An upper bound on the total inelastic cross-section as a function of the total cross-section

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Recently André Martin has proved a rigorous upper bound on the inelastic cross-section σ_{inel} at high energy which is one-fourth of the known Froissart-Martin-Lukaszuk upper bound on σ_{tot}. Here we obtain an upper bound on σ_{inel} in terms of σ_{tot} and show that the Martin bound on σ_{inel} is improved significantly with this added information.

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1. Introduction

The total cross-section $\sigma_{tot}(s)$ for two particles to go to anything at c.m. energy $\sqrt{s}$ must obey the Froissart-Martin bound,

$$\sigma_{tot}(s) \leq_{s\to\infty} C\,[\ln(s/s_0)]^2 \qquad (1)$$

proved at first from the Mandelstam representation by Froissart [1] and later from the basic principles of axiomatic field theory by Martin [2]. Of the two unknown constants the constant C was fixed by [3] to obtain,

$$\sigma_{tot}(s) \leq_{s\to\infty} 4\pi/t_0\,[\ln(s/s_0)]^2, \qquad (2)$$

where, $t = t_0$ is the lowest singularity in the t-channel. For many physically interesting cases such as $\pi\pi, KK, K\overline{K}, \pi K, \pi N, \pi\Lambda$ scattering $t_0 = 4m_\pi^2 - \epsilon$, ϵ being an arbitrary small positive constant, and m_π the pion-mass [4]. In some cases we can take $\epsilon = 0$, e.g. for pion-pion scattering if the D-wave scattering length is finite [5].It will be convenient to denote the right-hand side of the bound on $\sigma_{tot(s)}$ as

$$\sigma_{max}(s) = 4\pi/t_0\,[\ln(s/s_0)]^2. \qquad (3)$$

The Froissart-Martin bound has been seminal both to the development of the field of high energy theorems in axiomatic field theory (see e.g. the review [6])and to that of phenomenological models leading to accurate predictions of total and elastic cross sections before their experimental measurem ents [7]. Remarkably, one of us (A. M.) has recently obtained a bound on the total inelastic cross section at high energy [8],

$$\sigma_{inel}(s) \leq_{s\to\infty} \pi/t_0\,[\ln(s/s_0)]^2, \qquad (4)$$

which is one-fourth of the bound $\sigma_{max}(s)$ on the total cross-section, thus improving the simple bound $\sigma_{inel} \leq \sigma_{tot}$.

The present paper is inspired by Martin's bound on the inelastic cross-section. In fact T. T. Wu [9] by extending Martin's variational calculation to incorporate a given total cross-section and independently S.M .Roy and Virendra Singh [10], by exploiting their previous upper bound on the differential cross section in terms of elastic cross-section, [11],[12] realized that one could solve a more general problem: find a bound on the inelastic cross-section as a function of the value of the total cross-section. It is obvious that if the total cross section vanishes the inelastic cross section also vanishes. but it is also extremely plausible that if one maximizes the total cross section, the important partial wave amplitudes will be imaginary and maximal so that, from the unitarity condition, there is no room left for the inelastic cross section which will receive only negligible contributions from the tail of the partial wave distribution.

The net result exhibiting both these features is the bound we present in this paper,

$$\Sigma_{inel}(s) \leq_{s\to\infty} \Sigma_{tot}(s)\big(1 - \Sigma_{tot}(s)\big), \qquad (5)$$

where,

$$\Sigma_{tot}(s) \equiv \sigma_{tot}(s)/\sigma_{max}(s), \qquad (6)$$

and

$$\Sigma_{inel}(s) \equiv \sigma_{inel}(s)/\sigma_{max}(s). \qquad (7)$$

Maximizing wth respect to σ_{tot} we get the factor 1/4 announced at the beginning of this paper,i.e.

$$\sigma_{inel}(s) \leq_{s\to\infty} \sigma_{max}(s)/4. \qquad (8)$$

In Sec. 2 we summarise our notations and recall the basic results from axiomatic field theory. We then present two possible derivations of the bound on the inelastic cross-section in terms of total cross-section, the direct

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variational approach in Sec. 3, and the approach using the 1970 bound on the differential cross section in terms of the elastic cross-section [11], [12] in Sec. 4. Sec. 5 contains concluding remarks including directions for future work on high energy phenomenology.

