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$L^1(\mu, X)$ as a complemented subspace of its bidual

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Abstract. We show that for a Banach space X, if the space of X-valued Bochner integrable functions is complemented in some dual space, then it is complemented in the space of X-valued countably additive, μ-continuous vector measures.

Keywords. Bochner integrable functions; vector measures; L-ideals.

1. Introduction

Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and let X be a Banach space that is complemented in its bidual. In this note we consider the question "when is $L^1(\mu, X)$ complemented in its bidual?". It is well known that $L^1(\mu)$ is complemented (by a norm one projection) in its bidual. Lindenstrauss [6] had observed that X is complemented in its bidual iff it is complemented in some dual space. Hence being complemented in its bidual is a property inherited by complemented subspaces. Using this observation, the author has proved in [7] that if X has the Radon Nikodym property w.r.t μ and X is complemented in its bidual (which is clearly a necessary condition) then $L^1(\mu, X)$ is complemented in its bidual. Looking at our argument Emmanuele [3] has recently pointed out that the stage where we use the R.N.P by requiring $L^1(\mu, X) = cabv(\mu, X)$ (the space of countably additive X-valued measures on $\mathscr{A}$ of bounded variation that are absolutely continuous w.r.t μ) can be replaced by the requirement $L^1(\mu, X)$ is complemented in $cabv(\mu, X)$ to yield the same conclusion. This becomes a significant remark in view of the recent result of F Freniche and L Rodriguez-Piazza, that for any Banach lattice not containing a copy of c_0, $L^1(\mu, X)$ is complemented in $cabv(\mu, X)$ (see [3]).

In this note we point out that for an X that is complemented in its bidual, $L^1(\mu, X)$ is complemented in its bidual iff it is complemented in $cabv(\mu, X)$. As a consequence we show that if $L^1(\mu, X)$ is complemented in its bidual and $M \subset X$ is a reflexive subspace then $L^1(\mu, X|M)$ is complemented in its bidual. Our approach also gives easier proofs of some of the results from [3].

It should be noted that several authors [2, 8] have observed that if X has a copy of c_0, then $L^1(\mu, X)$ is not complemented in its bidual.

Main Result: Let K denote the Stone space of $L^\infty(\mu)$ and $\wedge$ denote the Gelfand map. Let $\hat{\mu}$ denote the measure defined on the Borel σ-field of K via the Gelfand map. It follows from the arguments given in [7] that the $\wedge$ map can be extended to $cabv(\mu, X)$ onto $rcabv(\hat{\mu}, X)$ in such a way that $L^1(\mu, X)$ gets mapped onto $L^1(\hat{\mu}, X)$. Therefore the requirement $L^1(\mu, X)$ is complemented in $cabv(\mu, X)$ is equivalent to $L^1(\hat{\mu}, X)$ being complemented in $rcabv(\hat{\mu}, X)$.

Theorem. *Let X be complemented in its bidual, then $L^1(\mu, X)$ is complemented in its bidual iff it is complemented in $cabv(\mu, X)$.*

Proof. From the argument given above it is clear that there is no loss of generality in assuming that μ is a finite, category measure on the Borel σ-field of a hyperstonean space K.

Suppose $L^1(\mu, X)$ is complemented in its bidual. Since $L^1(\mu, X) = L^1(\mu) \hat{\otimes} X$ and $L^1(\mu)^* = C(K)$ we have,

$$L^1(\mu, X)^{**} = \mathscr{L}(X, C(K))^*$$

(where $\mathscr{L}(X, C(K))$ denotes the space of operators), see [1].

Let $P: L^1(\mu, X)^{**} \to L^1(\mu, X)$ be a projection.

It is well known (see [5]) that

$$\mathscr{L}(X, C(K))^* = \mathscr{K}(X, C(K))^* \oplus \mathscr{K}(X, C(K))^{\perp}$$

(where $\mathscr{K}(X, C(K))$ denotes the space of compact operators).

Since $\mathscr{K}(X, C(K)) = C(K, X^*)$, and by Singer's Theorem, $C(K, X^*)^* = rcabv(X^{**})$.

Therefore

$$L^1(\mu, X) \subset rcabv(\mu, X) \subset L^1(\mu, X)^{**}.$$

Hence $P|rcabv(\mu, X) \to L^1(\mu, X)$ is the required projection.

The converse part of the proof proceeds in the same lines as the one in [7] where one now takes the compositon with a projection from $rcabv(\mu, X)$ into $L^1(\mu, X)$ instead of the equality of these objects used in our earlier proof, as also pointed out during the proof of Theorem 6 in [3].

COROLLARY 1

Suppose X is such that $L^1(\mu, X)$ is complemented in its bidual. Let $M \subset X$ be a reflexive subspace. Then $L^1(\mu, X|M)$ is complemented in its bidual.

