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Stochastic motion of a charged particle in a magnetic field: I Classical treatment

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Abstract. We study the dissipative, classical dynamics of a charged particle in the presence of a magnetic field. Two stochastic models are employed, and a comparative analysis is made, one based on diffusion processes and the other on jump processes. In the literature on collision-broadening of spectral lines, these processes go under the epithet of weak-collision model and Boltzmann–Lorentz model, respectively. We apply our model calculation to investigate the effect of magnetic field on the collision-broadened spectral lines, when the emitter carries an electrical charge. The spectral lines show narrowing as the magnetic field is increased, the narrowing being sharper in the Boltzmann–Lorentz model than in the weak collision model.

Keywords. Magnetic field-induced dynamics; weak collision-model; Boltzmann–Lorentz model; spectral lineshape; dissipative dynamics.

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1. Introduction

The problem of dissipative dynamics of a charged particle in the presence of a magnetic field pervades several areas of basic physics such as classical mechanics, electromagnetic theory and statistical physics. It also has ramifications in plasma physics [1], solid state physics, in particular, diamagnetism [2], and as we shall discuss here, in a branch of optical spectroscopy called "collision broadening" [3]. In its most elementary form, the problem involves reversible dynamics due to the Lorentz force, and its interplay with dissipative effects arising from "collisions". The term is put under quotes to imply that its usage is meant to be understood in a generalized sense to describe either real collisions between the test charge with other foreign particles or effects of interaction with other degrees of freedom, eg. phonons in solids.

Whatever be the source of collisions, its consequence is to make the velocity $\mathbf{v}(t)$ of the charged particle a stochastic process. In the present work, we shall assume this process to be a classical one and that it is a stationary Markov process [4]. Needless to say, the time-integral of $\mathbf{v}(t)$ i.e. the position vector $\mathbf{r}(t)$ is also a stochastic process (albeit a non-stationary one) which can be completely specified by the so-called characteristic function

$$\phi(\mathbf{k}, t) = \langle \exp(i\mathbf{k}\cdot\mathbf{r}(t)) \rangle, \tag{1}$$

where $\mathbf{k}$ is an arbitrary vector and the angular brackets denote the average over the probability function which defines the underlying stochastic process.

It is interesting to note that although the quantity $\phi(t)$ in (1) is introduced as an entirely mathematical object, it has the physical interpretation of the 'phase' of the electromagnetic radiation (of wave vector $\mathbf{k}$) from an emitter whose instantaneous position is $\mathbf{r}(t)$. Indeed, the frequency-Fourier transform of $\phi(t)$ yields the optical lineshape for velocity modulation in gas-phase spectroscopy [5]. This can be readily seen by writing (1) as

$$\phi(\mathbf{k},t) = \left\langle \exp i\mathbf{k}\cdot \int_0^t \mathbf{v}(t')\mathrm{d}t' \right\rangle, \tag{2}$$

if we assume that $\mathbf{v}(0) = 0$. Note that if the velocity $\mathbf{v}(t)$ were constant in time, the integral in eq. (2) would simply yield $\mathbf{v}t$. We may then perform the average (indicated by the angular brackets), over a stationary Maxwellian distribution of $\mathbf{v}$, thereby yielding a Gaussian in t. The Fourier-transform $\tilde{\phi}(\omega)$, in turn, is also a Gaussian in ω with a width proportional to $T^{1/2}$, T being the temperature characterizing the underlying velocity distribution. Therefore, the spectral line undergoes broadening as temperature increases, which is usually referred to as the Doppler broadening [3]. However, if the emitter suffers *velocity-changing collisions*, its 'effective' velocity is reduced, leading to a narrowing of the Doppler broadened line. This phenomenon is akin to the 'motional narrowing' effect in magnetic resonance [6]. One of our aims in the present paper is to investigate what influence, if any, does an external magnetic field have on the motional narrowing, when the emitter carries an electrical charge.

