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Weak*-extreme points of injective tensor product spaces

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ABSTRACT. We investigate weak*-extreme points of the injective tensor product spaces of the form $A \otimes_\epsilon E$, where A is a closed subspace of $C(X)$ and E is a Banach space. We show that if $x \in X$ is a weak peak point of A then $f(x)$ is a weak*-extreme point for any weak*-extreme point f in the unit ball of $A \otimes_\epsilon E \subset C(X, E)$. Consequently, when A is a function algebra, $f(x)$ is a weak*-extreme point for all x in the Choquet boundary of A; the conclusion does not hold on the Silov boundary.

1. Introduction

For a Banach space E we denote by E_1 the closed unit ball in E and by $\partial_e E_1$ the set of extreme points of E_1. In 1961 Phelps [**16**] observed that for the space $C(X)$ of all continuous functions on a compact Hausdorff space X every point f in $\partial_e (C(X))_1$ remains extreme when $C(X)$ is canonically embedded into its second dual $C(X)^{**}$. The question whether the same is true for any Banach space was answered in the negative by Y. Katznelson who showed that the disc algebra fails that property. A point $x \in \partial_e E_1$ is called weak*-extreme if it remains extreme in $\partial_e E_1^{**}$; we denote by $\partial_e^* E_1$ the set of all such points in E_1. The importance of this class for geometry of Banach spaces was enunciated by Rosenthal when he proved that E has the Radon-Nikodym property if and only if under any renorming the unit ball of E has a weak*-extreme point [**19**].

While not all extreme points are weak*-extreme the later category is among the largest considered in the literature. For example we have:

$$\text{strongly exposed} \subsetneq \text{denting} \subsetneq \text{strongly extreme} \subsetneq \text{weak}^*\text{-extreme}.$$

We recall that $x \in E_1$ is *not* a strongly extreme point if there is a sequence x_n in E such that $\|x \pm x_n\| \to 1$ while $\|x_n\| \nrightarrow 0$ (see [**3**] for all the definitions). We denote by $\partial_e^s E_1$ the set of strongly extreme points of E_1. It was proved in [**14**] that $e \in \partial_e^s E_1$ if and only if $e \in \partial_e^s E_1^{**}$ (see [**9**], [**13**], or [**17**] for related results). Examples of weak*-extreme points that are no longer weak*-extreme in the unit ball of the bidual were given only recently in [**6**].

In this paper we study the weak*-extreme points of the unit ball of the injective tensor product space $A \otimes_\epsilon E$, where A is a closed subspace of $C(X)$. Since $C(X) \otimes_\epsilon E$

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can be identified with the space $C(X,E)$ of E-valued continuous functions on X, equipped with the supremum norm, elements of $A\otimes_\epsilon E$ can be seen as functions on X. We are interested in the relations between $f\in\partial_e^*(A\otimes_\epsilon E)$ and $f(x)\in\partial_e^*E_1$, for all (some?) $x\in X$. Since any Banach space can be embedded as a subspace A of a $C(X)$ space no complete characterization should be expected in such very general setting. For example if A is a finite dimensional Hilbert space naturally embedded in $C(X)$, with $X=A_1^*$, and $\dim E=1$, then any norm one element f of $A\otimes_\epsilon E$ is obviously weak*-extreme however the set of points x where $f(x)$ is extreme is very small consisting of scalar multiples of a single vector in A_1^*. Hence in this note we will be primarily interested in the case when A is a sufficiently regular subspace of $C(X)$ and/or x is a sufficiently regular point of X.

