

# The Abelian Sandpile and Related Models

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## Abstract

The Abelian sandpile model is the simplest analytically tractable model of self-organized criticality. This paper presents a brief review of known results about the model. The abelian group structure of the algebra of operators allows an exact calculation of many of its properties. In particular, when there is a preferred direction, one can calculate all the critical exponents characterizing the distribution of avalanche-sizes in all dimensions. For the undirected case, the model is related to  $q \rightarrow 0$  state Potts model. This enables exact calculation of some exponents in two dimensions, and there are some conjectures about others. We also discuss a generalization of the model to a network of communicating reactive processors. This includes sandpile models with stochastic toppling rules as a special case. We also consider a non-abelian stochastic variant, which lies in a different universality class, related to directed percolation.

## 1 Introduction

It has been about 10 years since Bak, Tang and Wiesenfeld's landmark papers on self-organized criticality appeared [1, 2]. In this period, the concept of self-organized criticality has been invoked to describe a large variety of different systems such as forest-fires, earthquakes, punctuated equilibrium in biology, stock-market fluctuations etc.. I shall not attempt to review the very large number of papers inspired by Bak's justly influential ideas. A readable overview may be found in his recent book [3]. Instead I shall concentrate on one specific model: the abelian sandpile model. Even here, rather than provide a summary of all the papers on this problem, I will try to provide an overview, from a perhaps somewhat biased personal perspective. The review is not self-contained, and for details of arguments the reader will have to consult the original papers. Even so, I hope that it will be a useful introduction, and guide to literature, for students and others trying to learn about this subject for the first time. The paper also contains some unpublished material (mainly in section 5). Another recent review, similar in scope, is by Ivashkevich and Priezzhev [4]. Brief accounts of my own work on this model have appeared in conference proceedings earlier[5, 6].

The sandpile model was proposed as a paradigm of self-organized criticality (SOC). It is certainly the simplest, and best understood, *theorist's* model of SOC: it is a non-equilibrium system, driven at a slow steady rate, with local threshold relaxation rules, which in the steady state shows relaxation events in bursts of a wide range of sizes, and long-range spatio-temporal correlations, obtained without fine-tuning of any control parameters.

The Abelian sandpile model (ASM) is the name given to a particular subclass of the sandpile models that have a nice mathematical structure (an abelian group). The group structure allows analytical calculation of many of the properties of the model. The mathematical tractability of the model is certainly its main attraction. In addition, the model turns out to be related to several other models in statistical mechanics: the Potts model, the voter model, directed percolation, Takayasu aggregation model etc..

A different reason for the continued interest of physicists in the model is its intractability: in spite of its apparent simplicity, the original 2-dimensional BTW model has defied an analytical calculation of all critical exponents so far. The fact that the steady state for the ASM is well characterized, and allows exact calculation of averages of many physical quantities of interest ( and with only moderate effort), and that several critical exponents are known exactly, suggests that the model is 'soluble', and other exponents can also be determined analytically (if only we could figure out how!).

## 2 General properties of the Abelian Sandpile Model

The ASM is defined as follows [7]: we consider a lattice of  $N$  sites labelled by integers  $i = 1$  to  $N$ . At each site  $i$ , we define a nonnegative integer height variable  $z_i$  called the height of the sandpile, and a threshold value  $z_{ic}$ . If  $z_i < z_{ic}$ , for all  $i$ , the pile is said to be stable. The time-evolution of the sandpile is defined by the following two rules:

1. Adding a particle: Select one of the sites randomly, the probability that site  $i$  is picked being some given value  $p_i$ , and we add a grain of sand there. Clearly,  $\sum_i p_i = 1$ . On addition of the grain at site  $i$ ,  $z_i$  increases by 1. Height at other sites remains unchanged.
2. Toppling: If for any site  $z_i \geq z_{ic}$ , then that site is said to be unstable, it topples, and loses some sandgrains to other sites. This sandgrain transfer is defined in terms of an  $N \times N$  integer matrix  $\Delta$ . On toppling at site  $i$ , the configuration is updated according to the rule:

$$z_j \rightarrow z_j - \Delta_{ij}, \text{ for } j = 1 \text{ to } N. \quad (1)$$

If the toppling results in some other site becoming unstable, it is also toppled. The process continues until all sites become stable.

The matrix  $\Delta$  is assumed to have the following properties:

- i)  $\Delta_{ii} > 0$ , for all  $i$ .
- ii)  $\Delta_{ij} \leq 0$ , for all  $i \neq j$ .
- iii)  $\sum_j \Delta_{ij} \geq 0$ , for all  $i$ .

These conditions just ensure that on toppling at site  $i$ ,  $z_i$  must decrease, and height at other sites  $j$  can only increase, and there is no creation of sand in the toppling process. Some sand may get lost from the system if the toppling occurs at a boundary site. In fact, no stationary state of the sandpile is possible unless particles can leave the system. The model

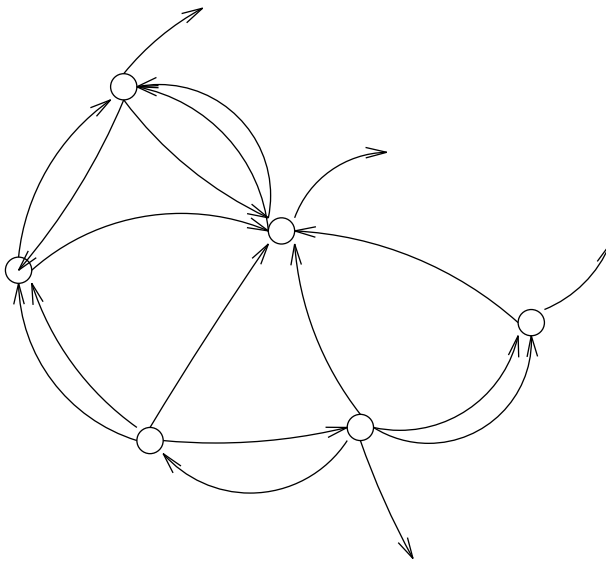


Figure 1: A graphical representation of the general ASM. Each node denotes a site. On toppling at any site, one particle is transferred along each arrow directed outward from the site.

can be represented by a directed graph on  $N$  vertices, where we draw  $(-\Delta_{ij})$  directed bonds from site  $i$  to site  $j$ , and  $(\sum_j \Delta_{ij})$  arrows from  $i$  to outside ( fig. 1).

Without loss of generality, we can assume that  $z_{ic} = \Delta_{ii}$ . Then if the site  $i$  is stable, we have  $0 \leq z_i < z_{ic}$ . We start with a stable configuration of the pile, and add a particle at random. If this leads to unstable site, it is relaxed using the toppling rule. If this makes some other sites unstable, they are toppled in turn until a stable configuration of pile is reached. then we add another grain, and repeat the process. After a large number of grains are added, the system loses memory of the initial state, and reaches a statistically stationary state. In this stationary state, relaxation after the addition of another grain typically involves a sequence of topplings. This is called an avalanche. The size of avalanche is a random variable. In many cases of interest, it seems to have a power law tail, which is a signal of existence of long-ranged correlations in the system.

We may use different measures to determine the size of an avalanche. These are the total number of topplings  $s$ , the number of distinct sites toppled  $s_d$ , the diameter of the region affected by avalanche  $R$ , the duration of the avalanche  $t$ . The probability that the avalanche has ‘size’  $x$  in the thermodynamic limit of large system sizes will be defined to vary as  $x^{-\tau}$ . The exponents  $\tau_s, \tau_d, \tau_r$  and  $\tau_t$  will be used for the size measures  $s, s_d, R, t$  respectively.

This model has a very important abelian property [7]. We define operators  $a_i$ , which act on the space of stable configurations of the model. If  $C$  is any stable configuration,  $a_i C$  is the stable configuration obtained by adding a particle at  $i$ , and relaxing the system. It is easy to check that given any unstable configuration with two or more unstable sites, we get the same configuration by toppling at an unstable site  $i$ , and then at unstable site  $i'$ , as we would get if we toppled first at  $i'$ , and then at  $i$ . Thus the topplings commute with other.

