

# Zero-temperature hysteresis in the random-field Ising model on a Bethe lattice

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**Abstract.** We consider the single-spin-flip dynamics of the random-field Ising model on a Bethe lattice at zero temperature in the presence of a uniform external field. We determine the average magnetization as the external field is varied from  $-\infty$  to  $+\infty$  by setting up the self-consistent field equations, which we show are exact in this case. The qualitative behaviour of magnetization as a function of the external field unexpectedly depends on the coordination number  $z$  of the Bethe lattice. For  $z = 3$ , with a Gaussian distribution of the quenched random fields, we find no jump in magnetization for any non-zero strength of disorder. For  $z \geq 4$ , for weak disorder the magnetization shows a jump discontinuity as a function of the external uniform field, which disappears for a larger variance of the quenched field. We determine exactly the critical point separating smooth hysteresis curves from those with a jump. We have checked our results by Monte Carlo simulations of the model on three- and four-coordinated random graphs, which for large system sizes give the same results as on the Bethe lattice, but avoid surface effects altogether.

## 1. Introduction

Recently, a simple model was introduced [1] for hysteresis in magnets, which incorporates interesting effects such as the return-point memory and Barkhausen noise [2]. In this model, Ising spins with a quenched random field at each site evolve by a zero-temperature single-spin-flip dynamics. The authors argued that in this model, if the external field is increased slowly, the steady-state magnetization as a function of the field has a jump discontinuity at some critical value of the field for a small disorder, but is a continuous function with no jump discontinuity for large disorder. This picture was supported by numerical simulations of the model on hypercubic lattices in two and higher dimensions. Subsequent work [3] studied in detail the transition from jump to no jump in magnetization at a critical value of the Gaussian disorder, and observed scaling behaviour in the neighbourhood of this critical disorder. However, the exact solution of this model in one dimension shows a jump discontinuity for Gaussian disorder neither for the ferromagnetic nor antiferromagnetic exchange couplings [4].

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In this paper, we extend the treatment of [4] to study this hysteresis model on a Bethe lattice. The Bethe lattice of coordination  $z$  is the formal infinite-size limit of a branching tree (the Cayley tree) where each spin has  $z$  nearest neighbours, and the statistical averages are calculated away from the ‘surface’ of the lattice. For  $z = 3$ , the lowest non-trivial coordination, we find no jump in magnetization for any non-zero disorder, if the quenched random fields have a Gaussian distribution—just as in the one-dimensional case. For coordination  $z > 4$ , we find there is a non-zero critical disorder where the macroscopic jump in magnetization in the hysteresis loop first disappears.

This is very surprising, as in all the models studied on the Bethe lattice that we are aware of, the qualitative behaviour of the solution has been found to be independent of the coordination number (so long as it is greater than 2). In particular, a Bethe lattice with finite coordination number  $z$  has the same critical behaviour as the mean-field theory, which corresponds to the limit of large coordination number  $z$  and coupling constant scaling as  $1/z$ . The reason why this unusual dependence on  $z$  shows up in this problem is not yet understood.

Our treatment is based on setting up self-consistent equations for some nearest-neighbour correlation function in the problem [5]. We can show that these self-consistent equations are exact in this case, although we are not aware of a rigorous proof that this happens in general for a Bethe lattice in the presence of quenched disorder. In fact, the presence of the disorder usually renders the problem analytically intractable. For example, for the Ising spin-glass problem on a Bethe lattice with random  $\pm J$  bonds, it has not been possible to determine exactly even the zero-temperature quantities such as the ground-state energy or the ground-state entropy [6, 7]. However, the qualitative behaviour of the system near the thermal critical point seems fairly well understood [8].

