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Exact chiral invariance at finite density on the lattice

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We propose a lattice action for the overlap Dirac matrix with nonzero chemical potential which is shown to preserve the chiral invariance on the lattice exactly. We further demonstrate it to arise from the Domain wall by letting the chemical potential count only the physically relevant wall modes.

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I. INTRODUCTION

Our world of strongly interacting particles has two light quarks with masses much smaller than Λ_{QCD}, the scale of the corresponding theory, Quantum Chromo Dynamics (QCD). While the mass of the strange quark is comparable to this scale, other quarks are heavy. It has been argued on the basis of the corresponding symmetries of the theory, called the chiral symmetries, that QCD may have a critical point in the temperature (T)-baryon number density (or equivalently, the baryonic chemical potential μ_B). Theoretical [1–3] as well as experimental searches for locating it are currently going on [4]. Its discovery would be exciting in many ways. Apart from becoming a new milestone in our understanding of the nature of the strongly interacting matter, it would also be unique compared to the other known phase diagrams in the way theory and experiment compliment in locating the critical point in it.

Due to the essentially non-perturbative nature of the problem, theoretical efforts based on a first principles approach employ lattice QCD techniques which have made successful predictions for many hadronic quantities such as the heavy meson decay constants. Indeed, it has also been very successful in application at finite temperature, leading to several interesting results for the RHIC and other heavy ion experiments [5]. But its foray in the finite density domain has been hampered by serious conceptual problems. In spite of the so-called 'fermion sign problem', which refers to the complex measure of the theory for nonzero chemical potential, many attempts have been made to explore nonzero μ_B, some claiming success [1, 2, 6] in locating it, while others arguing for a lack [7] of a critical point. Most of these computations use the staggered quarks, primarily since the critical point is thought to be related to chiral symmetry. Staggered quarks have a $U(1)_V \times U(1)_A$ chiral symmetry on the lattice, and a corresponding order parameter, the chiral condensate. Studying its variation with T or μ_B, one can look for a chiral phase transition. The critical point would then just be an end point of a line of first order phase transitions.

Staggered quarks, however, i) break the flavour and spin symmetry on the lattice, ii) have no flavour singlet axial symmetry and iii) have a stronger [8] 'rooting problem' at nonzero μ_B. Since the QCD critical point is expected [9] to exist if only if one has two light flavours, and the flavour singlet anomaly is mildly temperature dependent [10], it appears desirable to improve upon them. The overlap Dirac fermions [11], or the closely related domain wall fermions [12], offer such a possibility to improve. Indeed, the overlap quarks have all the symmetries of the continuum QCD and also have a an index theorem [13] as well, raising the hope that even the anomaly effects could be well treated. Unfortunately, adding the chemical potential turns out to be nontrivial for them. Bloch and Wettig [14] made a proposal to do so but it violates [15] the exact chiral invariance on lattice as does the simple addition [16] of a baryon number term. In this letter we propose an alternative which does have exact chiral invariance on lattice for any value of the lattice spacing and any chemical potential. It therefore can permit, in principle, the task of mapping out the T-μ_B phase diagram, assuming that the algorithmic developments can handle the fermion sign problem well.

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II. FORMALISM

The massless continuum QCD action can be written in a form where the chiral symmetry is manifest in terms of the fields appearing in the action:

$$\begin{aligned} S_{QCD} &= \int d^3x \; d\tau [\bar{\psi} \not{D} \psi - F^{\mu\nu} F_{\mu\nu}/4] \\ &= \int d^3x \; d\tau [\sum_{i=L,R} (\bar{\psi}_i \not{D} \psi_i) - F^{\mu\nu} F_{\mu\nu}/4] \; , \end{aligned} \tag{1}$$

where $\psi_L = (1-\gamma_5)\psi/2$ and $\psi_R = (1+\gamma_5)\psi/2$ with $\bar{\psi}_L = \bar{\psi}(1+\gamma_5)/2$ and $\bar{\psi}_R = \bar{\psi}(1-\gamma_5)/2$. The second term is the action for gluons which will play no role in our discussion below. We assume below that some usual convenient form has been chosen. Adding the canonical μN to the first line for investigating finite density effects is the same as adding $\mu \bar{\psi}_i \gamma^4 \psi_i$ to the second line, leaving the manifest chiral symmetry intact. We propose that addition of chemical potential on the lattice be done in this explicit chiral symmetry preserving manner as well, opting therefore for the overlap quarks.