Also, the process of addition of particles commutes with topplings. By repeated use of these properties, we conclude that the operators  $a_i$ 's commute with each other.

$$[a_i, a_j] = 0, \quad \text{for all } i \text{ and } j. \quad (2)$$

Adding  $\Delta_{ii}$  particles at site  $i$  will necessarily cause a toppling there, whatever the configuration. Thus, it is same as adding  $-\Delta_{ij}$  particles at all sites  $j \neq i$ . Thus the operators  $a_i$  satisfy the equations

$$\prod_{j=1}^N a_j^{\Delta_{ij}} = 1, \quad \text{for all } i. \quad (3)$$

Because of the randomness in where grains are added, the time evolution in this model is Markovian. As in standard Markov theory, the stable configurations of the sandpile can be divided into two classes: recurrent and transient. Let us denote by  $\mathcal{R}$  the space of recurrent configurations. Then  $\mathcal{R}$  is closed under multiplication by  $\{a_i\}$ . In  $\mathcal{R}$ , we can define an inverse operator  $a_i^{-1}$  for each  $i$ . Then restricted to  $\mathcal{R}$ , the operators  $\{a_i\}$  form a finite abelian group [7].

We define an equivalence relation in the space of all configurations ( stable, transient or unstable) by the property that  $\{z_i\}$  and  $\{z'_i\}$  are equivalent under toppling if there exist integers  $\{n_i\}$  such that

$$z_j = z'_j + \sum_i n_i \Delta_{ij}, \quad \text{for all } j. \quad (4)$$

If  $\{z_i\}$  are represented as lattice points in an N-dimensional euclidean space, the equivalent points form a superlattice. The basis-vectors of the superlattice are the rows of the matrix  $\Delta$ . It can be shown that there is exactly one recurrent configuration in each such equivalence class. Hence we get,  $|\mathcal{R}|$  equals the volume of a unit cell of this superlattice. Hence we get

$$|\mathcal{R}| = \text{Det} \Delta \quad (5)$$

The shape of the recurrent set  $\mathcal{R}$  is in general quite nontrivial. A simple case  $N = 2$  is shown in fig. 2.

The state of the system after addition of  $t$  grains to the system ("time"  $t$ ) is specified by a probability vector

$$|P(t)\rangle = \sum_{C \in \mathcal{R}} \text{Prob}(C, t) |C\rangle \quad (6)$$

This evolves according to the equation

$$|P(t+1)\rangle = \left( \sum_{i=1}^N p_i a_i \right) |P(t)\rangle \quad (7)$$

Since the operators  $\{a_i\}$  commute with each other, these can be simultaneously diagonalized. Let the vector  $|\psi\rangle$  be simultaneous eigenvector with

$$a_i |\psi\rangle = \exp(i\phi_i) |\psi\rangle, \quad \text{for all } i. \quad (8)$$

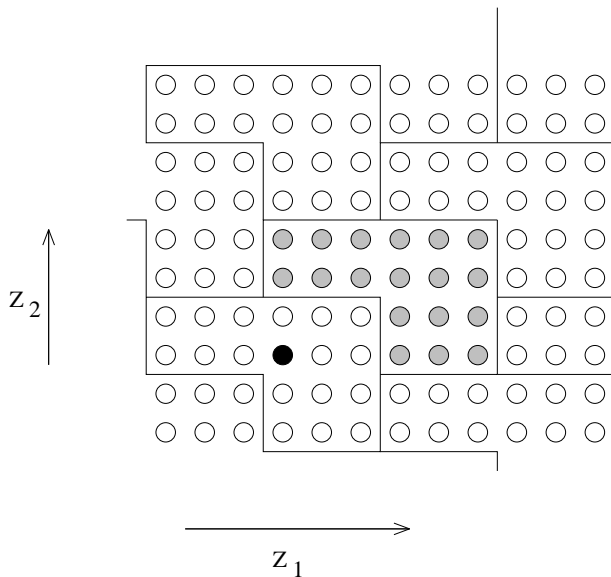


Figure 2: A tiling of the 2-dimensional lattice of all configurations  $(z_1, z_2)$  for a 2-site automata with  $\Delta = \begin{bmatrix} 6 & -2 \\ -3 & 4 \end{bmatrix}$  with copies of the recurrent set  $\mathcal{R}$ . The grey vertices are the recurrent configurations. The black circle marks the configuration  $(0, 0)$ .

Then  $\{\phi_i\}$  using eq.(3) are found to be expressible as

$$\phi_i = 2\pi \sum_j [\Delta^{-1}]_{ij} n_j, \quad \text{for all } i. \quad (9)$$

where  $\{n_i\}$ 's are some integers. Different choices of  $\{n_i\}$  give different eigenvectors. The choice of all  $n_i = 0$  gives  $a_i = 1$  for all  $i$ . This corresponds to the steady state. It follows that in the steady state all recurrent configurations occur with equal probability. This characterizes completely the steady state of the system. Since the time evolution operator is a linear combination of different  $a_i$ 's, that is also completely diagonalized. From Eq.(9) one can show that for a d-dimensional lattice of size L with nearest neighbor topplings, the longest relaxation time scales as  $L^d$ .

There is a simple algorithm (called the burning algorithm) to test if a configuration is recurrent or not. We scan the lattice, and recursively 'burn' any site  $i$  for which  $z_i \geq \sum'_j (-\Delta_{ji})$ , where the primed sum is over all unburnt sites  $j$ . If eventually, all sites are not burnt away, the configuration is transient. The unburnt sites form a forbidden subconfiguration (FSC). The simplest example of an FSC for an ASM on a d-dimensional hypercubical lattice is two adjacent sites with both having height 0. If the matrix  $\Delta$  is symmetric, and all sites burn away, the configuration is always recurrent. For non-symmetric  $\Delta$ , in which some sites have higher indegree than outdegree, there may be transient configuration that pass the burning test. Then one needs a more stringent test, called the 'script' test [8].

We can also determine some correlation functions in the steady state. If  $G_{ij}$  is the average number of topplings at site  $j$  due to a single particle added at  $i$  in the steady state,

the condition of mass balance gives [7]

$$G_{ij} = [\Delta^{-1}]_{ij} \quad (10)$$

The algebra of operators  $\{a_i\}$  has an interesting relationship to the familiar quantum mechanics. Here the set of basis vectors of the Hilbert space is the space  $\mathcal{R}$  of recurrent configurations, or equivalently, the space of all configurations  $\{z_i\}$ , with configurations equivalent under toppling identified. The operators  $a_i$  are translation operators in this space, ( like  $exp(ip)$  ) and they commute with each other. One can generalize these operators to define operators  $a_i(\epsilon)$ , which add a phase factor  $\epsilon$  each time a toppling occurs. This generalization preserves the abelian property. In fact, the evolution of  $a_i(\epsilon)$  with the parameter  $\epsilon$  is governed by a ‘quantum-Hamiltonian’ which is linear in the position operators  $\{z_i\}$  [9]. The nontrivial nature of avalanches is reflected in the fact that commutation relations between  $\{z_i\}$  and  $\{a_i\}$  are complicated, and difficult to write down.

