuum limit. Comparing them with the energy density of the corresponding ideal gas at T_c, $\epsilon_{SB} = \pi^2(N^2-1)T_c^4/15 = \pi^2 T_c^4$, one obtains for the latent heat,

$$\frac{\Delta\epsilon}{\epsilon_{SB}} = 0.6657 \pm 0.0162 \ . \tag{11}$$

Here we have used Δ_1 to get the result above. Using Δ_2 leads to 0.65 ± 0.10.

4. **SUMMARY AND DISCUSSION**

Establishing the order of the deconfinement phase transition for the $SU(4)$ lattice gauge theory at finite temperature is important for our understanding of the physics of the phase transition to quark-gluon plasma. Possible subtle influences, which the bulk transitions for the mixed action of eq. (2) (see Fig. 1) may have, necessitate an approach to bypass them. In this paper, we have attempted this by generalizing the idea of positive plaquette model for the $SU(2)$ case and simulating the action of eq. (3) to investigate the $SU(4)$ theory at $T \neq 0$ on $N_s^3 \times N_t$ lattice with $N_s \geq 2N_t$ and $N_t = 4$, 5, 6 and 8.

Various qualitative indicators, such as, the histograms and evolutions of the order parameter L, suggest a *first* order deconfinement phase transition for $SU(4)$ on $N_t = 4$ lattices with N_s varying up to 12. The linear growth of $\chi_{|L|}^{\max}$ with volume for $N_t = 4$, quantitatively confirms this finding. Increasing N_t to 5 and 6, one again finds a growth in $\chi_{|L|}^{\max}$ that is consistent with being linear in volume. Defining the transition coupling as the location at which the maximum of the above susceptibility occurs, one finds a significant shift in it as N_t is varied from 4 to 8. The amount of shift is consistent with the expectations of a physical deconfinement transition; a bulk transition would have had much smaller shifts. Indeed, the set of β_c for N_t =4, 5, 6 and 8 is consistent with the asymptotic scaling relation of eq.(7) when used in conjunction with the improved coupling of eq. (9), leading to a determination of the transition temperature $T_c = (18.5 \pm 0.5)\Lambda_L$.

The presence of latent heat at a first order phase transition necessarily means a discontinuity in the average plaquette on a finite lattice, which can however also be due to an unphysical bulk transition. We find that the plaquette discontinuity ΔP decreases with increase in N_t as approximately the fourth power of N_t. This indicates both a lack of a bulk transition and the presence of a first order deconfinement phase transition, since it suggests i) a vanishing discontinuity in the $N_t \rightarrow \infty$ limit ii) but in a manner that leaves the latent heat constant in physical units. We estimate the latent heat to be (0.666 ± 0.016) times the corresponding ideal gas energy density at T_c using the perturbative β-function in eq. (10). This value, obtained from our N_t =8 simulation, is expected to be close to the continuum limit since the two different estimates from eq. (10) coincide. Comparing with the corresponding value (using again perturbative β-function) for the $SU(3)$

theory, which is about 0.43 [12], one finds that the deconfinement transition grows stronger in nature as the number of colours N is increased. Whether this continues as N is increased further is at present not clear. However, already the known results for $N = 3$ and 4 are sufficient to cast doubt on the explanation proposed in Ref. [1]. Consequently, the mystery of the behaviour of the energy density and the deconfinement order parameter, L, mentioned in the introduction remains.

5. ACKNOWLEDGMENTS

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