This paper is organized as follows. In section 2, we define the model precisely. In section 3, we set up recursion relations on a Cayley tree for conditional probabilities that the spin at a given site at height  $r$  from the boundary is down, given that the spin on its parent, ‘upward’ neighbour on the tree is down. We are interested in the intensive quantities, such as magnetization or energy density on the tree far away from the boundary. These turn out to be independent of details of the boundary conditions, and we take this as the definition of Bethe approximation in our case. We obtain an explicit expression for magnetization as a function of external field for arbitrary distribution of the quenched random fields. In section 4, we describe a method of simulating spin systems on the Bethe lattice that is computationally efficient, and does not suffer from surface effects. We use this method to check the validity of our self-consistent equations for the case of Gaussian and rectangular distributions of quenched random fields. The agreement is found to be excellent. Section 5 contains some concluding remarks.

## 2. The model

We consider a lattice of  $N$  sites. Each site is labelled by an integer  $i = 1-N$ , and carries an Ising spin  $S_i$  ( $S_i = \pm 1$ ) which interacts with a finite number  $z$  of neighbouring spins with a ferromagnetic interaction  $J$ . There is a uniform magnetic field  $h$  which is applied externally. In addition, at each site  $i$ , there is a local quenched random field  $h_i$ . The fields  $\{h_i\}$  are assumed to be independent identically distributed random variables with a continuous probability distribution  $p(h)$ . This is the well known random-field Ising model

[9], and is described by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i S_j - \sum_i h_i S_i - h \sum_i S_i. \quad (1)$$

The zero-temperature single-spin-flip Metropolis–Glauber dynamics [10] is specified by the transition rates

$$\text{Rate } [S_i \rightarrow -S_i] = \begin{cases} \Gamma & \text{if } \Delta E \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where  $\Delta E$  is the change of energy of the system as a result of the spin-flip. We shall be interested in long timescales  $\gg \Gamma^{-1}$ . In this limit, the dynamical rule simplifies to the following: choose a spin at random, and flip it only if this process would lower the energy. Repeat until a stable configuration is obtained.

The problem of hysteresis which we address here is as follows. Start with a sufficiently large negative applied field  $h$ , so that in the stable configuration all spins are down ( $S_i = -1$ , for all  $i$ ) and increase the field slowly. At some value of  $h$ , the local field  $\ell_i$  at some site  $i$ , defined by

$$\ell_i = J \sum_j S_j + h_i + h \quad (3)$$

will become positive, and this spin would flip up. (The summation in (3) is over all the neighbours  $j$  of  $i$ ). This changes the effective field at the neighbours, and some of them may flip up, etc, causing an avalanche of flipped spins. We determine the total magnetization when the avalanche has stopped. Then we raise the applied field a bit more, and determine the magnetization in the stable state again. The process is continued until all the spins flip up. This generates the lower half of the hysteresis loop (plot of magnetization  $m(h)$  versus  $h$ ) in the situation where the applied field is varied very slowly, or equivalently, when the spins relax infinitely fast. The upper half of the hysteresis loop  $m_u(h)$  is obtained when the field  $h$  is decreased from  $+\infty$  to  $-\infty$ . This is related to the lower half of the loop  $m_\ell(h)$  by symmetry

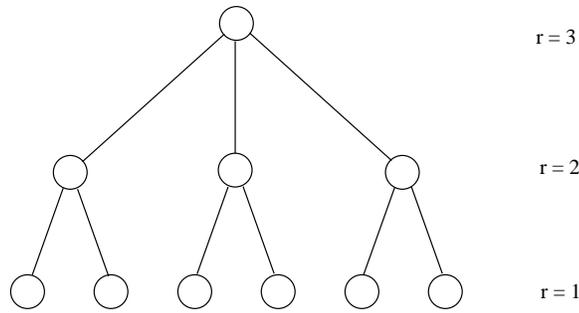
$$m_u(h) = -m_\ell(-h). \quad (4)$$

This corresponds to the zero frequency limit of a driving field oscillating sinusoidally in time with frequency  $\omega$ . Note that the limit of  $\omega \rightarrow 0$  is taken after the limit temperature  $T \rightarrow 0$ . If the limits are taken in the reverse order, the area of the hysteresis loop goes to zero as  $\omega$  goes to zero for all non-zero  $T$ .