The mathematical structure of the abelian group for a general finite  $N \times N$  matrix  $\Delta$  has also been investigated. In general, a finite abelian group  $\mathcal{G}$  can be expressed as a product of cyclic groups  $Z_{d_1} \times Z_{d_2} \times \dots Z_{d_g}$ , where  $g$  is the minimum number of generators of  $\mathcal{G}$ , and  $d_i$  is a multiple of  $d_{i+1}$ . To determine these integers  $\{d_i\}$ , we express the integer matrix  $\Delta$  in its normal form (this can be always done)

$$\Delta = ADB \quad (11)$$

where  $A$  and  $B$  are integer matrices of determinant  $\pm 1$ , and  $D$  is diagonal integer matrix with diagonal entries  $d_1, d_2, d_3, \dots$ . We can also choose  $A$  and  $B$  so that  $d_i$  is a multiple of  $d_{i+1}$ . The eigenvalues of  $D$  are precisely the cycle-lengths  $d_1, d_2, \dots, d_g$ . The remaining  $N - d_g$  eigenvalues of  $D$  are 1. The generators of these sub-groups of  $\mathcal{G}$  can be explicitly written down as a product of powers of  $\{a_i\}$ ’s in terms of the (non-unique) matrices  $A$  and  $B$ . The number of generators is a complicated function of  $\Delta$ . In the special case of two-dimensional BTW model, it can be shown that the the number of generators for a  $L \times L$  lattice is  $L$  [10].

### 3 Directed Abelian Sandpiles

The simplest of the sandpile models is when particle transfer occurs preferentially in one direction ( say provided by gravity). If there is no transfer in the reverse direction, the matrix  $\Delta$  becomes upper triangular. Then  $Det\Delta = \prod_{i=1}^N \Delta_{ii}$ . Thus all stable configurations are recurrent (there are no forbidden subconfigurations). Correspondingly, the measure of different configurations in the steady state is the simple product measure, and averages of various quantities in the steady state are easy to compute [11, 12].

Another simplification which occurs due to directedness is the fact that for most cases of interest( say, d-dimensional hypercubical lattice) each site can topple at most once in an avalanche. Thus we have no multiple topplings, and  $s = s_d$ . Also, the duration of the avalanche is equal to its longitudinal size. These simple facts reduce the number of unknown independent exponents in the model to 2.

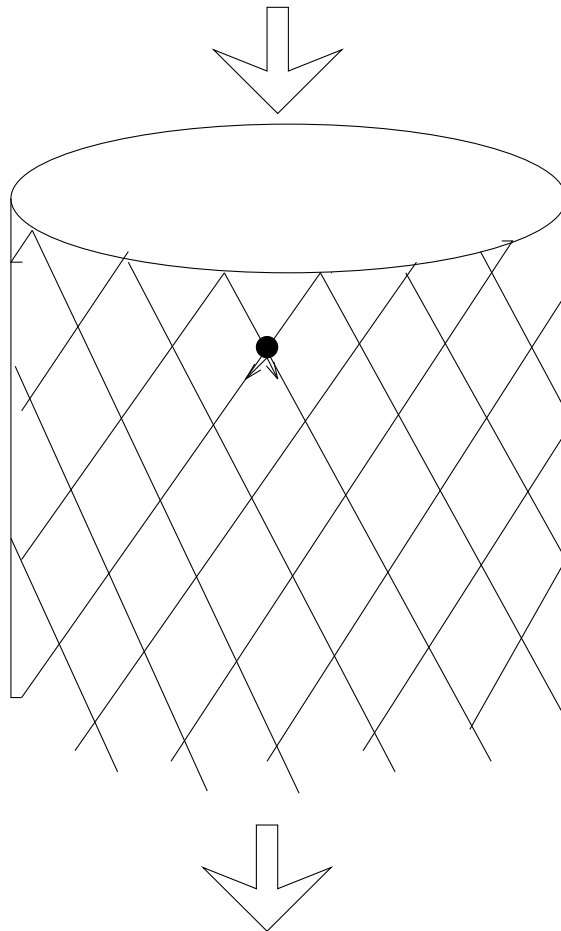


Figure 3: A two dimensional directed ASM. Particles are added on top, and removed from the bottom. On toppling at any site, two particles are transferred downward.

It is straight forward to determine the distribution of avalanche sizes in  $d$  dimensions. We consider the ASM on a  $d$ -dimensional hypercubical lattice. We define the longitudinal coordinate of site  $\mathbf{X} \equiv (x_1, x_2, \dots, x_d)$  as  $T = \sum x_i$ . We consider a finite lattice with  $0 \leq T < L$ . We assume periodic boundary conditions in the transverse direction. The pile will be assumed to be driven by particles added at top layer ( $T = 0$ ), and removed from the bottom layer ( $T = L - 1$ ). The matrix  $\Delta$  is assumed to have  $\Delta_{ii} = d$  for all sites  $i$ ,  $\Delta_{ij} = -1$ , if  $j$  is a nearest neighbor of  $i$  in the direction of increasing  $T$  (fig.3).

Let  $G_0(\mathbf{X}; \mathbf{Y})$  be the probability that if a particle is added at the site  $\mathbf{Y}$ , it will cause a toppling at the site  $\mathbf{X}$ . Since in the steady state, the height  $z_{\mathbf{X}}$  takes values  $0, 1 \dots (d - 1)$ . Thus, the probability that a site topples, given that  $r$  of its backward neighbors have toppled is  $r/d$ . This implies that  $G_0(\mathbf{X}; \mathbf{Y})$  satisfies a linear equation

$$G_0(\mathbf{X}; \mathbf{Y}) = \frac{1}{d} \left( \sum_{i=1}^d G_0(\mathbf{X} - \mathbf{e}_i; \mathbf{Y}) + \delta_{\mathbf{X}, \mathbf{Y}} \right) \quad (12)$$

Such stochastic processes are well studied in literature under the name of voter models [13]. The explicit solution of the above equation is easily written down:

$$G_0(\mathbf{X}; \mathbf{0}) = d^{-1-T} (T(\mathbf{X})! / \prod_i X_i!) \quad (13)$$

For large  $X$ , this has the familiar diffusive form

$$G_0 \sim T^{-\frac{d+1}{2}} \exp(-\text{Const.} R_{\perp}^2 / T), \quad (14)$$

where  $R_{\perp}$  is the length of the transverse component of  $X$ .

Let  $m(t)$  be the expected number of sites which topple on the surface  $T = t$  given that there is at least one toppling on this layer. The probability that the avalanche reaches upto layer  $T = t$ , by definition, varies as  $t^{1-\tau_t}$ . As the average flux out of this surface in the steady state must be 1 particle per avalanche, we get  $m(t) \sim t^{\tau_t-1}$ . As the average mass of an avalanche cluster of duration  $T$  is  $\int_0^T m(t) dt$ , we see that the number of sites toppled in the avalanche  $s$  scales as  $t^{\tau_t}$ . This immediately gives

$$\tau_s = \tau_d = 2 - 1/\tau_t \quad (15)$$

We define the 3-point function  $G(\mathbf{X}, \mathbf{Y}; \mathbf{0})$  as the probability that toppling occurs at both sites  $\mathbf{X}$  and  $\mathbf{Y}$ , when a particle is added at  $\mathbf{0}$ . We shall restrict ourselves to the case when  $T(\mathbf{X}) = T(\mathbf{Y})$ . In this case, this function also satisfies a difference equation, and the equations for 3-point function does not involve higher order  $n$ -point functions. It is easily shown that it satisfies also a linear equation

$$G(\mathbf{X}, \mathbf{Y}; \mathbf{0}) = \frac{1}{d^2} \sum_i \sum_j G(\mathbf{X} - \mathbf{e}_i, \mathbf{Y} - \mathbf{e}_j; \mathbf{0}); \quad \text{for } \mathbf{X} \neq \mathbf{Y}. \quad (16)$$

where  $\mathbf{e}_i$  and  $\mathbf{e}_j$  are the unit vectors of the  $d$ -dimensional lattice. For  $\mathbf{X} = \mathbf{Y}$ , we have the obvious condition that acts as boundary condition for the previous difference equation:

$$G(\mathbf{X}, \mathbf{X}; \mathbf{0}) = G_0(\mathbf{X}; \mathbf{0}) \quad (17)$$



These equations can be solved recursively [11]. Summing  $G(\mathbf{X}, \mathbf{Y}; \mathbf{0})$  over  $\mathbf{X}, \mathbf{Y}$ , we get  $\langle F^2 \rangle$ , where  $F$  is the number of site toppled at the layer  $T = t$ . It is found that this varies as  $t^{1/2}$  for  $d = 2$ . For  $d = 3$ , one finds  $\langle F^2 \rangle \sim t/(\log t)$ . This logarithmic correction factor to the power-law behavior has been checked in recent numerical simulations [14]. For  $d > 3$ , the variance varies as  $t$ . Thus we see that

$$\tau_t = 3/2, \text{ for } d = 2 \quad (18)$$

$$= 2, \text{ for } d \geq 3. \quad (19)$$

And the transverse size of the cluster at layer  $t$  varies as  $t^{1/2}$  in all dimensions. Using scaling relations, all other avalanche exponents can be determined. For the special case of two dimensions, the avalanche clusters have no holes, and the result  $\tau_t = 3/2$  can be obtained by using the fact that the left and right boundaries of the cluster can be thought of as annihilating random walkers on a line.