2. Basic Results from Axiomatic Field Theory

Let $F(s,t)$ be the elastic scattering amplitude for $ab \to ab$ at c.m. energy $\sqrt{s}$ and momentum transfer squared t and be normalized such that the differential cross-section is given by

$$\frac{d\sigma}{d\Omega}(s,t) = |\frac{F(s,t)}{\sqrt{s}}|^2 \tag{9}$$

with t being given in terms of the c.m. momentum k and the scattering angle θ by the relation,

$$t = -2k^2(1-\cos\theta). \tag{10}$$

Then, for fixed s larger than the physical $s-$channel threshold, $F(s;\cos\theta) \equiv F(s,t)$ is analytic in the complex $\cos\theta$ -plane inside the Lehmann-Martin ellipse [13], [2], with foci -1 and +1 and semi-major axis $\cos\theta_0 = 1 + t_0/(2k^2)$, where t_0 is independent of s. In fact, as mentioned already, $t_0 = 4m_\pi^2 - \epsilon$ for many interesting cases. Within the ellipse $F(s,t)$ has the partial wave expansion,

$$F(s,t) = \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(1+t/(2k^2)), \tag{11}$$

which converges absolutely and uniformly in t for $|t| < t_0$; hence $F(s,t)$ is analytic in t for $|t| < t_0$. Unitarity implies that,

$$Ima_l(s) \geq |a_l(s)|^2 \tag{12}$$

in the physical region. Further, [14] for fixed t in the region $|t| < t_0$, $F(s,t)$ satisfies dispersion relations in s with two subtractions. This implies, in particular, that the s-channel absorptive part for $0 \leq t < t_0$ has the convergent partial wave expansion,

$$A(s,t) \equiv ImF(s,t)$$
$$= \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)Ima_l(s)P_l(1+t/(2k^2)), \tag{13}$$

and obeys

$$\int_C^\infty dsA(s,t)/s^3 < \infty,\; 0 \leq t < t_0. \tag{14}$$

Hence, if we assume that $A(s,t)$ is continuous in s, there exist sequences of $s \to \infty$ such that

$$A(s,t) < Const.\frac{s^2}{\ln(s/s_0)},\; 0 \leq t < t_0. \tag{15}$$

For simplicity, in this paper, we deduce asymptotic bounds on $\sigma_{inel}(s)$ only for such sequences. Bounds on energy averages will be considered later to avoid this restriction.

3. Variational Bound on Inelastic Cross-section in terms of Total Cross-section

Since $\sigma_{inel} = \sigma_{tot} - \sigma_{el}$, this problem is equivalent to finding a lower bound on σel. Further,

$$\sigma_{el}(s) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|a_l(s)|^2$$
$$\geq \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(Ima_l(s))^2 \equiv \sigma_{el,im}(s). \tag{16}$$

So, it suffices to find a variational lower bound on $\sigma_{el,im}$. We vary the $Ima_l(s)$ subject to the unitarity constraints

$$Ima_l(s) \geq\ 0\,, \tag{17}$$

to a given value of,

$$\sigma_{tot}(s) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)Ima_l(s)\,, \tag{18}$$

and to the constraint

$$A(s,t_0) \equiv \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)Ima_l(s)P_l(1+t_0/(2k^2))$$
$$< Const.\frac{s^2}{\ln(s/s_0)}\,. \tag{19}$$

For simplicity, since we work at a fixed-s , we suppress the s-dependence of $Ima_l(s)$, $\sigma_{el,im}(s)$ and $\sigma_{tot}(s)$. Denoting ,

$$z_0 = 1 + t_0/(2k^2), \tag{20}$$

the lower bound on $\sigma_{el,im}$ is obtained by choosing,

$$Ima_l = \alpha\big(1 - P_l(z_0)/P_{L+r}(z_0)\big), for\; 0 \leq l \leq L\,, \tag{21}$$

and,

$$Ima_l = 0, for\; l > L\,, \tag{22}$$

with the constants $0 \leq r < 1, \alpha > 0$ and the positive integer L being fixed from the given value of σ_{tot} and the given upper bound on $A(s,t_0)$. We omit the straight forward proof which is by direct subtraction of a $\sigma_{el,im}$ with arbitrary partial waves obeying the given constraints from the variational result. After carrying out the summations over l, the constraint equations become,