An important feature of the above dynamics for ferromagnetic couplings is that if we start with any stable configuration, and then increase the external field and allow the system to relax, then in the relaxation process no spin flips more than once. Furthermore, the final stable configuration is the same whatever the order in which unstable spins are flipped. This property is called the ‘no passing property’ [11], and greatly simplifies the analysis.

### 3. Recursion relations on the Cayley tree

The standard approach for solving statistical mechanics problems on the Bethe lattice is to consider the problem on a Cayley tree, and consider behaviour deep inside the tree, i.e. far from the boundaries of the tree [12, 13]. If suitable care is taken to remove the effects of the boundary, all correlation functions deep inside the Cayley tree (say for the Ising model with an external field) are found to be the same as in the Bethe approximation. Thus, we



**Figure 1.** A Cayley tree of coordination number 3 and height 3.

may say that the Bethe lattice is the *deep interior* part of the Cayley tree. Here we shall use this approach. In section 4, a different approach is presented.

Consider a Cayley tree of height  $n$ . Each site of the tree has coordination number  $z$ , except the boundary sites which have coordination number 1. The level  $n$  consists of only one site  $O$ , called the central site. For  $r \geq 1$  the level  $(n - r)$  has exactly  $z(z - 1)^{r-1}$  sites (figure 1).

We start with the external field  $h$  large and negative, so that ground state of the system is with all spins down. Now, increase the external field to a finite value  $h$ , and flip up any spin for which the net local field is positive. As the same final stable configuration is attained, whatever the order in which spins are relaxed, we may start by first relaxing spins of level 1. Then we relax spins of level 2, then of level 3, etc. If a spin at level  $r$  is flipped up, we check all its descendants again for possible upward flips.

Let  $P_r$  be the conditional probability that a randomly chosen spin at level  $r$  is upturned in this scheme, given that its parent spin at level  $(r + 1)$  is kept down, and the spin and all its descendent spins are relaxed as far as possible. Let  $S_r$  be the spin at level  $r$ . We relax all the descendent spins of  $S_r$  first, keeping  $S_r$  down. In this process, each of the  $z - 1$  direct descendants of  $S_r$  at level  $(r - 1)$  is independently flipped up with probability  $P_{r-1}$ . Hence, the probabilities that  $z - 1, z - 2, \dots, 0$  of the children of  $S_r$  are flipped up in this relaxation process are  $P_{r-1}^{z-1}, (z - 1)P_{r-1}^{z-2}(1 - P_{r-1}), \dots, (1 - P_{r-1})^{z-1}$  respectively. Consider the case where  $s$  of the children are up: since the parent neighbour remains down for this part of the calculation, the net number of down neighbours is  $z - 2s$ , and hence, the spin  $S_r$  will flip up if the local field at this site exceeds  $(z - 2s)J - h$ . Let  $p_s(h)$  denote the probability that the local field at a randomly chosen site is large enough so that the spin will flip up if  $s$  of its children are up, and the uniform field is  $h$ . Clearly

$$\begin{aligned} p_s(h) &= \text{Prob that local field} \geq -h + zJ - 2sJ \\ &= \int_{-h+(z-2s)J}^{\infty} p(h_i) dh_i. \end{aligned} \quad (5)$$

Then it is easily seen, for example for  $z = 3$  that

$$(z = 3) : \quad P_r = P_{r-1}^2 p_2(h) + 2P_{r-1}(1 - P_{r-1})p_1(h) + (1 - P_{r-1})^2 p_0(h). \quad (6)$$