The overlap quarks have all the symmetries of the continuum QCD but also have a nonlocal action:

$$S_F = \sum_{n,m} \bar{\psi}_n \; aD_{ov,nm} \; \psi_m \; , \tag{2}$$

where the sum over n and m runs over all the space-time lattice sites, a is the lattice spacing, and the overlap Dirac matrix D_{ov} is defined by $aD_{ov} = 1 + \gamma_5 sgn(\gamma_5 D_W)$. sgn denotes the sign function. D_W is the standard Wilson-Dirac matrix on the lattice but with a negative mass term $M \in (0,2)$:

$$D_W(x,y) = (4-M)\delta_{x,y} - \sum_{i=1}^{4} [U_i^\dagger(x-\hat{i})\delta_{x-\hat{i},y} \frac{1+\gamma_i}{2} + \frac{1-\gamma_i}{2} U_i(x)\delta_{x+\hat{i},y}] \; . \tag{3}$$

The overlap Dirac matrix satisfies Ginsparg-Wilson relation [17], $\{\gamma_5, D\} = aD\gamma_5 D$ and has exact chiral symmetry on lattice. The corresponding infinitesimal chiral transformations [18] are

$$\delta\psi = i\alpha\gamma_5(1 - \frac{a}{2}D_{ov})\psi \quad \text{and} \quad \delta\bar{\psi} = i\alpha\bar{\psi}(1 - \frac{a}{2}D_{ov})\gamma_5 \; . \tag{4}$$

An alternate set of transformations, differing by terms of $\mathcal{O}(a)$ is

$$\delta\psi = i\alpha\gamma_5(1 - aD_{ov})\psi \quad \text{and} \quad \delta\bar{\psi} = i\alpha\bar{\psi}\gamma_5 \; . \tag{5}$$

Since one needs to write chiral projectors on the lattice to mimic eq.(1), we focus on eq.(5). The generators of the transformation in eq.(5) satisfy $\gamma_5^2 = 1$ and $\hat{\gamma}_5^2 \equiv [\gamma_5(1 - aD_{ov})]^2 = 1$. On the other hand, the corresponding ones in eq.(4) do not square to unity. One therefore defines the left-right projections for quark fields as $\psi_L = (1-\hat{\gamma}_5)\psi/2$ and $\psi_R = (1+\hat{\gamma}_5)\psi/2$, leaving the antiquark field decomposition as in the continuum above. Such a decomposition is commonly done for writing chiral gauge theories [19] on the lattice and is possible since ψ and $\bar{\psi}$ are independent fields in the Euclidean field theory. In analogy with the eq.(1) of continuum QCD, the action for the overlap quarks in presence of nonzero chemical potential may now be written down as

$$S = \sum_n \sum_{i=L,R} [\bar{\psi}_{n,i}(aD_{ov} + a\mu\gamma^4)\psi_{n,i}] \tag{6}$$

$$= \sum_n \bar{\psi}_n [(1 - a\mu\gamma^4/2)aD_{ov} + a\mu\gamma^4]\psi_n \; . \tag{7}$$