However, the compactness of the avalanche clusters is not a necessary condition for the validity of this result. The discussion above is easily adapted to other cases. For example, consider the 2-dimensional model, in which the toppling rule is that on toppling at site  $(x, y)$ , we transfer one particle each to the sites  $(x + 1, y - 1)$ ,  $(x, y - 1)$ , and  $(x + 1, y + 1)$ . For this model also, the avalanche exponents are same as for the simpler cases discussed in [11].

The  $d$ -dimensional directed ASM turns out to be equivalent to an  $(d-1)$ -dimensional aggregation model with uniform injection of particles proposed by Takayasu [15]. The  $d = 2$  case was proposed by Scheidegger as a model of river-networks [16]. In the Takayasu model, at each time step, one particle of mass 1 is injected at each site. Then each particle jumps to a neighbor chosen at random. If more than one particles jump to a site, they coalesce, and their masses add. After a long time, the distribution of masses tends to a limiting distribution. In a space-time picture, if we draw arrows along the direction of particle-diffusion, we get a directed spanning tree (which can be thought of a picture of a river network). There is a one-one correspondence between different trees, and configurations of the ASM. It is easily shown that on removing a randomly chosen site, the probability that removed subtree is  $S$ , is the same as the probability that the avalanche cluster of ASM is  $S$ , for all shapes and sizes of  $S$ . In particular, in the steady state of the Takayasu model, the probability that the mass at a randomly chosen site is  $m$  is exactly the same as the probability that there will be  $m$  topplings in the corresponding ASM in one avalanche.

In fact, we can easily adapt the analysis to even fractal lattices, so long as there is a translational invariance in the longitudinal direction. The prototype of such lattices is the so called tobleron lattice, obtained by a direct product of a linear chain with the Sierpinski gasket [17]. The details of calculation are omitted here. One finds that  $\langle F^2 \rangle \sim S(t)$ , where  $S(t)$  is the expected number of distinct sites visited by a random walker up to time  $t$  on the base fractal ( here the Sierpinski gasket). As  $S(t)$  is known to vary as  $t^{d_s/2}$ , where  $d_s$  is the spectral dimension of the base fractal, we get that in general

$$\tau_t = d_s/2 + 1, \text{ for } d_s \leq 2. \quad (20)$$

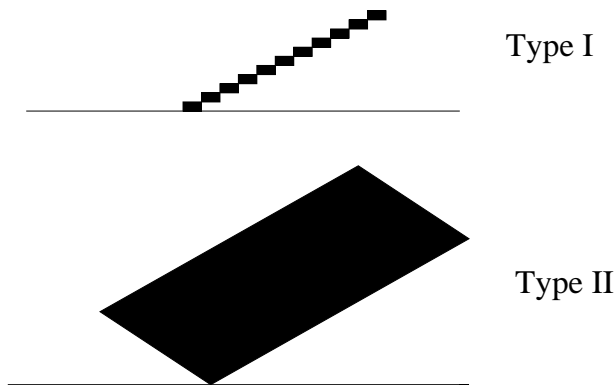


Figure 4: Space-time histories of the two types of avalanches for one-dimensional lattices. The x-direction is space, and y-direction is time. A box denotes a toppling event.

## 4 Undirected Sandpiles

Since for the directed ASM's, all the critical exponents can be determined exactly, these are more suited for pedagogical purposes than the original undirected BTW model. However, the undirected BTW model continues to tantalize researchers. It been studied a lot by theory, and simulations, and we shall now summarize our current understanding of it.

### 4.1 One-dimensional sandpiles

The simplest undirected lattice is the 1-dimensional linear chain. This simple case was already solved by BTW in their first paper [1]. In this case, it is easy to see that on a linear chain of size  $L$ , with  $z_{ic} = 2$ , and toppling to nearest neighbors, we get  $\det\Delta = L + 1$ . Only configurations with at most one site having height 0 are recurrent. Almost all avalanches are large, and the  $Prob(s < L^\alpha)$  goes to zero as  $L$  is increased for all  $\alpha < 2$ . The probability of avalanche having  $s$  topplings has the scaling form

$$Prob(s; L) \simeq L^{-2}f(s/L^2) \quad (21)$$

The detailed form of the scaling function  $f$  has been worked out by Ruelle and Sen [18].

A natural question is if this behavior is generic to one-dimensional models. We have studied this question by analysing the behavior of ASM's on more general one-dimensional graphs, with more complicated unit cells [19, 20]. Interestingly, it was found that the linear chain is atypical, and in general, on 1-dimesional lattices with more complicated unit cells, one finds two types of avalanches (fig. 4). In the first type of avalanches,  $s$  scales linearly with  $R$ . In the second type,  $s \sim R^2$ . Both these types occur with comparable frequencies. One cannot find a simple finite-size scaling form to describe the distribution of avalanche sizes, but needs a more complicated linear combination of two simple scaling forms (LC2SSF)

$$Prob_L(s) \sim L^{-1}F_1(s/L) + L^{-2}F_2(s/L^2) \quad (22)$$

Kadanoff et al had noted that simple finite-size scaling does not seem to work for a large class of 1-d models of SOC [21]. They proposed a multifractal description of avalanches, which corresponds to an integral over finite-size scaling forms with different scaling exponents. The LC2SSF form is simpler as it has only a finite sum, and not an integral. Recently, De Menech et al have argued that even in two dimensions, the simple finite-size scaling does not work for the distribution of avalanche sizes in the ASM [22]. This is an indication that the two dimensional avalanches may also be described by an LC2SSF form.

## 4.2 Sandpiles in very large dimensions

In large dimensions, one expects that the mean-field description of sandpiles will be correct. Mean-field descriptions of sandpiles have been tried in several different ways by different authors. The earliest treatment is by Tang and Bak [23]. For other approaches, see [4, 24, 25, 26]. The simplest treatment is to treat the sandpile model on a Bethe lattice. The undirected ASM on a Bethe lattice was solved exactly in [27].

The technique involves writing exact recursion equations for the probabilities of sub-configurations on a given part of a tree. For example, one finds that for a 3-coordinated tree, far away from the boundary, the heights  $z = 0, 1, 2$  occur with probabilities  $\frac{1}{12}, \frac{4}{12}, \frac{7}{12}$  respectively. One can also determine correlations between heights at different sites. It is found that the probability of an avalanche of size  $s \sim s^{-3/2}$ , for large  $s$ . The probability that the duration of avalanche is  $T$  varies as  $T^{-2}$  for large  $T$ . Multiple topplings are rare, and the probability of  $n$  topplings at the origin in the same avalanche decreases as  $\exp(-\exp(n))$ .

These exponents are the same as for critical percolation clusters on the Bethe lattice, and confirm our expectation that the avalanche process on this lattice propagates as a critical infection process. The mathematical treatment and the connection to the critical infection process is much easier to see if one considers a directed Bethe lattice, with equal number of arrows in and out at each site [28].