Given a value of  $h$ , we determine the quantities  $p_s(h)$ . Then, using equation (6), and the initial condition  $P_1 = p_1(h)$ , we can recursively determine  $P_r$  for all  $r \geq 2$ . For large  $r \ll n$ ,  $P_r$  tends to a fixed point  $P^*$  given by the self-consistent equation

$$P^* = \sum_{r=0}^{z-1} \binom{z-1}{r} P^{*r} (1 - P^*)^{z-1-r} p_r(h). \quad (7)$$

This is a polynomial equation in  $P^*$ , which can be solved in terms of  $\{p_s(h)\}$ . Finally, for the central site O at level  $n$ , there are  $z$  children, and a similar argument gives

$$\text{Prob}(S_O = +1) = \sum_{r=0}^z \binom{z}{r} P^{*r} (1 - P^*)^{z-r} p_r(h). \quad (8)$$

Substituting the value of  $P^*$ , from equation (7), we determine the probability that this spin  $S_O$  is up, and hence the average magnetization at this site.

The arguments above do not require that all the  $z$  descendent subtrees of O be of equal height. So long as O is sufficiently far from the boundary, we get the same conditional probability  $P^*$ , and hence the same value of magnetization. This proves that all sites ‘deep inside’ the tree have the same average magnetization.

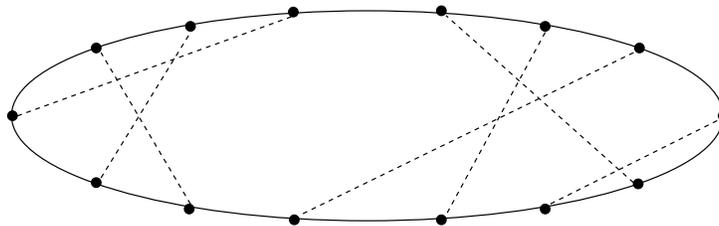
#### 4. Simulations

The derivation of our self-consistent equations assumes the existence of a unique thermodynamic state deep within the Cayley tree which is independent of boundary conditions. While this is quite plausible, uniqueness of the Gibbs state has been proved so far for the RFIM only in one dimension, and only for a bivariate distribution of the quenched field, and non-zero temperatures [14]. It seems desirable to have a direct check of these equations by numerical simulations which do not involve making any assumptions about the thermodynamic state.

While the procedure of the previous section treating the Bethe lattice as sites deep inside the Cayley tree is well known and conceptually simple, it is not suited for numerical simulations. Most of the sites of the Cayley tree are within a short distance from the surface, and cannot be used for averaging. Since the ‘bulk’ forms a negligible fraction of all possible sites, special care has to be taken to subtract the surface contribution. For our simulations, we used a different technique that is computationally efficient and gets rid of surface effects altogether. This technique has already been used to study spin systems on random graphs by Monte Carlo simulations [15].

Our simulation algorithm involves constructing a random graph having  $N$  sites such that each site has exactly  $z$  neighbours. The precise algorithm we used was as follows. Label the  $N$  sites by integers from 1 to  $N$ . We shall assume  $N$  is even in the following. Connect site  $i$  to site  $(i + 1)$  for all  $i$ . Site  $N$  is connected to site 1. This gives us a ring of  $N$  sites. Now, construct  $(z - 2)$  independent random pairing of  $N$  sites into  $N/2$  pairs, and add a bond for each of the paired sites. Thus, we get a graph in which each site has coordination number  $z$  (figure 2).

In this construction, all sites are on the same footing, and there is no ‘surface’. Unlike