Proof. It is sufficient to show that $L^1(\mu, X|M)$ is complemented in $cabv(\mu, X|M)$ in the category measure set-up. Clearly $X|M$ is complemented in its bidual.

It follows from Theorem 1 of [3] that what is required is to lift an $F \in cabv(\mu, X|M)$ to an element of $cabv(\mu, X)$. Fix $F \in cabv(\mu, X|M)$. Since $M \subset X^{**}$, $F \in cabv(\mu, X^{**}|M)$, by the proof of corollary 3 in [3], we get a lifting measure $\tilde{F} \in cabv(\mu, X^{**})$. Fix a $A \in \mathscr{A}$ and let $F(A) = \pi(x)$ where π is the quotient map and $x \in X$.

Since $\pi(\tilde{F}(A)) = F(A) = \pi(x)$ we get that $\tilde{F}(A) - x \in M$ and hence $\tilde{F}(A) \in X$.

Hence $\tilde{F}$ is a lifting for F.

COROLLARY 2

Let $(\Omega, \mathscr{A}, \mu)$ and $(\Omega', \mathscr{A}', \mu')$ be two finite measure spaces. Let γ denote the product measure on the product σ-field. Suppose X is complemented in its bidual. If $L^1(\gamma, X)$ is complemented in $cabv(\gamma, X)$ then $L^1(\mu, L^1(\mu', X))$ is complemented in $cabv(\mu, L^1(\mu', X))$.

Proof. The hypothesis implies that $L^1(\gamma, X)$ is complemented in its bidual. Therefore $L^1(\mu', X)$ is complemented in its bidual. Let $Y = L^1(\mu', X)$. Since $L^1(\mu, Y) = L^1(\gamma, X)$, this space is complemented in its bidual and hence by the Theorem, $L^1(\mu, Y)$ is complemented in $cabv(\mu, Y)$.

Remark. This provides a simple proof of Theorem 5 of [3] and our Corollary 1 improves on Corollary 4 of [3]. This also shows that one need not invoke the results of Freniche and Rodriguez-Piazza in the proof of Corollary 2 in [3].

Part of our motivation for looking at questions of this nature came from our interest in L-structure of Banach spaces. A Banach space X is said to be an L-ideal in its bidual, if there is an onto projection $P: X^{**} \to X$ (X is canonically embedded in X^{**}) such that $\|P(\Lambda)\| + \|\Lambda - P(\Lambda)\| = \|\Lambda\|, \forall \Lambda \in X^{**}$. For any positive measure μ, $L^1(\mu)$ is such a space and an interesting open question in this area is whether $L^1(\mu, X)$ is an L-ideal in its bidual whenever X is?. We refer to the monograph [4] for properties of these spaces and for the authorship of some of the results that we will be quoting.

The result we are going to present below and some of the preceding results indicate the possibility that $L^1(\mu, X)$ is a constrained subspace of its bidual, whenever X is an L-ideal in its bidual.

We need the following results from Chapter 4 of [4] due to D Li.

1) If X and Y are L-ideals in their biduals and $Y \subset X$ (isometrically) then $X|Y$ is an L-ideal in its bidual.

2) Let L be a Banach space such that L^* is injective and X, Y as in 1), then every operator $T: L \to X|Y$ factors through the quotient map $\pi: X \to X|Y$.

PROPOSITION

Let X and Y be L-ideals in their biduals with $Y \subset X$. Suppose Y has the RNP and $L^1(\mu, X)$ is constrained in its bidual, then $L^1(\mu, X|Y)$ is constrained in its bidual.

Proof. Note that because of 1), $X|Y$ is an L-ideal in its bidual. It follows from the arguments given in [7] that w.l.o.g, we can assume that μ is a finite measure.

It now follows from our Theorem that it is enough to show that $L^1(\mu, X|Y)$ is constrained in $cabv(\mu, X|Y)$. In view of the results from [3], what is required to show is that any $F \in cabv(\mu, X|Y)$ can be 'lifted' to a $\tilde{F} \in cabv(\mu, X)$. Let $F \in cabv(\mu, X|Y)$ and define a bounded linear operator

$T: L^1(|F|) \to X|Y$ by the formula $T([\chi_E]) = F(E)$ for any measurable set E where $|F|$ is the variation of F.

Since $L^\infty(|F|)$ is clearly an injective Banach space, it follows from the result 2) quoted above that $\exists \tilde{T}: L^1(|F|) \to X$ such that

$$\pi \circ \tilde{T} = T$$

Now define $\tilde{F}: \mathscr{A} \to X$ by $\tilde{F}(E) = \tilde{T}([\chi_E])$. From standard vector measure theory we know that $\tilde{F} \in cabv(\mu, X)$ and

$$\begin{aligned} \pi(\tilde{F}(E)) &= \pi(\tilde{T}[\chi_E]) \\ &= T([\chi_E]) = F(E). \end{aligned}$$

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