It is easy to verify that i) this action is invariant under the chiral transformation eq.(5) for all values of $a\mu$ and a and ii) it reproduces the continuum action in the limit of $a \to 0$ with $\mu \to \mu/M$ scaling. In order to obtain the order parameter for checking if the symmetry is spontaneously broken, one usually adds a linear breaking term. Adding a quark mass term, $am(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$, one obtains the order parameter valid for all T and μ on the lattice by taking a derivative of the log of the partition function with respect to am as,

$$\langle\bar{\psi}\psi\rangle = \lim_{m\to 0} \lim_{V\to\infty} \langle \mathrm{Tr} \; \frac{(1 - aD_{ov}/2)}{[aD_{ov} + (am + a\mu\gamma^4)(1 - aD_{ov}/2)]} \rangle \; . \tag{8}$$

The only real eigenvalues of aD_{ov} are 0 or 2, with only the former contributing to the order parameter. Defining a matrix $K_{ov} = D_{ov}(1 - aD_{ov}/2)^{-1}$, such that $\{\gamma_5, K_{ov}\} = 0$, the order parameter can be written in a form more analogous to the continuum :

$$\langle \bar\psi\psi \rangle = \langle \mathrm{Tr}\ \frac{1}{aK_{ov} + am + a\mu\gamma^4} \rangle\ . \tag{9}$$

Although the discussion above is for a single flavour of quark, i.e, $U(1)_L \times U(1)_R$ symmetry, its generalization to N_f flavours is straightforward. Indeed, since it relies only on the spin-structure, the flavour index as well as the corresponding generator matrices just carry through.

III. DOMAIN WALL FERMIONS

The action in eq.(7) was obtained by demanding the left-right symmetry to be explicit. A physically more intuitive way to arrive it is in the domain wall formalism. Since only the massless domain wall modes are physical although there are many massive unphysical modes, the appropriate way to introduce μ is as a Lagrange multiplier for the number of these massless modes. The domain wall action of [20] then for nonzero chemical potential, μ, is

$$\begin{aligned} S \ &= \sum_{x,x'} \sum_{s,s'=1}^{N_5} a^4 \bar\psi(x,s) \Big[-\delta_{x,x'} \left(P_- \delta_{s',s+1} + P_+ \delta_{s',s-1} \right) + (\frac{a_5}{a} D_W(x,x') + \delta_{x,x'}) \delta_{s,s'} \\ &+ \ a\mu\ \gamma_4\ \delta_{x,x'} \left(\delta_{s,1}\delta_{s',1} P_- + P_+ \delta_{s,N_5}\delta_{s',N_5} \right) + am\ \delta_{x,x'} \left(\delta_{s,1}\delta_{s',N_5} P_+ + P_- \delta_{s,N_5}\delta_{s',1} \right) \Big] \psi(x',s')\ , \end{aligned} \tag{10}$$

where $P_\pm = (1 \pm \gamma_5)/2$ and N_5 and a_5 are the number of sites and the lattice spacing in the fifth direction respectively. The physically relevant 4D massless fermion field is identified with the fermion fields at the boundaries of the fifth dimension as,

$$\psi = P_- \psi_1 + P_+ \psi_{N_5} \quad , \quad \bar\psi = \bar\psi_1 P_+ + \bar\psi_{N_5} P_-\ . \tag{11}$$

It is convenient to visualize the five dimensional action in terms of the fields η_i localized on four dimensional branes existing at each site i along the fifth dimension, as in [21]. Defining a transfer matrix, $T = (1 + a_5 H_W P_+)^{-1}(1 - a_5 H_W P_-)$, between pairs of neighboring branes, where $H_W = \gamma_5 D_W$, the five dimensional action can be rewritten in terms of these fields as,

$$\begin{aligned} S \ &= \sum_{x,s} \Big[\bar\eta_1 \left(P_- - maP_+ + a\mu \left(a_5 H_W P_- - 1 \right)^{-1} \gamma_4 P_- \right) \eta_1 - \bar\eta_{N_5} T^{-1} \left(P_+ - maP_- - a\mu \left(a_5 H_W P_+ + 1 \right)^{-1} \gamma_4 P_+ \right) \eta_1 \\ &- \ \bar\eta_1 T^{-1} \eta_2 + \sum_{s=2}^{N_5-1} \left(\bar\eta_s \eta_s - \bar\eta_s T^{-1} \eta_{s+1} \right) + \bar\eta_{N_5} \eta_{N_5} \Big]\ . \end{aligned} \tag{12}$$