$$\sigma_{tot}\, k^2/(4\pi\alpha) =$$
$$(L+1)^2 - (P'_{L+1}(z_0) + P'_L(z_0))/P_{L+r}(z_0)\,, \tag{23}$$

and

$$A(s,t_0)\frac{k}{\alpha\sqrt{s}} = P'_{L+1}(z_0) + P'_L(z_0) - \frac{(L+1)^2P_L^2(z_0) - (z_0^2-1)(P'_L(z_0))^2}{P_{L+r}(z_0)}, \tag{24}$$

and the bound on σ_{el} becomes,

$$k^2/(4\pi\alpha^2)\,\sigma_{el} \geq k^2/(4\pi\alpha^2)\,\sigma_{el,im} \geq (L+1)^2 - 2(P'_{L+1}(z_0) + P'_L(z_0))/P_{L+r}(z_0) + \frac{(L+1)^2P_L^2(z_0) - (z_0^2-1)(P'_L(z_0))^2}{(P_{L+r}(z_0))^2}. \tag{25}$$

At high energies, using $s/\sigma_{tot} \to \infty$, the two constraint equations yield easily that $L = O(\sqrt{s}\ln(s/s_0))$; we may therefore set $r = 0$ and use the following approximations for the Legendre polynomials,

$$\begin{aligned} P_L(z_0) &= I_0(\xi)(1+O(L/s)), \xi \equiv (2L+1)\sqrt{(z_0-1)/2} \\ P'_L(z_0) &= (1/2)L\sqrt{s/t_0}I_1(\xi)(1+O(L/s)), \\ I_\nu(\xi) &= \frac{\exp\xi}{\sqrt{2\pi\xi}}(1-(4\nu^2-1)/(8\xi)+...), \xi\to\infty, \\ for \quad & s\to\infty,\ L/\sqrt{s}\to\infty,\ L/s\to 0\,, \end{aligned} \tag{26}$$

where the $I_\nu(\xi)$ denote the modified Bessel functions. We then have,

$$\sigma_{tot}\,k^2/(4\pi\alpha) \approx L^2 - \frac{I_1(\xi)}{I_0(\xi)}L\sqrt{s/t_0} \tag{27}$$

$$\approx L^2(1+O(\sqrt{s}/L)), \tag{28}$$

$$A(s,t_0)\frac{k}{\alpha\sqrt{s}} \approx I_1(\xi)L\sqrt{s/t_0} + L^2\frac{I_1(\xi)^2 - I_0(\xi)^2}{I_0(\xi)} \tag{29}$$

$$\approx I_0(\xi)\frac{L\sqrt{s}}{2\sqrt{t_0}}(1+O(\sqrt{s}/L)). \tag{30}$$

The asymptotic bounds on elastic and inelastic cross-sections become, with these approximations,

$$k^2/(4\pi\alpha^2)\,\sigma_{el} \geq L^2 - 2L\sqrt{s/t_0}\frac{I_1(\xi)}{I_0(\xi)} + L^2(1-(\frac{I_1(\xi)}{I_0(\xi)})^2), \tag{31}$$

and

$$k^2/(4\pi\alpha)\,\sigma_{inel} \leq (1-2\alpha))(L^2 - L\sqrt{s/t_0}\frac{I_1(\xi)}{I_0(\xi)}) + \alpha L^2(\frac{I_1(\xi)}{I_0(\xi)})^2. \tag{32}$$

We now use the assumed upper bound on $A(s,t_0)$ to evaluate L, α for high energies. We have,

$$Const.s/(\sigma_{tot}\ln(s/s_0)) = I_0(\xi)\frac{\sqrt{s}}{2L\sqrt{t_0}}(1+O(\sqrt{s}/L)), \tag{33}$$

which yields ,

$$\frac{L}{\sqrt{s}} = (1/(2\sqrt{t_0}))\ln(\frac{s}{s_0^2\sigma_{tot}})(1+O(\ln(s/s_0))^{-1}) \tag{34}$$

$$\alpha = \frac{\sigma_{tot}(s)}{\hat{\sigma}_{tot}(s)}(1+O(\ln(s/s_0))^{-1}) \tag{35}$$

where,

$$\hat{\sigma}_{tot}(s) \equiv 4\pi/t_0\,[\ln(\frac{s}{s_0^2\sigma_{tot}})]^2. \tag{36}$$