It was proved in [**5**] that

$$f\in\partial_e^sC(X,E)_1 \iff [f(x)\in\partial_e^sE_1,\text{ for all } x\in X]. \tag{1.1}$$

It follows from the arguments given during the proof of Proposition 2 in [**6**] that for a function $f\in(A\otimes_\epsilon E)_1$ we have

$$[f(x)\in\partial_e^*E_1 \text{ for all } x\in X \text{ with } \delta_x\in\partial_eA_1^*]\Longrightarrow\ f\ \in\partial_e^*(A\otimes_\epsilon E)_1,$$

where we denote by δ_x the functional on A of evaluation at the point x. In this paper we obtain a partial converse of the above result (Theorem 1). Our proof also shows that if $f\in\partial_e^s(A\otimes_\epsilon E)_1$ then $f(x)\in\partial_e^sE_1$ for any weak peak point x (see Def. 1), extending one of the implications of (1.1). It follows that when A is a function algebra then any weak*-extreme point of A_1 is of absolute value one on the Choquet boundary ChA (and hence on its closure, the Shilov boundary) and consequently is a strongly extreme point [**17**]. Since we have concrete descriptions of the set of extreme points of several standard function algebras (see e.g. [**12**], page 139 for the Disc algebra) one can give easy examples of extreme points that are not weak*-extreme. Recently several authors have studied the extremal structure of the unit ball of function algebras ([**1**], [**15**], [**18**]). It follows from their results that the unit ball has no strongly exposed or denting points. Our description that strongly extreme and weak*-extreme points coincide for function algebras and are precisely the functions that are of absolute value one on the Shilov boundary completes that circle of ideas. We also give an example to show that the weak*-extreme points of $(A\otimes_\epsilon E)_1$ in general need not map the Shilov boundary into ∂_eE_1. Considering the more general case of the space of compact operators $\mathcal{K}(E,F)$ (we recall that under assumptions of approximation property on E or F^*, $\mathcal{K}(E,F)$ can be identified with $E^*\otimes_\epsilon F$) we exhibit weak*-extreme points $T\in\mathcal{K}(\ell^p)_1$ for $1<p\neq 2<\infty$ for which T^* does not map unit vectors to unit vectors.

Our notation and terminology is standard and can be found in [**3**], [**4**], or [**11**]. We always consider a Banach space as canonically embedded in its bidual. By $E^{(n)}$ we denote the n-th dual of E. By a function algebra we mean a closed subalgebra of a $C(X)$ space separating the points of X and containing the constant functions; we denote the Choquet boundary of A by ChA.

2. The result

As noticed earlier, for $A\subset C(X)$ and a point $f\in\partial_e^*(A\otimes_\epsilon E)_1$ we may *not* have $f(x)\in\partial_e^*E_1$ for all $x\in X$ even in a finite dimensional case. Hence we need to define a sufficiently regular subset of X in relation to A.

Definition 1. *A point $x_0 \in X$ is called a weak peak point of $A \subset C(X)$ if for each neighborhood U of x_0 and $\varepsilon > 0$ there is $a \in A$ with $1 = a(x_0) = \|a\|$ and $|a(x_0)| < \varepsilon$ for $x \in X \backslash U$; we denote by $\partial_p A$ the set of all such points in X.*

There are a number of alternative ways to describe the set $\partial_p A$. If $x_0 \in X$ is a weak peak point of $A \subset C(X)$, μ is a regular Borel measure on X annihilating A, and a_γ is a net in A convergent almost uniformly to 0 on $X \backslash \{x_0\}$ and such that $a_\gamma(x_0) = 1$ then $\mu(\{x_0\}) = \lim_\gamma \int_X a_\gamma = 0$. Hence if $a^* \in A^*$ and ν_1, ν_2 are measures on X representing a^* we have $\nu_1(\{x_0\}) = \nu_2(\{x_0\})$, consequently $\nu \overset{\chi_{\{x_0\}}}{\longmapsto} \nu(\{x_0\})$ is a well defined functional on A^*.

