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Principles of maximally classical and maximally realistic quantum mechanics

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Abstract. Recently Auberson, Mahoux, Roy and Singh have proved a long standing conjecture of Roy and Singh: In $2N$-dimensional phase space, a maximally realistic quantum mechanics can have quantum probabilities of no more than $N+1$ complete commuting cets (CCS) of observables coexisting as marginals of one positive phase space density. Here I formulate a stationary principle which gives a nonperturbative definition of a maximally classical as well as maximally realistic phase space density. I show that the maximally classical trajectories are in fact exactly classical in the simple examples of coherent states and bound states of an oscillator and Gaussian free particle states. In contrast, it is known that the de Broglie–Bohm realistic theory gives highly nonclassical trajectories.

Keywords. Maximally realistic quantum theory; phase space Bell inequalities; maximally classical trajectories in realistic quantum theory.

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1. Realistic quantum mechanics of de Broglie and Bohm

A definition of 'physical reality' useful in discussing the nature of quantum reality was given by Einstein, Podolsky and Rosen [1]. A physical quantity has reality in a given state, if its value can be predicted with certainty without disturbing the state. In ordinary quantum mechanics the state is supposed to be completely specified by the state vector. Hence any observable of which the state is not an eigenstate has no physical reality: the state only specifies the probabilities of obtaining particular values in an experiment to measure that observable. Since it is assumed that noncommuting observables are not measurable in the same context, even this probabilistic description is inherently context dependent. For example,

$$|\psi(\vec{x},t)|^2 \mathrm{d}\vec{x}$$

is the probability of 'observing' position to be in $\mathrm{d}\vec{x}$ if position were measured. It is not the probability of position 'being' in $\mathrm{d}\vec{x}$ independent of observation. In fact, the same state vector also yields

$$|\tilde{\psi}(\vec{p},t)|^2 \mathrm{d}\vec{p}$$

as the probability of observing momentum to be in the interval $\mathrm{d}\vec{p}$ in a different experiment where momentum is measured. But the joint probability for position and momentum is not specified.

The earliest and best known realistic quantum theory is due to De Broglie and Bohm [2] (dBB) in which position always has physical reality. The state vector supplemented by the instantaneous position is the complete description of the state of the system. Here $\vec{x} = (\vec{x}_1, \ldots, \vec{x}_N)$ denotes the configuration space coordinate which evolves according to

$$\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = \frac{1}{m_i}\vec{\nabla}_i S(\vec{x}(t), t), \tag{3}$$

where m_i denotes the mass of particle i, and the Schrödinger wave function is given by

$$\langle \vec{x} | \psi(t) \rangle \equiv R \exp(iS), \tag{4}$$

with R and S real functions of $(\vec{x}, t)$. dBB show that if we start at $t = 0$ with an ensemble of particles whose position density coincides with $|\psi(\vec{x}, 0)|^2$ at $t = 0$, and evolves with time according to (1), then the position density coincides with $|\psi(\vec{x}, t)|^2$ at any arbitrary time t. Thus, the phase space density is

$$\rho_{\mathrm{dBB}}(\vec{x}, \vec{p}, t) = |\psi(\vec{x}, t)|^2 \delta\left(\vec{p} - \vec{\nabla} S(\vec{x}, t)\right) \tag{5}$$

whose marginal (i.e., integral over momentum) reproduces the position probability density at arbitrary time

$$\int \rho_{\mathrm{dBB}}(\vec{x}, \vec{p}, t)\, \mathrm{d}\vec{p} = |\psi(\vec{x}, t)|^2. \tag{6}$$

The momentum and other variables besides position, however, do not have the same favoured status as position. As Takabayasi [3] pointed out the dBB phase space density does not yield the correct quantum momentum density, i.e.,

$$\int \rho_{\mathrm{dBB}}(\vec{x}, \vec{p}, t) \mathrm{d}\vec{x} \neq |\tilde{\psi}(\vec{p}, t)|^2.$$

To overcome this problem dBB introduce a measurement interaction whose purpose is to convert the pre-existing momentum prior to measurement into one whose distribution agrees with the quantum distribution. In contrast, for position, the value observed is the same as the pre-existing value. 'Momentum' therefore has not the same reality as 'position'. A second problem that has received much attention is that the dBB trajectories are so different from the classical trajectories that they are sometimes called surrealistic [4]!

The answer to the first problem above is in the construction of maximally realistic quantum mechanics [5,6] which treats position and momentum symmetrically. We present in this paper the answer to the second problem in the form of a stationary principle. It provides a nonperturbative criterion to choose a 'maximally classical' quantum mechanics from the infinite set of 'maximally realistic' ones.