## 4.3 Equivalence to Spanning Trees and to $q \rightarrow 0$ Potts' Model

An ASM with a given symmetric toppling matrix  $\Delta$  can be represented by an undirected graph with  $N + 1$  vertices, corresponding to the  $N$  vertices of the ASM, and one additional vertex called the sink. We construct this graph by putting  $(-\Delta_{ij})$  edges between the vertices  $i$  and  $j$ , and  $\sum_j \Delta_{ij}$  edges between site  $i$  and the sink. From the well-known matrix-tree theorem [29], it follows that the number of spanning trees on this graph =  $\text{Det } \Delta$ , which is also  $|\mathcal{R}|$ .

A one-to-one correspondence between the recurrent configurations of ASM and the spanning trees can be set up by using the order in which ‘fire’ propagates in the burning algorithm in a configuration  $C$  to construct the spanning tree corresponding to  $C$  [30].

In the burning algorithm, a site with height 0 is burnt only after all its neighbors are burnt. In the corresponding spanning tree, such a site is a leaf-site. The fraction of sites which have  $z_i = 0$  thus gets related to the fraction of sites that are leaf sites in a large spanning tree [31]. For a square lattice, the latter fraction is calculable, and gives the concentration of sites with  $z = 0$  on the square lattice BTW model as  $f_0 = \frac{2}{\pi^2}(1 - \frac{2}{\pi})$ .

One can easily calculate the fraction of sites of higher coordination numbers in the spanning tree problem, but that is not directly related to probabilities of heights in the ASM case. The calculation of the latter is much more difficult, and requires somewhat sophisticated combinatorial graph theory, and the calculation has only been done for the square lattice so far [32].

One can also calculate the probability that two sites at distance  $R$  from each other both have heights 0. This can be done by a straightforward generalization of the 1-site problem [31]. It is found that in  $d$ -dimensions, this probability for large  $R$  varies as  $A + BR^{-2d}$ . It is expected that similar  $R^{-2d}$  tails will be seen in the joint probabilities for other heights, but the calculation is more difficult. The  $R^{-4}$  behavior of correlations is also seen for the surface sites in the two-dimensional case [33].

The spanning tree problem is well-known to be equivalent to the resistor network problem, and Fortuin and Kasteleyn showed that it can be considered as the  $q \rightarrow 0$  limit of the  $q$ -state Potts' model [35]. Thus, the undirected ASM is equivalent to the  $q = 0$  Potts model, and not to the  $q = 1$  Potts model, which might have been expected on the basis of its connection to the percolation problem.

#### 4.4 Exponents of the Two dimensional Model

The equivalence to the equilibrium 2-dimensional Potts model allows one to use known exponents of the latter to predict exponents for the ASM. At its critical point, the  $q$ -state Potts model shows additional symmetry: the conformal invariance. The central charge corresponding to  $q = 0$  is  $c = -2$ . For this value of central charge, one can look up the Kac table for other exponents of the theory. We find that the fractal dimension of chemical paths in the spanning trees problem in two dimensions is  $5/4$  [36, 37]. Since the outward propagation of activity in the avalanches, is like the spread of fire in the burning algorithm, we identify this with the exponent relating the distance to time of avalanches  $T \sim R^{5/4}$ . In terms of exponents, this gives the relation

$$(\tau_t - 1) = \frac{4}{5}(\tau_r - 1) \quad (23)$$

Another result which can be derived exactly is the distribution of avalanche sizes that are formed by adding a particle at the vertex of a wedge of angle  $\theta$  with open boundaries[38]. In this case, there is only one wave of toppling. Hence probability of avalanche reaching distance  $R$  is same as the propagator  $\Delta^{-1}$ . This is easily evaluated by using complex coordinates. One finds that the probability decreases as  $R^{-x}$ , with  $x = \pi/\theta$ . Using the compactness of clusters, one finds that for a wedge of angle  $\theta$  the value of corner exponent  $\tau_{corner}$  is  $1 + \frac{\pi}{2\theta}$ . For  $\theta = 2\pi$ , we get  $\tau_{corner} = 5/4$ . One may expect that adding a particle at the end of an open half-line should give rise to fewer topplings than adding a particle in bulk. This would suggest that  $\tau_d \leq 5/4$  in two dimensions. Unfortunately, it is difficult to prove such an inequality because of the many more recurrent states in the wedge problem than in the bulk case, and it is difficult to compare the probabilities of different events.

## 4.5 Waves of toppling

An avalanche in the ASM can be broken into a sequence of sub-avalanches. Let us call the site where a new particle is added  $O$ . Since the topplings can be done in any order, we topple once at  $O$ , and then topple any other unstable sites. This disturbance spreads as a wave, called the first wave of toppling. If  $O$  is still unstable, we topple once again at  $O$  and let other sites relax, constituting the second wave of toppling. This process is continued until  $O$  becomes stable. Thus, an avalanche is broken into a sequence of waves of toppling [34]. It was shown by Priezzhev that the set of all waves is in one-to-one correspondence with all two-rooted spanning trees. From the fact that in two dimensions, the propagator  $\Delta^{-1}$  has a logarithmic dependence on distance  $R$ , he showed that in an ensemble in which all waves have equal weight ( an avalanche with  $n$  waves is counted  $n$  times), one gets

$$Prob(\text{wave with } s \text{ topplings}) \sim 1/s, \text{ for } s \gg 1. \quad (24)$$

Using the fact that the distribution of sizes of last wave is related to the distribution of number of sites disconnected from a tree if a randomly chosen site is removed from a spanning tree, and the known fractal dimension of the paths along spanning trees, it can be shown that the probability of the size of the last wave is  $s$  varies as  $s^{-11/8}$  for large  $s$  [39].

## 4.6 Recent Developments

Priezzhev et al [40] developed a scaling picture of avalanches in the two-dimensional undirected ASM, based on the decomposition of avalanches into waves of topplings. The basic picture of Priezzhev et al depends on observation that the progress of an avalanche typically depends on a fast expansion, and then slow contraction. In the contraction phase, if the  $k$ -th wave is of size  $s_k$ , the next wave is smaller, with  $s_k - s_{k+1}$  being of order  $s_k^x$ , with  $x < 1$ . A cluster of size largest wave-size  $s_d$ , would typically have  $n_c \sim s_d^{1-x}$  waves, and hence  $s \sim s_d^{2-x}$ . Using some plausibility arguments for the value  $x = 3/4$ , these authors conjecture that the exact values of the exponents are  $\tau_d = 5/4$ ,  $\tau_s = 6/5$ , and  $\tau_t = 7/5$ .

Paczuski and Boettcher [41] have questioned some of the assumptions used by Priezzhev et al, in particular, the fact that quite often  $s_{k+1} > s_k$ . [Some of these objections have been taken into account in a revised version of their arguments by Ktitarev and Priezzhev [42].] Paczuski and Boettcher studied the conditional probability  $Prob(s_{k+1}|s_k)$  that the  $(k+1)$ -th wave is of size  $s_{k+1}$ , given that the previous wave was of size  $s_k$  by simulations. They presented numerical evidence that this quantity varies as  $s_{k+1}^{-a}$  for  $s_{k+1} \ll s_k$ , and as a different power for  $s_{k+1} \gg s_k$ . They also proposed that this conditional probability is only a function of the ratio of  $s_{k+1}$  and  $s_k$ , and has the scaling form

$$Prob(s_{k+1} = s' | s_k = s) \simeq (1/s')F(s'/s) \quad (25)$$

However, note that if the scaling variable is  $s'/s$ , then the change in  $s$  is of order  $s$ , hence necessarily  $x = 1$ . This makes  $\tau_s = \tau_d = 1$ .

In a recent paper De Menech et al [22] have argued that avalanches that reach the boundary scale differently than those that don't. This leads to a breakdown of the simple finite-size scaling theory usually assumed in deriving scaling relations, and in data analysis.