It is well-known that a stationary Markov process such as $\mathbf{v}(t)$ has an underlying probability function $P(\mathbf{v},t)$ that obeys the following master equation

$$\frac{\partial P(\mathbf{v},t)}{\partial t} = \int \mathrm{d}\mathbf{v}' [P(\mathbf{v}',t) W(\mathbf{v}'|\mathbf{v}) - P(\mathbf{v},t) W(\mathbf{v}|\mathbf{v}')], \tag{3}$$

where $W(\mathbf{v}'|\mathbf{v})$ denotes the probability per unit time that $\mathbf{v}$ jumps (instantaneously) from $\mathbf{v}'$ to $\mathbf{v}$ [7]. The general solution for $P(\mathbf{v},t)$ is not available in an operationally useful form, except in the following two extreme situations:

(i) *The diffusion process.* In this case the velocity $\mathbf{v}(t)$ is assumed to describe Brownian motion such that the effect of a collision is to alter the velocity of the particle by a 'small' amount. Mathematically, $W(\mathbf{v}'|\mathbf{v})$ can be approximated by a second order Kramers–Moyal expansion (in velocity moments) yielding a Fokker–Planck equation for $P(\mathbf{v},t)$ [4, 5]. In the literature on collision broadening, such a process goes by the name of weak collision model (WCM), and we shall, henceforth employ this nomenclature for describing the diffusion process. In the sequel, we shall use $P(\mathbf{v},t)$ in the presence of a magnetic field to investigate the characteristic function $\phi(t)$, and from that, compute the spectral line shape $\tilde{\phi}(\omega)$.

(ii) *The Boltzmann–Lorentz model.* The BLM is a 'strong collision model', in contrast to the WCM, wherein the collisions are viewed to alter the direction of the velocity vector by arbitrary angles, keeping the magnitude fixed [7, 8]. Unlike the WCM, the stochastic process $\mathbf{v}(t)$ is now a jump process, and is applicable to a situation of gas-phase spectroscopy in which the emitter is a small particle that suffers collisions with much heavier buffer-gas particles [5, 8]. In yet another interpretation, borrowed from the classical kinetic theory, the BLM describes scattering of the test particle from frozen-in scatterers distributed randomly in space. The BLM, like the WCM, enables one to obtain a closed-form solution of the master equation (3), which can then be used to calculate $\phi(t)$. We should emphasize that in the WCM, the particle is continually under the influence of dissipative terms. On the other hand, in the BLM, the trajectory of the particle evolves freely, albeit under a magnetic field, until this evolution is disrupted by the next collision event.

The plan of the paper is to investigate the motion of a charged particle in the presence of a magnetic field in both the WCM and the BLM and make a comparative study of the velocity-correlation, the characteristic function and the spectral line shape. The idea is to explore whether the magnetic field, like temperature or pressure, can be used as a control parameter in the study of collision broadening of spectra. The paper is organized as follows. In § 2, we introduce the WCM and the BLM for describing stochastic motion of a charged particle in the presence of a magnetic field, and make a special reference to the respective velocity auto correlation functions. The results for the characteristic function $\phi(t)$ and the line shape function $\tilde{\phi}(\omega)$ for both the WCM and the BLM are presented in § 3. We discuss in the concluding § 4, how the magnetic field can be used to tune the motional narrowing effect, and also offer some comments on possible quantum generalizations of our results.

2. Two distinct models for stochastic dynamics

a. *The weak collision model.* As introduced in § 1, the weak collision model (WCM) describes the diffusion process or the Brownian motion of a tagged particle in a medium (heatbath). Here we analyze the motion when the particle is charged and is under the influence of an external magnetic field. Most of the results are well known in the literature [9, 10], but we still summarize them in order to make a meaningful comparison, later, with the results derived in the Boltzmann–Lorentz model (BLM).