2. Maximally realistic quantum mechanics

The earliest attempt at a phase space description of quantum mechanics symmetric in position and momentum is due to Wigner [7]. Appropriate marginals of the Wigner distribution function $W(\vec{x},\vec{p})$ reproduce the correct quantum probability densities for position and momentum. However, there exist states for which the Wigner function is not positive over all phase space. Hence the Wigner function cannot be given a probability interpretation. However, Cohen *et al* [8] showed that there exist positive phase space densities $\rho(\vec{x},\vec{p})$ whose marginals reproduce the quantum position and momentum probability densities, and constructed the most general densities with these properties. In a series of investigations, Roy and Singh [5,6] went much further; they constructed 'maximally realistic' causal quantum mechanics in which a single positive $\rho(\vec{x},\vec{p})$ has marginals which reproduce not just the quantum position and momentum probability densities but also the quantum probability densities for $(N-1)$ other complete commuting sets (CCS) of observables, where N is the dimension of the configuration space. Further, they conjectured that $N+1$ is the maximum number of CCS whose probability densities can be reproduced as marginals of a single positive $\rho(\vec{x},\vec{p})$. Recently, Auberson *et al* [9,10] have proved this conjecture and also constructed the most general phase space density with these properties. Recall first the one-dimensional construction of Roy and Singh,

$$\rho(x,p,t) = |\psi(x,t)|^2\,|\tilde{\psi}(p,t)|^2\delta\left(\int_{-\infty}^{p}\mathrm{d}p'|\tilde{\psi}(p',t)|^2 - \int_{-\infty}^{x}\mathrm{d}x'|\psi(x',t)|^2\right). \tag{5}$$

Its positivity is obvious and the integrals over position and momentum can be readily verified to reproduce respectively the quantum momentum and position probability densities. For $N=2$, Roy and Singh construct a causal quantum mechanics in which the quantum probability densities corresponding to three different CCS of observables, e.g., $(X_1,X_2),(P_1,X_2),(P_1,P_2)$ are simultaneously realized. Explicitly, the positive phase space density

$$\rho(\vec{x},\vec{p},t) = |\psi(x_1,x_2,t)|^2\,|\psi(p_1,x_2,t)|^2|\psi(p_1,p_2,t)|^2\delta(A_1)\delta(A_2), \tag{6}$$

where

$$A_1 \equiv \int_{-\infty}^{p_1}|\psi(p_1',x_2,t)|^2\mathrm{d}p_1' - \int_{-\infty}^{x_1}|\psi(x_1',x_2,t)|^2\mathrm{d}x_1', \tag{7}$$

$$A_2 \equiv \int_{-\infty}^{p_2}|\psi(p_1,p_2',t)|^2\mathrm{d}p_2' - \int_{-\infty}^{x_2}|\psi(p_1,x_2',t)|^2\mathrm{d}x_2', \tag{8}$$

reproduces as marginals the correct quantum probability densities $|\psi(x_1,x_2,t)|^2$, $|\psi(p_1,x_2,t)|^2$ and $|\psi(p_1,p_2,t)|^2$. (For notational simplicity we have omitted the tildas denoting Fourier transforms).

The maximally realistic phase space densities given above are by no means unique. The positivity conditions and the marginal conditions allow an infinity of phase space densities. We need a new physical principle to choose a phase space density candidate for a realistic theory. We shall propose here the principle of 'maximal classicality' to be the new principle.

3. Construction of maximally classical and maximally realistic quantum mechanics

I propose a maximal classicality criterion, nonperturbative in Planck's constant. I start from the most general maximally realistic positive phase space density constructed by

Auberson *et al* [9,10] with $N+1$ marginals coinciding with quantum probability densities. The strategy is to determine the unknown function appearing in this construction so as to obtain a phase space density as close to classical as possible. Here I illustrate the procedure for $N=1$ choosing the two marginal conditions to be obeyed as

$$\int \rho(x,p,t)\mathrm{d}x = |\tilde{\psi}(p,t)|^2, \tag{9}$$

$$\int \rho(x,p,t)\mathrm{d}p = |\psi(x,t)|^2. \tag{10}$$

The most general positive phase space density obeying these conditions is [9,10]

$$\rho(x,p,t) = \rho_0(x,p,t)\,(1+\lambda(t)h(x,p,t)), \tag{11}$$

where $\rho_0(x,p,t)$ is a particular density obeying the marginal conditions

$$\rho_0(x,p,t) = |\psi(x,t)|^2|\tilde{\psi}(p,t)|^2, \tag{12}$$

and $h(x,p,t)$ is the most general function such that $\rho_0(x,p,t)h(x,p,t)$ has zero integral over x and p. This last condition implies that there exist nonnegative $a(t)$ and $b(t)$ such that

$$-a(t) \le h(x,p,t) \le b(t).$$

It also implies that h must be of the form [9,10]

$$h(x,p,t) = ((1-K)f)(x,p,t), \tag{13}$$

$$\begin{aligned} Kf(x,p,t) = &\int \mathrm{d}p|\tilde{\psi}(p,t)|^2 f(x,p,t) \\ &+\int \mathrm{d}x|\psi(x,t)|^2 f(x,p,t) \\ &-\int\int \mathrm{d}x\mathrm{d}p\,\rho_0(x,p,t)f(x,p,t), \end{aligned} \tag{14}$$