On the other hand if $\chi_{\{x_0\}} \in A^{**}$ then $\mu(\{x_0\}) = 0$, for any annihilating measure μ, and x_0 is a weak peak point. To justify the last claim notice that A_1 is weak*-dense in A_1^{**} so $\chi_{\{x_0\}}$ is in the weak*-closure of the set $K = \{f \in A_1 : f(x_0) = 1\}$. Let U be an open neighborhood of x_0 and $A_{X\backslash U}$ be the space of all restrictions of the functions from A to $X \backslash U$. We define the norm on $A_{X\backslash U}$ as sup on $X\backslash U$. Let $K_{X\backslash U}$ be the set of restrictions of the functions from K and $cl\left(K_{X\backslash U}\right)$ be the norm closure of $K_{X\backslash U} \subset A_{X\backslash U}$. If $0 \notin clK_{X\backslash U}$ then there is $G \in \left(A_{X\backslash U}\right)^*$, represented by a measure η on $X\backslash U$ and separating $K_{X\backslash U}$ from 0:

$$\operatorname{Re} G(h) > a > 0 = \chi_{\{x_0\}}(\mu), \text{ for all } h \in clK_{X\backslash U}.$$

The measure η extends G to a functional on A so K is functionally separated from 0 in A contrary to our previous observation. Hence $0 \in clK_{X\backslash U}$ so there is a function in K that is smaller then ε outside U which means that x_0 is a weak peak point.

The concept of weak peak points is well known in the context of function algebras where $\partial_p A$ coincides with the Choquet boundary ([**8**], p. 58). For more general spaces of the form $A_0 \overset{df}{=} \{f_0 a \in C(X) : a \in A\}$, where $A \subset C(X)$ is a function algebra and f_0 a nonvanishing continuous function on X we have $ChA \subseteq \partial_p A_0$. Spaces of these type appear naturally in the study of singly generated modules and Morita equivalence bimodules in the operator theory [**2**].

Theorem 1. *Let E be a Banach space, X a compact Hausdorff space, and A a closed subspace of $C(X)$. If $f \in A \otimes_\epsilon E$ is a weak* (strongly) extreme point of the unit ball then $f(x)$ is a weak*(strongly) extreme point of the unit ball of E for any $x \in \partial_p A$.*

In particular if $f \in (C(X,E))_1$ is a weak(strongly) extreme point then $f(x)$ is a weak*(strongly) extreme point of E_1 for all $x \in X$.*

We first need to show that for a weak peak point $x_0 \in X$ there exists a function in A not only peaking at x_0 but that is also almost real and almost positive.

Lemma 1. *Assume X is a compact Hausdorff space, A is a closed subspace of $C(X)$, and x_0 is a weak peak point of A. Then for each neighborhood U of x_0 and $\varepsilon > 0$ there is $g \in A$ such that*

$$\begin{aligned} \|g\| &= 1 = g(x_0), \\ |g(x)| &< \varepsilon, \ \textit{for all } x \in X\backslash U, \ \textit{and} \\ \left\|\operatorname{Re}^+ g - g\right\| &< \varepsilon, \end{aligned} \tag{2.1}$$

where $\operatorname{Re}^+ z = \max\{0, \operatorname{Re} z\}$.

PROOF. Put $U_1 = U$ and let $g_1 \in A$ be such that

$$\|g_1\| = 1 = g_1(x_0) \text{ and } |g_1(x)| < \varepsilon \text{ for } x \notin U_1.$$

Put $U_2 = \{x \in U_1 : |g_1(x) - 1| < \varepsilon\}$ and let $g_2 \in A$ be such that

$$\|g_2\| = 1 = g_2(x_0) \text{ and } |g_2(x)| < \varepsilon \text{ for } x \notin U_2.$$