Hence, we have the lower bound on elastic cross-sections,

$$\sigma_{el}(s) \geq \frac{(\sigma_{tot}(s))^2}{\hat{\sigma}_{tot}(s)}(1+O(\ln(s/s_0))^{-1}). \tag{37}$$

Note that $\hat{\sigma}_{tot}(s)$ can be replaced by $\sigma_{max}(s)$ for $s \to \infty$, except in the unrealistic case $\sigma_{tot} \to 0, fors \to \infty$ which leads to a small inelastic cross-section $\sigma_{inel}(s) \to 0$. Hence, using equation (37), the upper bound on the inelastic cross-section valid in all cases is ,

$$\sigma_{inel}(s) \leq_{s\to\infty} \sigma_{tot}(s)\big(1-\Sigma_{tot}(s)\big), \tag{38}$$

which leads to the announced bound on $\sigma_{inel}(s)/\sigma_{max}(s)$, given by equation (5).

4. Upper Bound on Inelastic Cross-section from an Upper Bound on Differential Cross-section in terms of Elastic Cross-section

We show here that the inelastic cross-section bound can also be derived as a corollary of an upper bound on the differential cross section in terms of the elastic cross-section, established by two of us [11] many years ago,

$$\frac{d\sigma}{dt}(s,t=0) \leq_{s\to\infty} \frac{\sigma_{el}(s)}{4t_0}[\ln(\frac{s}{s_0^2\sigma_{el}})]^2. \tag{39}$$

This bound can also be written as [12],

$$\sigma_{tot}[1+(\frac{ReF(s,t=0)}{ImF(s,t=0)})^2] \leq_{s\to\infty} \frac{4\pi\sigma_{el}}{t_0\sigma_{tot}}[\ln(\frac{s}{s_0^2\sigma_{el}})]^2. \tag{40}$$

If the real part $ReF(s,t=0)$ is unknown we have the weaker bound,

$$\begin{aligned} \sigma_{tot} &\leq_{s\to\infty} \sqrt{\frac{4\pi}{t_0}}\big[\sqrt{\sigma_{el}}\big(\ln(\frac{s}{s_0^2\sigma_{tot}}) - \ln(\frac{\sigma el}{\sigma_{tot}})\big)\big] \\ &\leq_{s\to\infty} \sqrt{\sigma_{el}\hat{\sigma}_{tot}} + (2/e)\sqrt{\frac{4\pi\sigma_{tot}}{t_0}}\,, \end{aligned} \tag{41}$$

where, in the last line we have used the elementary inequality,$\sqrt{x}\ln x \geq -2/e, for\ 0 < x < 1$. This equation yields a lower bound on $\sigma_{el}(s)$ for any asymptotic behaviour of $\sigma_{tot}(s)$. As noted in the last section,for deducing an upper bound on $\sigma_{inel}(s)$, it suffices to assume that σ_{tot} does not vanish for $s \to \infty$. In particular, if $\sigma_{tot}(s) > 16\pi/(e^2 t_0)$, we have

$$\sigma_{tot}(\sqrt{\sigma_{tot}} - (2/e)\sqrt{\frac{4\pi}{t_0}}\,)^2 \;\leq_{s\to\infty}\; \sigma_{el}\hat{\sigma}_{tot} \approx_{s\to\infty}\; \sigma_{el}\sigma_{max} \quad (42)$$

and hence the upper bound on the inelastic cross-section,

$$\sigma_{inel} \leq_{s\to\infty} \sigma_{tot}\big[1 - \Sigma_{tot}(1 - (2/e)\sqrt{\frac{4\pi}{t_0\sigma_{tot}}}\,)^2\big], \quad (43)$$

which yields the desired bound (5) on the inelastic cross-section if $\sigma_{tot}(s) \to \infty, for\ s \to \infty$.

5. Conclusion

We have derived an asymptotic upper bound on the inelastic cross-section in terms of the total cross-section which improves Martin's recent bound [8] when $\sigma_{tot}(s) \sim C(\ln(s/s_0))^2$. Varying $\sigma_{tot}(s)$ over its allowed range we recover Martin's result $\sigma_{inel} < \sigma_{max}/4$ for some sequences of $s \to \infty$ mentioned before.For applications to high energy phenomenology, it is desirable to remove the unknown scale factor s_0 in these bounds, as well as the restriction to special sequences of $s \to \infty$. One way forward is to derive bounds on energy averages of $\sigma_{inel}(s)$ given energy averages of $\sigma_{tot}(s)$ and $A(s, t_0)$. One of us now has definitive results on the analogous problem of finding bounds on energy averages of the inelastic cross-section, as well as of the total cross-section [15].