where f is an arbitrary function. The positivity of $\rho(x,p,t)$ is ensured by a choice of $\lambda(t)$ such that

$$-1/b(t) \le \lambda(t) \le 1/a(t).$$

Note that Kf is a sum of a function of (x,t) and another function of (p,t) and that

$$Kf = K^2 f. \tag{15}$$

This implies that the maximal reality conditions (i.e., the marginal conditons) on $\rho(x,p,t)$ are equivalent to the simple condition

$$Kh = 0. \tag{16}$$

Suppose that at $t=0$ we have some ensemble of phase space points consistent with the quantum marginal conditions, with $\lambda(0)=1$ and $h=(1-K)f$ with some $f(x,p,0)$. Evolving these phase space points with the classical Hamiltonian equations gives $\rho_{\mathrm{cl}}(x,p,t)$. Let us define

$$\rho_{\rm cl}(x,p,t) = \rho_0(x,p,t)\left(1+h_{\rm cl}(x,p,t)\right), \tag{17}$$

and so exact classicality for arbitrary t would mean

$$(h-h_{\rm cl})(x,p,t) = 0, \tag{18}$$

$$\lambda(t) = 1. \tag{19}$$

While these equations hold at $t=0$ by definition, in general we expect them to break down at nonzero time since classical motion may not ensure agreement with quantum marginal conditions. Instead we define maximal classicality to mean that $h-h_{\rm cl}$ is just a sum of a function of (x,t) and a function of (p,t), and $\lambda(t)$ is as close to unity as is allowed by the positivity of ρ. That is,

$$(1-K)(h-h_{\rm cl}) = 0, \tag{20}$$

$$\lambda(t) = \min(1/a(t),1). \tag{21}$$

The two equations for h coming from maximal reality and maximal classicality are solutions of the single variational principle

$$\delta\int\int \mathrm{d}x\mathrm{d}p\ \rho_0((Kh)^2+((1-K)(h-h_{\rm cl}))^2) = 0. \tag{22}$$

The explicit solution for h is

$$h = (1-K)h_{\rm cl}. \tag{23}$$

Thus the maximally classical and maximally realistic phase space density is now completely determined. We work out simple examples in the next section.

4. Explicit examples of maximally classical and maximally realistic phase space densities

(i) *Coherent state and ground state of harmonic oscillator*. For a coherent state of an oscillator of mass m and frequency ω, a simple calculation yields

$$\begin{aligned}\rho_0(x,p,t) = {}& (1/(\pi\hbar))\exp[-2a(x-A_0\cos(\omega t+\phi_0))^2 \\ & -(1/(2a\hbar^2)(-m\omega A_0\sin(\omega t+\phi_0)-p)^2],\end{aligned} \tag{24}$$

where A_0 and ϕ_0 are the amplitude and phase parameters for the coherent state. The ground state corresponds to $A_0=0$. For the simple initial condition

$$\rho(x,p,0) = \rho_{\rm cl}(x,p,0) = \rho_0(x,p,0),$$

a straightforward calculation yields $h=h_{\rm cl}=0$ and

$$\rho(x,p,t) = \rho_{\rm cl}(x,p,t) = \rho_0(x,p,t).$$

That is, the maximally classical phase space density is in fact exactly classical, consistent with exactly classical trajectories! This holds also for the ground state. In contrast, the de Broglie–Bohm theory gives

$$p_{\text{dBB}} = -m\omega A_0 \sin(\omega t + \phi_0),$$

which corresponds to classical motion only for the centre of the wave packet, $x = A_0 \cos(\omega t + \phi_0)$. For the ground state $p_{\text{dBB}} = 0$!

(ii) *Gaussian states of a free particle.* Again, $\rho_0(x,p,t)$ is readily calculated and we start from the simple initial condition

$$\rho(x,p,0) = \rho_{\text{cl}}(x,p,0) = \rho_0(x,p,0).$$

A simple calculation yields ρ_{cl}, and thence,

$$h_{\text{cl}} = -1 + \sigma(t)\exp[-\alpha(x - pt/m)^2 + (\alpha/\sigma^2(t))(x - \beta t/m)^2], \tag{25}$$

where

$$\sigma(t) = \sqrt{1 + (\alpha\hbar t/m)^2}.$$

Though h_{cl} is non-zero, we obtain $Kh_{\text{cl}} = 0$, and hence

$$\rho(x,p,t) = \rho_{\text{cl}}(x,p,t),$$

consistent with exactly classical trajectories. In contrast, the dBB theory gives momenta depending on time (except for the centre of the wave packet $x = pt/m$),

$$p = \beta(1 + (xt/(\beta m))(\alpha\hbar)^2)/\sigma^2(t).$$

It will be interesting to compute the maximally classical phase space densities and trajectories in the examples where the dBB trajectories have been dubbed ‘surrealistic’ [4].

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