Put $U_3 = \{x \in U_2 : |g_2(x) - 1| < \varepsilon\}$. Proceeding this way we choose a sequence $\{g_n\}_{n\geq 1}$ in A. Fix a natural number k such that $k > \frac{1}{\varepsilon}$ and put

$$g = \frac{1}{k}\sum_{j=1}^{k} g_j.$$

We clearly have $\|g\| = 1 = g(x_0)$ and $|g(x)| < \varepsilon$ for $x \notin U$. Let $x \in U$, then either x belongs to all of the sets U_j, $j \leq k$, in which case $|g(x) - 1| < \varepsilon$, or there is a natural number $p < k$ such that $x \in U_p \backslash U_{p+1}$. In the later case we have

$$\begin{aligned}
\left| g(x) - \frac{p-1}{k} \right| &= \frac{1}{k}\left| \sum_{j=1}^{k} g_j - (p-1) \right| \\
&\leq \frac{1}{k}\begin{pmatrix} (|g_1(x) - 1| + \ldots + |g_{p-1}(x) - 1|) \\ +|g_p(x)| + (|g_{p+1}(x)| + \ldots + |g_k(x)|) \end{pmatrix} \\
&\leq \frac{p-1}{k}\varepsilon + \frac{1}{k} + \frac{k-p}{k}\varepsilon < \varepsilon.
\end{aligned}$$

Hence $\left\|\operatorname{Re}^+ g - g\right\| < \varepsilon$. □

We are now ready to finish the proof of the Theorem.

PROOF. Suppose $f(x_0)$ is not a weak*-extreme point. Then by [**9**] there is a sequence e_n in E_1 and $e^* \in E_1^*$ such that $\|f(x_0) \pm e_n\| \leq 1 + \frac{1}{n}$ and $e^*(e_n) \nrightarrow 0$. By the Lemma there is a sequence g_n in A such that

$$\begin{aligned}
\|g_n\| &= 1 = g_n(x_0), \\
|g_n(x)| &< \frac{1}{n}, \text{ if } \|f(x) - f(x_0)\| \geq \frac{1}{n}, \text{ and} \\
\left\|\operatorname{Re}^+ g_n - g_n\right\| &< \frac{1}{n},
\end{aligned} \tag{2.2}$$

Hence

$$\begin{aligned}
\|f(x) \pm g_n(x) e_n\| &\leq \max\begin{Bmatrix} \sup_{\|f(x) - f(x_0)\| \geq \frac{1}{n}} \{\|f(x)\| + |g_n(x)| \|e_n\|\}, \\ \sup_{\|f(x) - f(x_0)\| < \frac{1}{n}} \{\|f(x) \pm g_n(x) e_n\|\} \end{Bmatrix} \\
&\leq \max\left\{1 + \frac{1}{n}, \frac{1}{n} + \left\|f(x_0) \pm \operatorname{Re}^+ g_n(x) e_n\right\| + \frac{1}{n}\right\} \\
&\leq 1 + \frac{3}{n}.
\end{aligned}$$

Therefore $\|f \pm g_n e_n\| \to 1$ but $(\delta(x_0) \otimes e^*)(g_n e_n) = g_n(x_0) e^*(e_n) \nrightarrow 0$. This contradiction shows that $f(x_0)$ is a weak*-extreme point.

The same line of arguments shows that $f(x)$ is strongly extreme for any strongly extreme $f \in (A \otimes_\epsilon E)_1$. □

Since for a function algebra A the Choquet boundary ChA coincides with $\partial_p A$ ([**8**], p. 58) and the Shilov boundary ∂A is equal to the closure of ChA we have:

COROLLARY 1. *Let A be a function algebra, E a Banach space and $f \in (A \otimes_\epsilon E)_1$ a weak*-extreme point. Then*

$$f(x) \in \partial_e^* E_1, \quad \text{for } x \in ChA, \text{ and } \|f(x)\| = 1, \text{ for } x \in \partial A.$$

REMARK 1. *Theorem 1 is not valid for the spaces $WC(X,E)$ of E-valued continuous functions with E quipped with the weak topology. Even a strongly extreme point of $WC(X,E)_1$ need not assume extremal values at all points of X [**13**].*

We next give an example of a function algebra A and a 3-dimensional space E showing that a weak*-extreme point $f \in (A \otimes_\epsilon E)_1$ need not take extremal values on the entire Shilov boundary. Since E is finite dimensional this function f maps the Choquet boundary into the set of strongly extreme points but f is not a strongly extreme point.