The fermion fields η_i can be integrated out successively, resulting in a partition function of the form,

$$\mathcal{Z}_{5D} \equiv \int \mathcal{D}U \mathrm{e}^{-\mathrm{S_G}}\ \mathrm{J}\ \ \mathrm{det\, D^{(5)}(ma, a\mu)}\ , \tag{13}$$

where J is the Jacobian for the transformation from ψ-fields to η-fields and the five dimensional determinant $D^{(5)}$ is given by

$$\mathcal{D}^{(5)} = \det \Big[P_- - maP_+ + a\mu \left(a_5 H_W P_- - 1 \right)^{-1} \gamma_4 P_- - T^{-N_5} \left(P_+ - maP_- - a\mu \left(a_5 H_W P_+ + 1 \right)^{-1} \right) \gamma_4 P_+ \Big] . \tag{14}$$

In order to obtain the overlap matrix, the contribution of the bulk five dimensional modes needs to be removed from the partition function. Following [22], we introduce pseudo-fermions and obtain the partition function of interest as

$$\mathcal{Z}_{QCD}(T,\mu) = \int \mathcal{D}U\ \mathrm{e}^{-\mathrm{S_G}}\ \frac{\det \mathrm{D^{(5)}(ma, a\mu)}}{\det \mathrm{D^{(5)}(ma = 1, a\mu = 0)}} \tag{15}$$

where $\det D^{(5)}(ma = 1, a\mu = 0)$ is the contribution from the pseudo-fermions. Taking first $a_5 \to 0$ limit and then $N_5 \to \infty$, the ratios of determinants turns out to be $\det[D_{ov} + (1 - D_{ov}/2)(ma + a\mu\gamma^4)]$, where the dimensional μ and m have been scaled by a factor of M to match with the continuum limit. A little algebra shows that γ^4 can be commuted through in the determinant above to yield the same overlap matrix of eq.(7) with exact chiral symmetry on the lattice.

IV. DISCUSSION

The action in eq.(7) leads to an overlap fermion determinant which is identical to that in the recent work [23] with fermionic sources in the overlap formalism of [24]. The main difference is, however, in the necessity of sources in [23] to define the chiral symmetry. Indeed, the chiral symmetry transformation there is local, defined as rotation of the sources while our eq. (5) is nonlocal, defined as the rotation of quark fields. The left-right symmetry is in-built in the formalism there whereas we needed to introduce the left-right projections in form of L- and R-fields to do so. Our new fermion matrix is linear in μ_B, similar to an earlier proposal by us [16]. This leads us to expect it to have the same good as well as bad properties. In particular, only its first derivative with respect to $a\mu_B$ is nonzero, all the rest being zero. As a consequence, the coefficients of the Taylor expansion of the baryonic susceptibility in μ_B, needed in estimating the location of the QCD critical point simplify considerably. On the other hand, the corresponding free theory itself has a μ_B^2-divergence in the baryonic susceptibility in the continuum limit. It is easy to see that the successful prescription [25] for local actions of introducing μ_B as the fourth component of a constant Abelian gauge field to remove the divergence, will not work in this case since the nonlocal overlap Dirac matrix intertwines all four momentum components in a covariant manner. We have recently shown in case of the staggered fermions that the free theory divergence can be subtracted [26] out successfully. In particular, the resultant ratios of the Taylor coefficients appear to be in good agreement with those where the subtraction is effected analytically by a change of action. While further investigations of the finite cut-off and finite volume effects are needed to establish it firmly, it may be hoped that a similar subtraction scheme will work for our above overlap quarks as well. Of course, it would be clearly desirable to modify even the action (7), without loss of its chiral symmetry, such that it has no μ_B^2-divergences.

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