From analysis of their simulation data, these authors conclude that the events which reach the boundary occur roughly with probability  $L^{-1/2}$  on a lattice of size  $L$ . This suggests that the probability  $Prob(R \geq x)$  decreasing as  $x^{-1/2}$  for  $R < L$ , and hence gives  $\tau_r = 3/2$ , and  $\tau_d = 5/4$ , same as the value conjectured by Priezzhev et al [40]. However, in this range, there are few waves per avalanche, and De Menech et al argue that  $s \simeq s_d$  for  $s_d < L^2$ .

The avalanches which reach the boundary are only a small fraction ( $L^{-1/2}$ ). De Menech et al found that on the average each of these takes about  $L^{1/2}$  more waves before it is stopped. Thus, for these avalanches, each contraction is of order  $L^{3/2}$ , which is also consistent with the Priezzhev et al proposal that  $x = 3/4$ . For these avalanches,  $L^2 < s < L^{5/2}$ . If we argue that the probability that  $s$  is of order  $L^{5/2}$  is of order  $L^{-1/2}$ , this gives  $\tau_s = 6/5$ , as argued by Priezzhev et al. However, as the behavior for the full range of  $s$  cannot be described by a single power law, these exponents are only effective exponents.

## 5 The Abelian Distributed Processors Model

We would like to distinguish between the properties of the ASM that are specific to the details of the model, and those that hold for a larger class of SOC models. This has motivated us to study variations of the ASM, which keep some of its abelian structure intact. This is described below. For a different generalization, see [43, 44]. In particular, we show that the some sandpile models with stochastic toppling rules ( e.g. the Manna model [45]) still satisfies the abelian property [46]. The stochasticity in toppling rules models phenomenologically the variation of grain sizes, roughness etc. that are found in real experiments on granular media[47]. In the next section, we shall describe a different variation which makes the model non-abelian.

Consider a network of  $N$  finite-state automata  $P_i$ ,  $i = 1$  to  $N$ . The number of internal states may be different for different processors. Each processor is provided with an input stack, which can store messages until they are read, and some output channels to communicate to some of the other computers. At quasi-regular intervals, each computer checks its own input stack. If the stack is empty, it waits with no change of state. If the input stack is not empty, it pops a message from the stack ( according to some fixed local protocol, say first-in-first-out), and reads it. At each processor  $P_i$ , there is a rule table  $\mathcal{R}_i$  which tells what will be the final state  $\sigma_{fin}$  of the processor  $P_i$ , and what output messages and to which processor(s) are generated, given that initial state of the processor  $\sigma_{ini}$ , and the message read [48].

An input may generate none, one or more output messages depending on the initial state of the processor. We may assume, for simplicity, that each message consists of a single letter from a finite alphabet. The rule table, as well as, the processor speed can be different at different nodes of the network. Again, for simplicity, we ignore transmission delays, and an output generated will be assumed to reach the input stack of the receiver instantly. Some messages may also be sent ‘outside’, i.e. to computers outside the system ( say, a human).

In a quiescent state of the network, all the input stacks are empty, and the configuration of the full network is specified by  $\{\sigma_i\}$ , where  $\sigma_i$  is the internal state of each processor  $P_i$ . To begin a computation, we select an input message  $m_{in}$  at random, drawn from some

given fixed distribution and put the message in the input stack of one of the processors  $P_i$ , also chosen at random. This will cause the processor to change its internal state, and may generate messages to other processors. These new processors are now activated, and the computation goes on. Eventually, the system would again get into a state where all input stacks are empty, and no further processing occurs. This will be taken as the end of the computation.

We restrict our attention to systems where all computations halt after a finite number of steps. Once, an outside observer finds that the computation has stopped [49], he/she can reactivate the system by choosing again a site at random, and sending a message to its input stack.

The evolution of this network from one quiescent state to another is clearly Markovian, and after a large number of messages have been processed, it reaches a stochastic stationary state. This state is characterized by its probability distribution  $Prob(\{\sigma_i\})$ , of different quiescent states  $\{\sigma_i\}$ . Sending a message to a processor initiates a *computation*. The *size* of computation may be measured in terms of number of messages generated, or of processors affected, etc.. In the context of self-organized criticality, we would be interested in systems where such distributions show power-law tails.

In general, the result of such a computation will depend on the order in which different messages are received at a particular processor. In information science, one is often interested in ensuring that the network computation is independent of the speeds of different processors. This often requires setting up of controls by which some messages have to wait till some others have been read etc.. In the following, we restrict ourselves to computations in which the final result is always independent of the order in which inputs are received.

We assume that the rule-tables at each of the processors  $P_i$  are such that for all initial states  $\sigma_i$ , and all input messages  $m$  and  $m'$ , the final state  $\sigma_f$  and the set  $\{o_j\}$  of output messages generated is the same as would result if the messages were read in the reverse order ( first  $m'$ , and then  $m$ ). Thus we assume the abelian property at the level of individual processors. The validity of the abelian property can be checked by examining the rule table for each processor. Clearly, a network in which each component processor is abelian, is also abelian. We shall call such networks Abelian Distributed Processors (ADP).

As in the ASM case, we can define mutually commuting operators  $a_{i,m}$  corresponding to adding the message  $m$  to the processor  $P_i$  in a quiescent state  $C$  of the network, so that the resulting quiescent state is  $a_{i,m}C$ . Restricted to the set of recurrent states of the network, these operators define an Abelian group. Clearly, the ASM is a special case of ADP model, where the state  $\sigma_i$  is specified by the height variable, a message is a sandgrain, and rule table are the toppling rules of ASM. Many of the results obtained for the ASM are immediately generalized to the ADP case. It is useful to illustrate the generality of this structure with the help of some examples.

## 5.1 Block-renormalized Sandpiles

Consider the ASM, say on a square lattice with the usual toppling rule to nearest-neighbors. We try real space renormalization on this system, and group the sites together into blocks of size  $2 \times 2$  and think of each block as a single processor (fig. 5). The internal state of this

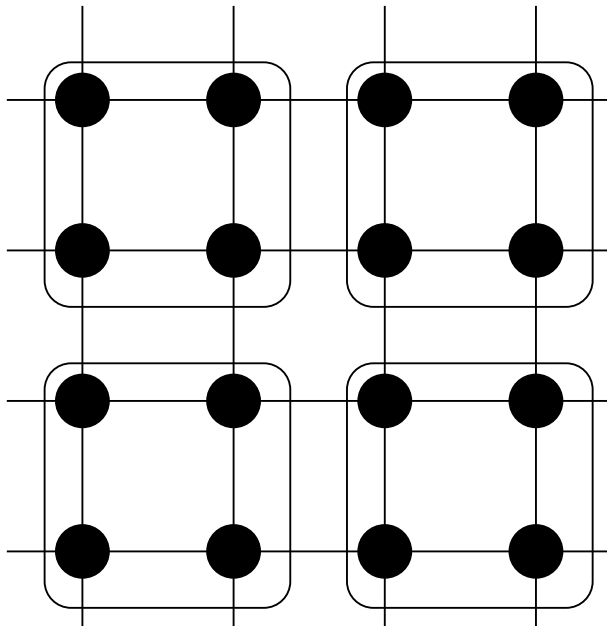


Figure 5: Block -renormalization of the ASM on a square lattice.

block is characterized by four integers  $(z_1, z_2, z_3, z_4)$ . Adding a grain could be to any of the four sites in the box. Accordingly, we define that an input message to the box is a single letter from a four letter alphabet. Because of forbidden configuration, the different possible internal states of the box are only 192, and not  $4^4 = 256$ . The rule table for transitions for each box is easily written down from the toppling rules of the original model. It is easy to see that this block renormalized ASM still satisfies the abelian property, and is an ADP model.

There has been some recent interest in applying real-space renormalization group techniques to ASM's ( on regular lattices [50], and also on fractals [51, 52]. While these seem to give rather good estimate of exponents, but the technique involves ad hoc uncontrolled approximations. The fact that abelian property is preserved under renormalization in the ADP model may help in devising more reliable approximation schemes [53].