EXAMPLE 1. *Put*

$$Q = \left\{(z,w,0) \in \mathbb{C}^3 : |z|^2 + |w|^2 \le 1\right\} \cup \left\{(0,w,u) \in \mathbb{C}^3 : \max\{|w|,|u|\} \le 1\right\},$$

and $B = convQ$. Let $\|\cdot\|$ be the norm on $\mathbb{C}^3$ such that B is its unit ball. Note that $(z,w,0)$ is an extreme point of B iff $|w| \neq 1$ and $|z|^2 + |w|^2 = 1$. Put $E = \left(\mathbb{C}^3, \|\cdot\|\right)$,

$$X \overset{df}{=} \{0\} \times \{1\} \times \mathbb{D} \cup \{(\sin t, \cos t, 0) : 0 \le t \le \pi\},$$

$f_0 : X \to E_1$, $f_0(x) \overset{df}{=} x$, and

$$A = \{h \in C(X) : h(0,1,\cdot) \in A(\mathbb{D})\},$$

where $A(\mathbb{D})$ is the disc algebra. We have

$$ChA = \{0\} \times \{1\} \times \partial\mathbb{D} \cup \{(\sin t, \cos t, 0) : 0 < t < \pi\}.$$

The function f_0 is in $A \otimes_\epsilon E$ and takes extremal values on the Choquet boundary of A so it is a weak-extreme points of $(A \otimes_\epsilon E)_1$. However $f_0(0,\pm 1,0) = (0,\pm 1,0)$ are not extreme points of E_1^* while $(0,\pm 1,0)$ are in the Shilov boundary of A.*

Since E is finite dimensional clearly the function f_0 maps the Choquet boundary of A into the set of strongly extreme points of E_1. We next show that f is not a strongly extreme point.

Let $g_n \in A$ be such that

$$\begin{aligned} \|g_n\| &= 1 = g_n\left(\sin\frac{1}{n}, \cos\frac{1}{n}, 0\right), \\ g_n(\sin t, \cos t, 0) &= 0, \text{ for } \frac{2}{n} < t \le 1, \text{ and} \\ g_n(0,1,z) &= 0, \text{ for } z \in \mathbb{D}. \end{aligned}$$

Put $f_n = (0,0,g_n) \in A \otimes_\epsilon E$. We have

$$(f_0 \pm f_n)(a,b,c) = \begin{cases} (0,1,c) & \text{for} \quad (a,b,c) \in \{0\} \times \{1\} \times \partial\mathbb{D} \\ (a,b,\pm g_n) & \text{for} \quad (a,b,c) \in \{(\sin t, \cos t, 0) : 0 \le t \le \pi\}. \end{cases}$$

Hence $\|f \pm f_n\| \to 1$ but $\|f_n\| \nrightarrow 0$ so f is not a strongly extreme point.

In the next Proposition we consider a more general setting of compact operators. For a Banach space E we denote by $\mathcal{L}(E)$ the space of all linear bounded maps on E, by $\mathcal{K}(E)$ the set of all compact linear maps, and by $S(E)$ the set of unit vectors in E. Since $\mathcal{K}(E, C(X))$ can be identified with $C(X, E^*)$ our result on weak*-extreme points taking weak*-extremal values can be interpreted as follows

$$T \in \partial_e^* \mathcal{K}(E, C(X))_1 \Longrightarrow T^* \left(\partial_e^* C(X)_1^*\right) \subset \partial_e^* E_1^*.$$

Thus more generally one can ask whether $T^*(\partial_e^* F_1^*) \subset \partial_e^* E_1^*$ for any $T \in \partial_e^* \mathcal{K}(E, F)_1$. We give a class of counter examples with the help of the following proposition.