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where, $t = t_0$ is the lowest singularity in the t-channel. For many physically interesting cases such as $\pi\pi, KK, K\overline{K}, \pi K, \pi N, \pi\Lambda$ scattering $t_0 = 4m_\pi^2 - \epsilon$, ϵ being an arbitrary small positive constant, and m_π the pion-mass [4]. In some cases we can take $\epsilon = 0$, e.g. for pion-pion scattering if the D-wave scattering length is finite [5]. It will be convenient to denote the right-hand side of the bound on $\sigma_{tot(s)}$ as

$$\sigma_{max}(s) = 4\pi/t_0\,[\ln(s/s_0)]^2. \tag{3}$$

In equation (3) s_0 is unknown. However, if one assumes that the total and elastic cross-sections are increasing beyond a certain energy, or if one works with cross-sections averaged over a certain energy interval, one can, using fixed t dispersion relations, fix the scale [6]. A reasonable guess is that s_0 lies between the square of the pion mass and the square of the nucleon mass. This means an uncertainty of ± 10% at the present energy of the LHC. The Froissart-Martin bound has been seminal both to the development of the field of high energy theorems in axiomatic field theory (see e.g. the review [7])and to that of phenomenological models leading to accurate predictions of total and elastic cross sections before their experimental measurements [8]. Remarkably, one of us (A. M.) has recently obtained a bound on the total inelastic cross section at high energy [9],

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The present paper is inspired by Martin's bound on the inelastic cross-section. In fact T. T. Wu [10] by extending Martin's variational calculation to incorporate a given total cross-section and independently S.M .Roy and Virendra Singh [11], by exploiting their previous upper bound on the differential cross section in terms of elastic cross-section, [12],[13] realized that one could solve a more general problem: find a bound on the inelastic cross-section as a function of the value of the total cross-section. It is obvious that if the total cross section vanishes the inelastic cross section also vanishes. but it is also extremely plausible that if one maximizes the total cross section, the important partial wave amplitudes will be imaginary and maximal so that, from the unitarity condition, there is no room left for the inelastic cross section which will receive only negligible contributions from the tail of the partial wave distribution.

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$$\geq \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(Im a_l(s))^2 \equiv \sigma_{el,im}(s). \tag{16}$$

So, it suffices to find a variational lower bound on $\sigma_{el,im}$. We vary the $Im a_l(s)$ subject to the unitarity constraints

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$$A(s,t_0) \equiv \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)Im a_l(s)P_l(1+t_0/(2k^2))$$
$$< Const.\frac{s^2}{\ln(s/s_0)}\ . \tag{19}$$

For simplicity, since we work at a fixed-s , we suppress the s-dependence of $Im a_l(s)$, $\sigma_{el,im}(s)$ and $\sigma_{tot}(s)$. Denoting ,

$$z_0 = 1 + t_0/(2k^2), \tag{20}$$

the lower bound on $\sigma_{el,im}$ is obtained by choosing,

$$Im a_l = \alpha\big(1 - P_l(z_0)/P_{L+r}(z_0)\big), for\ 0 \leq l \leq L\ , \tag{21}$$

and,

$$Im a_l = 0, for\ l > L\ , \tag{22}$$

with the constants $0 \leq r < 1, \alpha > 0$ and the positive integer L being fixed from the given value of σ_{tot} and the given upper bound on $A(s,t_0)$. We omit the straight forward proof which is by direct subtraction of a $\sigma_{el,im}$ with arbitrary partial waves obeying the given constraints from the variational result. After carrying out the summations over l, the constraint equations become,

$$\sigma_{tot}\, k^2/(4\pi\alpha) = (L+1)^2 - (P'_{L+1}(z_0) + P'_L(z_0))/P_{L+r}(z_0)\,, \quad (23)$$