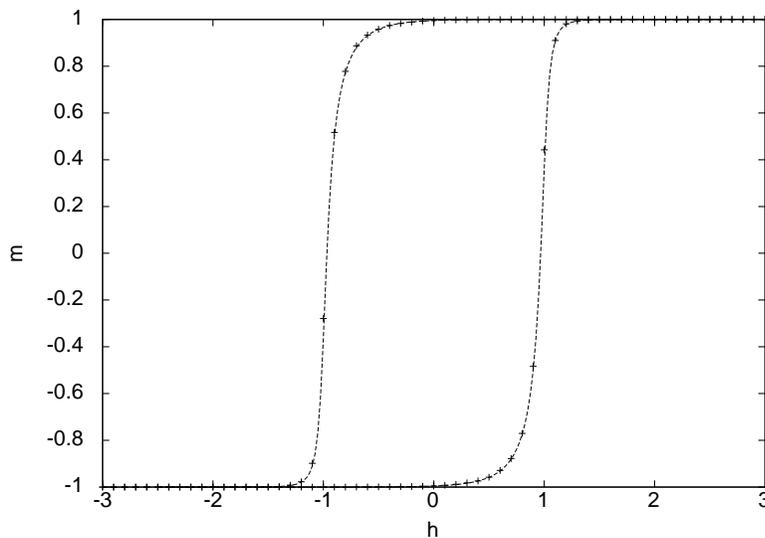


**Figure 2.** An example of a random graph with coordination number 3. Dotted curves indicate the random pairs.

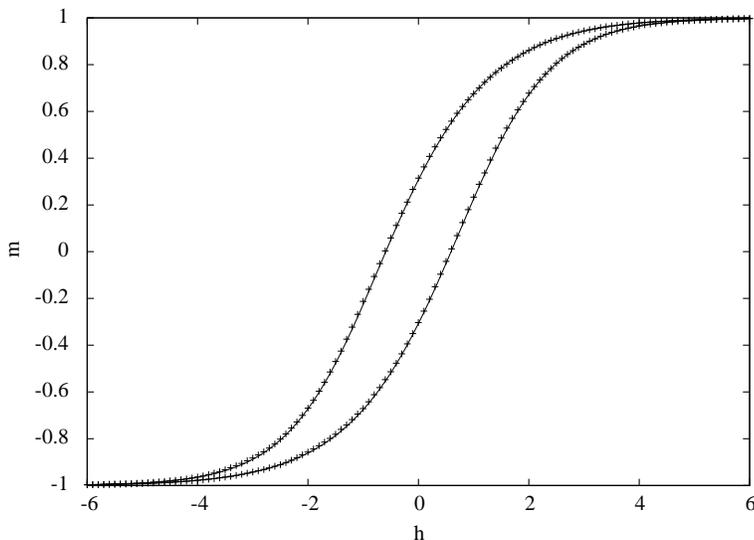
the Cayley tree, this graph has loops. However, it is easy to see that there are typically very few small loops. For example, for  $z = 3$  the probability that sites  $i$ ,  $(i + 1)$  and  $(i + 2)$  form a loop of length 3 is the probability that site  $i$  is paired with  $(i + 2)$ , and equals  $1/(N - 1)$ . Thus the expected number of loops of size 3 in a  $z = 3$  graph of  $N$  sites tends to 1 for large  $N$ . Similarly, it can be shown that the expected number of loops of length 4 is two for large  $N$ . In general, the average number of loops of length  $\ell$  increases as  $\lambda^\ell$  with  $\lambda = z - 1$  for the random graph with coordination number  $z$ , and is a negligible fraction of all sites that belong to any loop of length  $\leq \ell$  for  $\ell \ll \log N / \log \lambda$  [16].

If the smallest loop going through a given site is of length  $\leq (2d + 1)$ , then it follows that up to a distance  $d$  from that site, the lattice looks like a Bethe lattice. Hence, our random lattice would look like a Bethe lattice for  $z = 3$  at almost all sites for a distance  $\lesssim \log_2 N$ . This, in turn, can be shown to imply that in the thermodynamic limit  $N \rightarrow \infty$ , the free energy per site on our lattice for classical statistical mechanical models with short-range interactions (say nearest neighbour only) are the same as in the Bethe–Peierls approximation.