PROPOSITION 1. *Let E be an infinite dimensional Banach space such that $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$. If $T \in \mathcal{K}(E)_1$ then $T^*(\partial_e E_1^*) \not\subset S(E^*)$.*

We recall that a closed subspace M of a Banach space E is an M-ideal if there is a projection $P \in \mathcal{L}(E^*)$ such that $\ker P = M^\perp$ and $\|P(e^*)\| + \|e^* - P(e^*)\| = \|e^*\|$, for all $e^* \in E^*$ (see [**11**] for an excellent introduction to M-ideals).

PROOF. Since $\mathcal{K}(E)$ is an M-ideal it follows from Corollary VI.4.5 in [**11**] that E^* has the Radon-Nikodym property and hence the IP (see [**10**]). Also since $\mathcal{K}(E)$ is a proper M-ideal it fails the IP. It therefore follows from Theorem 4.1 in [**10**] that there exists a net $\{x_\alpha^*\} \subset \partial_e E_1^*$ such that $x_\alpha^* \to x_0^*$ in the weak*-topology with $\|x_0^*\| < 1$. Suppose $T^*(\partial_e E_1^*) \subset S(E^*)$. Since T^* is a compact operator by going through a subnet if necessary we may assume that $T^*(x_\alpha^*) \to T^*(x_0^*)$ in the norm. Thus $1 = \|T^*(x_0^*)\| < 1$ and the contradiction gives the desired conclusion. □

EXAMPLE 2. *Banach spaces E for which $\mathcal{K}(E)$ is an M ideal in $\mathcal{L}(E)$ have been well extensively studied. Chapter VI of* [**11**] *provides several examples including $E = \ell^p$, $1 < p < \infty$, as well as properties of these spaces. It was observed in* [**6**] *that for $p \neq 2$ there are weak*-extreme points in the space $\mathcal{K}(\ell^p)_1$. It follows from the last proposition that the adjoint of these weak*-extreme points do not even map extreme points to unit vectors.*

A strongly extreme point remains extreme in all the dual spaces of arbitrary even order. A weak*-extreme point remains extreme in the second dual but may not be extreme in the fourth dual. Hence the property of remaining extreme in all the duals of even order is placed between the strong and the weak* type of extreme points. It would be interesting to describe that property in terms of the original Banach space alone. A procedure for generating extreme points which have this property but are not strongly extreme was described in [**6**].

PROPOSITION 2. *Let X be a compact Hausdorff space, A a closed subspace of $C(X)$, and E a Banach space. Suppose $x_0 \in X$ is a weak peak point and $f \in A \otimes_\epsilon E$ is an extreme point in the unit ball of all the duals of even order. Then $f(x_0)$ is an extreme point of the unit ball of all the duals of E of even order.*

PROOF. Since the space $A \otimes_\epsilon E^{**}$ can be canonically embedded in $(A \otimes_\epsilon E)^{**}$ [**7**] we have, for any natural number n

$$A \otimes_\epsilon E^{(2n)} \subset (A \otimes_\epsilon E^{(2n-2)})^{**} \subset (A \otimes_\epsilon E)^{(2n)}.$$

If $f \in A \otimes_\epsilon E$ is an extreme point of $(A \otimes_\epsilon E)^{(2n+2)}$ then it is a weak*-extreme point of $(A \otimes_\epsilon E)^{(2n)}$, as it also belongs to $A \otimes_\epsilon E^{(2n)}$ it is a weak*-extreme point of $A \otimes_\epsilon E^{(2n)}$. Hence by our theorem $f(x_0)$ is an extreme point of $E_1^{(2n)}$. □

The next proposition characterizes strongly extreme points in terms of ultrapowers.

PROPOSITION 3. *An element e of a Banach space E is a strongly extreme point of the unit ball E_1 if and only if $(e)_{\mathcal{F}}$ is an extreme point of $(E_{\mathcal{F}})_1$.*

PROOF. If $e \notin \partial_e^s E_1$ then there is a sequence $\{e_n\}_{n\geq 1} \subset E_1$ with $\|e \pm e_n\| \to 1$ and $\inf_{n\in\mathbb{N}} \|e_n\| > 0$. Thus $\|(e)_{\mathcal{F}} \pm (e_n)_{\mathcal{F}}\| = 1$ and $\|(e_n)_{\mathcal{F}}\| \neq 0$ so $(e)_{\mathcal{F}}$ is not an extreme point.