## 5.2 Eulerian Walkers and Related Models

We may also consider sandpile models with periodic toppling rules. Both the threshold and the transfer rules may vary from one toppling to next in a deterministic manner *independently* at each processor. For example, consider a sandpile model on a square lattice with toppling rules as follows: stable heights of the pile are 0 and 1. If height is greater than 1, two particles are transferred to nearest neighbors. The transfer is to two neighbors in the vertical direction if in the previous toppling at that site was to the neighbors in the horizontal direction. And vice versa.

The simplest model of this type is the Eulerian walkers model [54]. In this model, we



imagine a walker on a lattice. at each site, there is an arrow marking the direction of the last exit of a walker from that site. A walker is randomly dropped on to a site. At each site, he resets the arrow to a new direction according to fixed rules,( say by rotating it clockwise by  $90^\circ$  if the lattice is the square lattice), and then takes a step in the direction of the arrow. At the boundary, some steps lead to the walker leaving the system. At this stage, a new walker is introduced in the system.

In the steady state, the directions of arrows in the system develop long-range correlations. In fact, it can be shown that in the quiescent state after a long time, all the arrow directions will form a directed spanning tree. This implies that one can set up a one-to-one correspondence between the recurrent configurations of the Eulerian walkers model, and of the corresponding ASM. On closed graphs with no sink sites, the walker eventually settles into cyclic state. It can be shown that this limit cycle which visits each bond exactly once. Such a path is called the Eulerian path, and hence the name of the model. Several properties of this model can be determined exactly [54]. For example, it is found that on a  $d$ -dimensional lattice with an initially random configuration of arrows, the mean square deviation of the walker from his initial position increases with time  $t$  as  $t^{\frac{2}{d+1}}$  for  $d \leq 2$ , and as  $t$  for  $d > 2$ . However, the distribution of the number of steps taken by the walker before he leaves the lattice is not known.

An interesting model in this class of models with periodic toppling rules is obtained if we consider lots of Eulerian walkers on a lattice, with the rule that a walker arriving at a site just waits there unless the number of walkers waiting at that site exceeds  $q$ . If it does,  $q$  walkers leave in  $q$  successive directions, and the arrow is reset. The ASM corresponds to  $q = 4$ , and Eulerian walkers to  $q = 1$  [54]. The case  $q = 2$  on a Bethe lattice of coordination number 4 has recently been studied by Shcherbakov [55]. The techniques used to study the ASM on the Bethe lattice are not easily adapted to this model as identification of all the forbidden subconfigurations, and the propagation of avalanches is more complicated.

The interesting point about these variations is that the particle addition operators  $a_i^+$  in this case satisfy the same algebra as the square lattice ASM [eqs (3)], but the structure and sizes of avalanches is quite different. In particular, for the Eulerian walkers case, on a lattice of size  $L$ , in most cases, a walker takes  $O(L^2)$  steps before leaving the system. Since this is the most natural analogue of number of topplings in the ASM, in this case, there are no small avalanches.

### 5.3 Abelian Stochastic Sandpile Models

The periodic toppling rules of the previous example are most simply realized by assuming the existence of a finite number  $k$  of distinct rule-tables  $R_1, R_2, \dots, R_k$  at each site at each site, and an infinite sequence  $\{r_i\}$  with  $1 \leq r_i \leq k$ , which tells which of the  $k$  rule-tables to follow for the  $i$ -th toppling. If the sequence of integers  $\{r_i\}$  is periodic, we get the ADP model with periodic update rules. We may however take more complicated sequences without losing the abelian character. In particular, we may take this sequence to be generated by a pseudo-random number generator of a large period. A system evolving this way, may equally well be described as undergoing evolution with stochastic update rules.

Let us consider a square lattice. The stable heights at each site are assumed to have

a value 0 or 1. If the height at a site exceeds 1, that site topples according to one of the following rules:

R1: Two particles transferred to north-south neighbors.

R2: two particles transferred to east-west neighbors.

Each site has a local pseudo-random number generator, which gives a random single bits (0 or 1) on each request. Before each toppling, a new random bit is drawn from this local generator. If the output is 0, rule R1 is used, else R2. As the output sequence from each generator is actually deterministic, this model belongs to the ADP class. This model is equivalent to a model of sandpiles with stochastic update rules first discussed by Manna [45]. He studied cases where the critical height is always 2, but to which neighbor the transferred particles go was decided randomly. In his first model, the two particles move in opposite directions with equal probability, which is equivalent to the ADP discussed above. In the second model, for each of the two particles, one of the four neighbors of the site is chosen at random, and the particle transferred there.

We define particle addition operators  $a_{i,j}$  corresponding to addition of a particle at site  $(i, j)$  and relaxation as before. It is easy to see that these operators still commute:

$$[a_{i,j}, a_{k,l}] = 0, \quad \text{for all } i, j, k, l. \quad (26)$$

If the internal states of different random number generators are not accessible to outside observer, the topplings are effectively stochastic. Applying  $a_i$  to a basis vector, say  $|C\rangle$ , of the system does not lead to a unique configuration, but a *linear combination of basis vectors*. But the operators  $a_i$  still commute. The closure relations between the operators are similarly modified. It is easy to see that for Manna's first stochastic model, these become

$$a_{i,j}^2 = (1/2)(a_{i+1,j}a_{i-1,j} + a_{i,j+1}a_{i,j-1}) \quad (27)$$

for all  $1 \leq i, j \leq L$ , and assuming that  $a_{i,j} = 1$ , if  $i$  or  $j$  equals 0 or  $L$ . If we consider one dimensional representations of this commutative algebra, we can think of  $a$ 's as complex numbers. However, not much can be said at present about the nontrivial solutions of this set of coupled quadratic equations. Even the number of nontrivial solutions is not easy to determine. In general,  $N$  simultaneous quadratic equations have  $2^N$  solutions. But several of these may be trivial (all  $a$ 's zero). For example, we can easily check that for a 2X2 square, the number of nontrivial solutions of the 4 coupled quadratic equations in 4 variables is only 3.

Looking at the original problem, it is easy to see that it has some FSC's: For example, it is easy to show that a 2X2 square with all sites unoccupied is an FSC. Clearly, there are other larger FSC's. For the small 2X2 square discussed above, the number of recurrent configurations is easily seen to be 15. Thus the number of nontrivial solutions  $\{a_i\}$  is no longer equal to the number of recurrent configurations in this stochastic generalization of the ASM.

In this respect, Manna's second model is more tractable. In this case, each of the two particles is moved randomly to one of the four neighbors. It follows that in this case the operators  $\{a_{i,j}\}$  satisfy the following equation:

$$a_{i,j}^2 = \frac{1}{16}[a_{i+1,j} + a_{i-1,j} + a_{i,j+1} + a_{i,j-1}]^2 \quad (28)$$

These are again reducible to  $L^2$  coupled quadratic equations for the simultaneous eigenvalues of these operators. But in this case, we can reduce them to  $L^2$  linear equations

$$\eta_{i,j}a_{i,j} = \frac{1}{4}[a_{i+1,j} + a_{i-1,j} + a_{i,j+1} + a_{i,j-1}] \quad (29)$$

where  $\eta_{i,j} = \pm 1$ . There are  $2^{L^2}$  different choices of the  $L^2$  different  $\eta$ 's. For each such choice, we get a set of simultaneous eigenvalues of  $\{a_{i,j}\}$ . Thus, we get the full set of eigenvalues for these operators. Correspondingly, one can show that the all empty configuration is recurrent in this case. Hence all configurations are recurrent.

It may be noted that the equations (29) look like the wave-equation of quantum-mechanical particle in a random potential. This provides an intriguing connection to the well-known Anderson localization problem [56]. Here  $a_{i,j}$  act like the wavefunction, and  $\eta_{i,j}$  is the random potential. Of course, in this case, it is an inhomogeneous linear equation (due to presence of boundary terms), and so involves a weighted sum over all the eigenvalues of the homogeneous case. The sum over the eigenvalues of  $\{a_{i,j}\}$ 's is the average over different realizations of the potential in the Anderson problem.