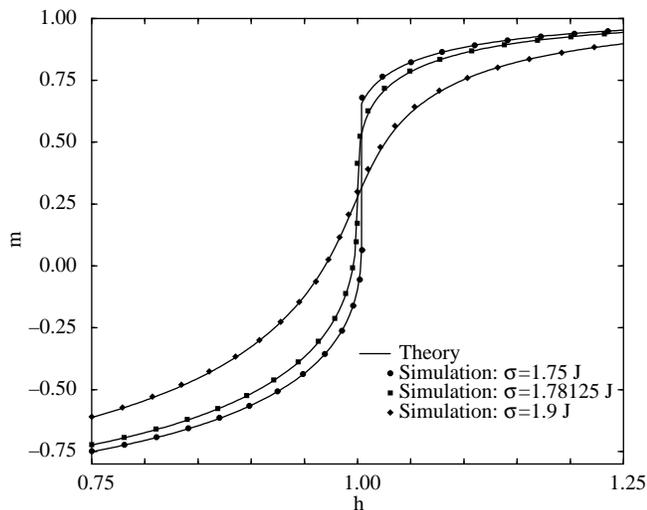
In our simulations, we used  $N = 10^6$ . We used simple scanning to decide which spins to be flipped at the next time step. The dotted curves in figures 3 and 4 show the results of a simulation for  $z = 3$  for quenched Gaussian random fields with mean 0 and variance  $\sigma = 1$  and  $\sigma = 3$  respectively. The lower and upper halves of the hysteresis loop were obtained separately in the simulation. Also shown in the figures are the results of equations (7), (8). The statistical errors of the simulation are quite small. Different runs, with different realizations of quenched fields give results which are indistinguishable at the scale of the graph. The agreement with the theoretical calculation is excellent. For much smaller values of disorder  $\sigma \lesssim 0.1$ , the hysteresis loops are very approximately rectangular. In this case, the value of the coercive field is governed by the largest realized value of quenched local field, which shows noticeable sample-to-sample fluctuations. As noted above, for  $z = 3$  the hysteresis loop is smooth for all values of the disorder greater than zero: the quadratic



**Figure 3.** Hysteresis loop on the Bethe lattice of coordination number 3. The case shown is for standard deviation of quenched random field  $\sigma = J$ . The result of simulation for  $N = 10^6$  spins (points) is in good agreement with our theoretical result (full curve).



**Figure 4.** Hysteresis loop on the Bethe lattice for  $z = 3$  and  $\sigma = 3J$ . The result of simulation for  $N = 10^6$  spins (points) is in good agreement with our theoretical result (full curve).



**Figure 5.** Magnetization curves for the Bethe lattice of coordination number 4 in the increasing field.

equation (6) has only one stable solution.

For  $z = 4$ , we do find a transition. At small disorder, the hysteresis loop has a jump: one large event flips a large, finite fraction of the spins in the thermodynamic limit. Figure 5 shows the analytical and simulation results for  $\sigma = 1.75$ ; there is a large jump in the magnetization at  $h = 1.0037$ . The critical value of the disorder is very slightly larger than this ( $\sigma_c^{(z=4)} = 1.78126$ ) and above the critical disorder the hysteresis loop is smooth. The critical field  $h_c = 1$  at the critical disorder for  $z = 4$  and we conjecture for larger  $z$  as well. This simple result follows from the observation that at  $h = 1$ ,  $P = \frac{1}{2}$  is always a

fixed point for equation (7) for all  $z$ :  $\sigma_c(z)$  may be determined by making this fixed point a double root. This makes the transition a traditional saddle-node transition (the lower branch merges with an ‘unstable’ branch of the self-consistent equation). The critical exponents are thus the same as that for the infinite-range mean-field model (which also undergoes a saddle-node transition in its self-consistent equation). We have checked that this same pattern also occurs for  $z = 5$  (where it gives  $\sigma_c = 2.58201$ ), and conjecture that it gives the correct critical point for all  $z > 3$ ; in  $z = 3$  the coalescence between the stable and unstable branches of the  $M(h)$  curves never occurs.