If $(e)_{\mathcal{F}} \notin \partial_e (E_{\mathcal{F}})_1$ then there is $0 \neq (e_n)_{\mathcal{F}} \in (E_{\mathcal{F}})_1$ with $1 = \|(e)_{\mathcal{F}} \pm (e_n)_{\mathcal{F}}\| = \lim_{\mathcal{F}} \|e \pm e_n\|$. Thus for every $\varepsilon > 0$ the set $\{n \in \mathbb{N}: \|e \pm e_n\| \leq 1 + \varepsilon\}$ is none empty as an element of $\mathcal{F}$. Hence there exists a sequence $\{k_n\}$ such that $\|e \pm e_{k_n}\| \to 1$ but $\|e_{k_n}\| \nrightarrow 0$ so e is not a strongly extreme point. $\square$

References

[1] P. Beneker and J. Wiegerinck, *Strongly exposed points in uniform algebras,* Proc. Amer. Math. Soc. 127 (1999) 1567-1570.

[2] D. Blecher and K. Jarosz, *Isomorphisms of function modules, and generalized approximation in modulus*, Trans. Amer. Math. Soc. 354 (2002), 3663-3701

[3] R. D. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodym property*, **LNM 993**, Springer, Berlin 1983.

[4] A. Browder, *Introduction to Function Algebras*, W. A. Benjamin, New York 1969.

[5] P. N. Dowling, Z. Hu and M. A. Smith, *Extremal structure of the unit ball of $C(K, X)$*, Contemp. Math., 144 (1993) 81-85.

[6] S. Dutta and T. S. S. R. K. Rao, *On weak**-extreme points in Banach spaces, preprint 2001.

[7] G. Emmanuele, *Remarks on weak compactness of operators defined on injective tensor products*, Proc. Amer. Math. Soc., 116 (1992) 473-476.

[8] T. Gamelin, *Uniform Algebras*, Chelsea Pub. Comp., 1984.

[9] B. V. Godun, Bor-Luh Lin and S. L. Troyanski, *On the strongly extreme points of convex bodies in separable Banach spaces*, Proc. Amer. Math. Soc., 114 (1992) 673-675.

[10] P. Harmand and T. S. S. R. K. Rao, *An intersection property of balls and relations with* M-ideals, Math. Z. 197 (1988) 277-290.

[11] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Springer LNM No 1547, Berlin 1993.

[12] K. Hoffman, *Banach spaces of analytic functions*, Dover 1988.

[13] Z. Hu and M. A. Smith, *On the extremal structure of the unit balls of Banach spaces of weakly continuous functions and their duals*, Trans. Amer. Math. Soc. 349 (1997) 1901-1918.

[14] K. Kunen and H. P. Rosenthal, *Martingale proofs of some geometric results in Banach space theory,* Pacific J. Math. 100 (1982) 153-175.

[15] O. Nygaard and D. Werner, *Slices in the unit ball of a uniform algebra,* Arch. Math. (Basel) 76 (2001) 441-444.

[16] R. R. Phelps, *Extreme points of polar convex sets*, Proc. Amer. Math. Soc. 12 (1961) 291-296.

[17] T. S. S. R. K. Rao, *Denting and strongly extreme points in the unit ball of spaces of operators,* Proc. Indian Acad. Sci. (Math. Sci.) 109 (1999) 75-85.

[18] T. S. S. R. K. Rao, *Points of weak-norm continuity in the unit ball of Banach spaces, J. Math. Anal. Appl., 265 (2002) 128-134.*

[19] H. Rosenthal, *On the non-norm attaining functionals and the equivalence of the weak*-**KMP** with the **RNP***, Longhorn Notes, 1985-86.

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