## 6 A Non-abelian Stochastic Sandpile Model

This model is a generalization of the directed ASM by making the toppling rules stochastic in a different way. The stochasticity is parameterized by a real parameter  $p$ , with  $0 \leq p \leq 1$ . For simplicity, let us define it on the same lattice as in section 3.2. The height at the site  $(i, j)$  of the square lattice is  $h(i, j)$ . If on adding a particle, the height  $h(i, j)$  exceeds 1, the site becomes *unstable*. With probability  $p$ , the height at the unstable site decreases by 2, and one particle is transferred to each downward neighbor. with probability  $(1 - p)$ , no transfer takes place. In either case, the site becomes stable for the next time step. At any time, only sites where at least one particle was added at the last step can become unstable [57].

The case  $p = 1$  corresponds to the directed ASM. For  $p \neq 1$ , the model is no longer abelian, as adding two particles at a site one by one may have a different effect than if they are added together. Also, if  $p \neq 1$ , in the steady state of the system, height at a site can take arbitrarily large values with a small probability. Interestingly, this introduction of stochasticity is found to be a relevant perturbation, and it changes the avalanche exponents. We determine these exactly in terms of the critical exponents of directed percolation. A recent discussion of different universality classes of sandpile models may be found in [58].

For any site, we can define two parameters  $p_1$  and  $p_2$ , which give the probabilities that a toppling occurs at the site if 1 or 2 of the sites above it have toppled. The evolution of avalanches in this model is as in the Domany-Kinzel (DK) model of directed percolation with the two probabilities  $(p_1, p_2)$  [59]. In our case, we have  $p_2 = p$  for all sites, and  $p_1 = p\rho$ ,  $\rho$  being the probability that the height at the site is nonzero. In the DK model, there are two phases: for low values of  $p_1, p_2$ , all percolation clusters are finite, but for  $p_1, p_2$  large enough, there is a finite probability of an infinite cluster. In our case, if no percolation occurs, there is a build up of particles in the top layers, and no steady state exists. [ This occurs if  $p < p^*$ , where  $p^*$  is the critical probability of directed site percolation on a square lattice.] Also,  $(p_1, p_2)$  cannot be in the supercritical phase, because then each layer would

lose more particles than it gains from the layer above. It follows that the steady state values of  $(p_1, p_2)$  must be very near the critical line of the DK model.

Thus the local evolution of avalanches in our model must be same as that of critical cluster in the DK model. But for the critical clusters in the DK model, it is known that expected number of sites in the  $\ell$ -th layer increases as a power of  $\ell$ . This cannot be satisfied because of the particle conservation, mean outflux of particle per layer per avalanche is 1. The resolution of these conflicting requirements is that the system in the steady state has a  $p_1$  value which varies with  $\ell$ , as

$$p_1(\ell) = p_{1c} - A\ell^{-1/\nu_{\parallel}} \quad (30)$$

where  $\nu_{\parallel}$  is the exponent characterizing the rate at which the longitudinal correlation length diverges away from critical point ( $\xi_{\parallel} \sim (\delta p)^{-\nu_{\parallel}}$ ). Thus the calculation of distribution of avalanche-sizes reduces to finding statistics of surface avalanches when the concentration profile has a power-law dependence away from surface. In the present case, these exponents depend on the amplitude  $A$ . The requirement of mass balance in the steady state thus fixes the value of  $A$ , and hence determines the avalanche distribution in terms of exponents of the directed percolation problem.

Thus the model with  $p \neq 1$  is a different universality class than the abelian case  $p = 1$ . In detail, one finds [57] that in  $d$ -dimensions ( $1 < d < 5$ ) the probability that avalanche stops at layer  $\ell$  varies as  $\ell^{-\tau_t}$ , where

$$\tau_t = 1 + (d - 1)\nu_{\perp}/\nu_{\parallel} - \beta/\nu_{\parallel}. \quad (31)$$

Here  $\beta$  and  $\nu_{\perp}$  are the standard exponents of directed percolation. and the probability that avalanche has exactly  $s$  topplings varies as  $s^{-\tau_s}$ , with  $\tau_s = 2 - 1/\tau_t$ .

For  $d = 2$ , using the known estimates of the DP exponents, we get  $\tau_s \simeq 1.313$  and  $\tau_t \simeq 1.473$ . These differ by only about 2% from the values  $\tau_s = 4/3$ , and  $\tau_t = 3/2$  for the case  $p = 1$ . The upper critical dimension changes from the value  $d_u = 3$  for  $p = 1$  to  $d_u = 5$  for  $p \neq 1$ . For all  $d \geq 5$ , we recover the mean-field values  $\tau_s = 3/2$ ,  $\tau_t = 2$ .

Thus all the exponents characterizing avalanche clusters for  $p \neq 1$ , are expressed in terms of exponents of directed percolation, but they are different from the exponents of directed percolation clusters because of the particle conservation in the model. The undirected version of this stochastic model remains unsolved even in  $1 + 1$  dimensions.

## 7 Concluding Remarks

The sandpile is, of course, a non-equilibrium system driven by the slow addition of grains. One often finds in literature the statement that non-equilibrium systems are qualitatively different from equilibrium ones, because one cannot write a ‘well-behaved’ hamiltonian with short-ranged interactions that will show the often long-ranged correlations observed in the non-equilibrium steady states (NESS’s). But for the ASM, the NESS is characterized by a Boltzmann measure corresponding to a rather well-known equilibrium statistical mechanics model (the spanning trees or the  $q=0$  Potts model). For the directed ASM’s, the even simpler hamiltonian  $H=0$  is adequate.

I think that the conventional wisdom is not really wrong here, because the transformation of variables needed to go from the height variables of the ASM to the spanning tree is quite complicated, and nonlocal. It preserves the probabilities of different configurations in the steady state, but the simple local rules of toppling in the ASM become complicated (and ill-understood) nonlocal evolution rules in the spanning tree description. A description directly in terms of Potts model spins is not possible because of the formal nature of the  $q \rightarrow 0$  limit.

Another oft-discussed question is related to the fact that in the ASM, the time-scale of sand addition is assumed to be much longer than the time-scale of individual topplings. In fact, the existence of two very widely separated time-scales has been argued to be one of the defining characteristics of self-organized critical systems [60]. But, as we have seen, the directed ASM in  $d$ -dimensions is equivalent to the Takayasu aggregation model. In the latter model, also driven, there seems to be no separation of time-scales: aggregation and diffusion both occur at comparable rates! In addition, the distribution of avalanche-sizes in the ASM simply translates to the distribution of masses in the steady state of the second model.

If we accept the existence of two time-scales as one of the defining characteristics of SOC, we find ourselves in the uncomfortable situation that the prototypical model of self-organized criticality is ‘equivalent’ to a model that does not show self-organized criticality. In this case, we cannot even resort to the argument about change of variables being complicated, as the avalanche distribution of ASM simply corresponds to the mass distribution in the Takayasu model. We conclude that the existence of two widely separated timescales should not be considered as a necessary condition for self-organized criticality. A long-time scale can be defined by the typical separation between suitably defined rare ‘burst’ events in the system. In the Takayasu model, these burst events are formation of large mass clusters.

Before concluding, let me make a small list of questions which are not quite well understood yet.

Can one obtain the results for spanning trees, without using the heavy machinery of conformal field theory? In particular, the result that tortuosity exponent of paths on spanning trees in two dimensions is  $5/4$  should have a direct ‘combinatorial’ derivation.

What are the exponents of the undirected ASM in two dimensions?

Is it true that fractal dimension of avalanche clusters is same as space dimension for all  $d \leq 4$ ?

Is there a systematic way to exactly compute the probabilities of avalanches of sizes  $1, 2, \dots$  in the undirected ASM?

What are the time-dependent correlations in avalanche sizes?

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