## 5. Discussion

It is natural to compare the zero-temperature hysteresis on the Bethe lattice with the corresponding infinite-range mean-field result obtained in the limit of large coordination number when the ferromagnetic coupling is taken to be  $J/N$ , the same for all pairs of sites. In this case the mean-field solution is given by [1]

$$m = \operatorname{erf} \left[ \frac{Jm + h}{\sqrt{2\sigma^2}} \right]. \quad (9)$$

For  $\sigma < \sigma_c = \sqrt{2/\pi}$ , the above equation has two solutions  $m_\ell^*(h)$  and  $m_u^*(h)$  which are related to each other by the symmetry  $m_\ell^*(-h) = -m_u^*(h)$ . These correspond to the two halves of the hysteresis loop for increasing and decreasing fields respectively.

For  $\sigma > \sigma_c$ , equation (9) has a single-valued real solution  $m^*(h)$  which is an odd function of  $h$ . Thus there is no hysteresis for  $\sigma > \sigma_c$ . The remanence goes to zero continuously as  $\sigma \rightarrow \sigma_c$  from below. For  $\sigma < \sigma_c$ , there is a discontinuity in magnetization at a critical field  $h_c$ . The value of  $h_c$ , and the magnitude of the jump in magnetization both tend to zero continuously as  $\sigma \rightarrow \sigma_c$ . This lack of hysteresis for  $\sigma > \sigma_c$  is an artefact of the hard-spin infinite-range model: our Bethe lattice has hysteresis at all values of  $\sigma$  (as does the infinite-range model with continuous spins in a double-well potential [1]).

For  $z > 3$  the behaviour as one approaches  $\sigma_c$  from below is similar to the infinite-range model, and the corresponding critical exponents  $\beta$  and  $\delta$  will be the same [1].

So far, we have discussed the case when the quenched random fields have an unbounded distribution. For bounded distributions of the quenched random field, one can get jumps in magnetization even for  $z = 3$ . Consider, for example, the case when  $\{h_i\}$  have a uniform rectangular distribution between  $-h_{\max}$  and  $+h_{\max}$  [17]. If we start with all spins down, and increase the field slowly, clearly nothing happens for  $h < 3J - h_{\max}$ . If  $h$  exceeds this value, then the spin with the largest value of  $h_i$  will flip up. If  $h_{\max} < J$ , this will make the net local field at the neighbours positive. These spins will flip up, which in turn flips their neighbours, etc. Thus, for  $h_{\max} < J$ , the magnetization  $m(h)$  jumps discontinuously from  $-1$  to  $+1$ , as  $h$  cross  $3J - h_{\max}^\dagger$ . If  $J < h_{\max} < 2J$ , one can show that same thing occurs as the system is, on the average, unstable for creating such a ‘nucleus’ of up spins. However, for  $h_{\max} > 2J$ , this particular instability is absent, and the magnetization is a continuous function of  $h$ . Note that the magnetization jump goes *discontinuously* to zero. If the distribution of quenched fields  $p(h_i)$  has delta functions, in addition to a continuous part, clearly this will lead to discontinuities in the  $m(h)$  curve. Any other singularities of  $p(h_i)$ , say at  $h_i = \alpha$  lead to singularities in  $m(h)$  for  $h = \alpha \pm 3J$ ,  $\alpha \pm J$ .

An interesting open question, which we have not been able to answer so far is to characterize all possible ‘metastable’ states on the Bethe lattice. Are all of these obtainable

<sup>†</sup> This value of the coercive field turns out to be dependent on boundary conditions, and is different if there are some boundary spins with fewer number of neighbours.

as solutions of self-consistent equations of the type discussed above? For example, can one calculate the magnetization when the external field is first increased monotonically from  $-\infty$  to a value  $H_1$ , and then reduced to a value  $H_2 < H_1$ ? Further study of such questions would perhaps help our understanding of the more general question of hysteretic dynamics of systems with many metastable states.

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