and

$$A(s,t_0)\frac{k}{\alpha\sqrt{s}} = P'_{L+1}(z_0) + P'_L(z_0) - \frac{(L+1)^2 P_L^2(z_0) - (z_0^2-1)(P'_L(z_0))^2}{P_{L+r}(z_0)}, \quad (24)$$

and the bound on σ_{el} becomes,

$$k^2/(4\pi\alpha^2)\,\sigma_{el} \geq k^2/(4\pi\alpha^2)\,\sigma_{el,im} \geq (L+1)^2 - 2(P'_{L+1}(z_0) + P'_L(z_0))/P_{L+r}(z_0) + \frac{(L+1)^2 P_L^2(z_0) - (z_0^2-1)(P'_L(z_0))^2}{(P_{L+r}(z_0))^2}. \quad (25)$$

At high energies, using $s/\sigma_{tot} \to \infty$, the two constraint equations yield easily that $L = O(\sqrt{s}\ln(s/s_0))$; we may therefore set $r = 0$ and use the following approximations for the Legendre polynomials,

$$\begin{aligned} P_L(z_0) &= I_0(\xi)(1+O(L/s)), \xi \equiv (2L+1)\sqrt{(z_0-1)/2} \\ P'_L(z_0) &= (1/2)L\sqrt{s/t_0}I_1(\xi)(1+O(L/s)), \\ I_\nu(\xi) &= \frac{\exp\xi}{\sqrt{2\pi\xi}}(1-(4\nu^2-1)/(8\xi)+...), \xi \to \infty, \\ for \quad & s \to \infty,\ L/\sqrt{s} \to \infty,\ L/s \to 0\,, \end{aligned} \quad (26)$$

where the $I_\nu(\xi)$ denote the modified Bessel functions. We then have,

$$\sigma_{tot}\, k^2/(4\pi\alpha) \approx L^2 - \frac{I_1(\xi)}{I_0(\xi)}L\sqrt{s/t_0} \quad (27)$$

$$\approx L^2(1+O(\sqrt{s}/L)), \quad (28)$$

$$A(s,t_0)\frac{k}{\alpha\sqrt{s}} \approx I_1(\xi)L\sqrt{s/t_0} + L^2\frac{I_1(\xi)^2 - I_0(\xi)^2}{I_0(\xi)} \quad (29)$$

$$\approx I_0(\xi)\frac{L\sqrt{s}}{2\sqrt{t_0}}(1+O(\sqrt{s}/L)). \quad (30)$$

The asymptotic bounds on elastic and inelastic cross-sections become, with these approximations,

$$k^2/(4\pi\alpha^2)\,\sigma_{el} \geq L^2 - 2L\sqrt{s/t_0}\frac{I_1(\xi)}{I_0(\xi)} + L^2(1-(\frac{I_1(\xi)}{I_0(\xi)})^2), \quad (31)$$

and

$$k^2/(4\pi\alpha)\,\sigma_{inel} \leq (1-2\alpha))(L^2 - L\sqrt{s/t_0}\frac{I_1(\xi)}{I_0(\xi)}) + \alpha L^2(\frac{I_1(\xi)}{I_0(\xi)})^2. \quad (32)$$

We now use the assumed upper bound on $A(s,t_0)$ to evaluate L, α for high energies. We have,

$$Const.s/(\sigma_{tot}\ln(s/s_0)) = I_0(\xi)\frac{\sqrt{s}}{2L\sqrt{t_0}}(1+O(\sqrt{s}/L)), \quad (33)$$

which yields ,

$$\frac{L}{\sqrt{s}} = (1/(2\sqrt{t_0}))\ln(\frac{s}{s_0^2\sigma_{tot}})(1+O(\ln(s/s_0))^{-1}) \quad (34)$$

$$\alpha = \frac{\sigma_{tot}(s)}{\hat{\sigma}_{tot}(s)}(1+O(\ln(s/s_0))^{-1}) \quad (35)$$

where,

$$\hat{\sigma}_{tot}(s) \equiv 4\pi/t_0\,[\ln(\frac{s}{s_0^2\sigma_{tot}})]^2. \quad (36)$$

Hence, we have the lower bound on elastic cross-sections,

$$\sigma_{el}(s) \geq \frac{(\sigma_{tot}(s))^2}{\hat{\sigma}_{tot}(s)}(1+O(\ln(s/s_0))^{-1}). \quad (37)$$

Note that $\hat{\sigma}_{tot}(s)$ can be replaced by $\sigma_{max}(s)$ for $s \to \infty$, except in the unrealistic case $\sigma_{tot} \to 0, for s \to \infty$ which leads to a small inelastic cross-section $\sigma_{inel}(s) \to 0$. Hence, using equation (37), the upper bound on the inelastic cross-section valid in all cases is ,

$$\sigma_{inel}(s) \leq_{s\to\infty} \sigma_{tot}(s)\big(1 - \Sigma_{tot}(s)\big), \quad (38)$$

which leads to the announced bound on $\sigma_{inel}(s)/\sigma_{max}(s)$, given by equation (5).

4. Upper Bound on Inelastic Cross-section from an Upper Bound on Differential Cross-section in terms of Elastic Cross-section

We show here that the inelastic cross-section bound can also be derived as a corollary of an upper bound on the differential cross section in terms of the elastic cross-section, established by two of us [12] many years ago,

$$\frac{d\sigma}{dt}(s,t=0) \leq_{s\to\infty} \frac{\sigma_{el}(s)}{4t_0}[\ln(\frac{s}{s_0^2\sigma_{el}})]^2. \quad (39)$$

This bound can also be written as [13],

$$\sigma_{tot}[1+(\frac{ReF(s,t=0)}{ImF(s,t=0)})^2] \leq_{s\to\infty} \frac{4\pi\sigma_{el}}{t_0\sigma_{tot}}[\ln(\frac{s}{s_0^2\sigma_{el}})]^2. \quad (40)$$

If the real part $ReF(s,t=0)$ is unknown we have the weaker bound,

$$\sigma_{tot} \leq_{s\to\infty} \sqrt{\frac{4\pi}{t_0}}\big[\sqrt{\sigma_{el}}\big(\ln(\frac{s}{s_0^2\sigma_{tot}}) - \ln(\frac{\sigma el}{\sigma_{tot}})\big)\big]$$
$$\leq_{s\to\infty} \sqrt{\sigma_{el}\hat{\sigma}_{tot}} + (2/e)\sqrt{\frac{4\pi\sigma_{tot}}{t_0}}\,, \quad (41)$$

where, in the last line we have used the elementary inequality,$\sqrt{x}\ln x \geq -2/e, for\ 0 < x < 1$. This equation yields a lower bound on $\sigma_{el}(s)$ for any asymptotic behaviour of $\sigma_{tot}(s)$. As noted in the last section,for deducing an upper bound on $\sigma_{inel}(s)$, it suffices to assume that σ_{tot} does not vanish for $s\to\infty$. In particular, if $\sigma_{tot}(s) > 16\pi/(e^2 t_0)$, we have

$$\sigma_{tot}(\sqrt{\sigma_{tot}} - (2/e)\sqrt{\frac{4\pi}{t_0}}\,)^2 \leq_{s\to\infty} \sigma_{el}\hat{\sigma}_{tot}$$
$$\approx_{s\to\infty} \sigma_{el}\sigma_{max} \quad (42)$$

and hence the upper bound on the inelastic cross-section,

$$\sigma_{inel} \leq_{s\to\infty} \sigma_{tot}\big[1 - \Sigma_{tot}(1 - (2/e)\sqrt{\frac{4\pi}{t_0\sigma_{tot}}}\,)^2\big], \quad (43)$$

which yields the desired bound (5) on the inelastic cross-section if $\sigma_{tot}(s)\to\infty, for\ s\to\infty$.

5. Conclusion

We have derived an asymptotic upper bound on the inelastic cross-section in terms of the total cross-section which improves Martin's recent bound [9] when $\sigma_{tot}(s) \sim C(\ln(s/s_0))^2$. Varying $\sigma_{tot}(s)$ over its allowed range we recover Martin's result $\sigma_{inel} < \sigma_{max}/4$ for some sequences of $s\to\infty$ mentioned before.For applications to high energy phenomenology, it is desirable to remove the unknown scale factor s_0 in these bounds, as well as the restriction to special sequences of $s\to\infty$. One way forward is to derive bounds on energy averages of $\sigma_{inel}(s)$ given energy averages of $\sigma_{tot}(s)$ and $A(s,t_0)$. One of us now has definitive results on the analogous problem of finding bounds on energy averages of the inelastic cross-section, as well as of the total cross-section [16].

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