The WCM is characterized by a Langevin equation for a Brownian particle of mass m and charge q in a fluid at temperature T, under the influence of a uniform magnetic field $\mathbf{B}$:

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{v}\times\mathbf{B}) - m\gamma\mathbf{v} + \mathbf{f}(t), \tag{4}$$

where γ is the friction coefficient and $\mathbf{f}(t)$ is a stationary Gaussian white noise with zero mean and correlation given by

$$\langle f_i(t) f_j(t')\rangle = \Gamma\delta_{ij}\delta(t-t'), \tag{5}$$

where the indices i and j denote cartesian co-ordinates and Γ is a positive constant which is related to the friction coefficient γ by the so-called fluctuation-dissipation relationship

$$\Gamma = 2m\gamma k_{\mathrm{B}}T. \tag{6}$$

Equation (6) ensures that $\mathbf{v}(t)$, and its moments and correlations starting from the respective initial values, approach equilibrium, asymptotically (as $t\to\infty$), governed by a Maxwellian distribution at temperature T.

While the Langevin approach is based on the equation of motion of a dynamical variable, in the present instance the velocity $\mathbf{v}(t)$, a completely equivalent picture is provided by the Fokker–Planck equation for the function $P(\mathbf{v}, t)$ which defines the conditional probability that the velocity is $\mathbf{v}(t)$ at time t, given that the velocity is $\mathbf{v}_0$ at $t=0$. This equation reads [10]

$$\frac{\partial P(\mathbf{v},t)}{\partial t}=\gamma\nabla_{\mathbf{v}}(\mathbf{v}P(\mathbf{v},t))-\frac{q}{m}(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}}P(\mathbf{v},t)+\frac{\Gamma}{2m^2}\nabla_{\mathbf{v}}^2P(\mathbf{v},t) \tag{7}$$

with

$$P(\mathbf{v},t=0)=\delta(\mathbf{v}-\mathbf{v}_0). \tag{8}$$

As mentioned earlier, eq. (7) for $\mathbf{B}=0$ is a limiting case of the integro-differential equation (3). It is straightforward to show, from either the Langevin equation (4) or the Fokker–Planck equation (7) that the correlation function of the velocity is [10]

$$\langle v_i(0)v_j(t)\rangle=\left(\frac{k_{\mathrm{B}}T}{m}\right)\mathrm{e}^{-\gamma|t|}[\delta_{ij}\cos\omega_{\mathrm{c}}t-\varepsilon_{ijl}b_l\sin\omega_{\mathrm{c}}t+b_ib_j(1-\cos\omega_{\mathrm{c}}t)], \tag{9}$$

where ω_{c} is the so-called cyclotron frequency defined as

$$\omega_{\mathrm{c}}=qB/m(B=|\mathbf{B}|), \tag{10}$$

$b_k=B_k/B$ are the direction cosines of the magnetic field $\mathbf{B}$ and ε_{ijl} is the fully antisymmetric tensor of rank 3. A special case of eq. (9) is the auto-correlation function

$$C(t)\equiv\langle\mathbf{v}(0)\cdot\mathbf{v}(t)\rangle=\left(\frac{k_{\mathrm{B}}T}{m}\right)\mathrm{e}^{-\gamma|t|}(1+2\cos\omega_{\mathrm{c}}t). \tag{11}$$

b. *The Boltzmann–Lorentz model.* In this approach, the velocity $\mathbf{v}(t)$ is taken to be a jump process; its magnitude is assumed to remain unaltered due to collisions – only the direction changes at random [8]. We may therefore view the velocity vector as a matrix $\mathbf{V}$ in the 'stochastic' space spanned by the states (Ω), where the set $\{\Omega\}$ specifies the Euler angles of the orientation of the velocity. In this space, the matrix $\mathbf{V}$ is diagonal with its elements being given by the possible values of the velocity. Thus, following the notation of ref. [8]

$$(\Omega|(\mathbf{V})|\Omega_0)=\hat{u}_{\Omega_0}v\delta(\Omega_0-\Omega),. \tag{12}$$

where $\hat{u}_{\Omega_0}$ is the unit vector in the direction of $\mathbf{v}$.

Given an initial velocity at time $t=0$, the velocity at time t is obtained in terms of the average of the time-evolution operator $U(t)$

$$(\Omega|(\mathbf{V}(t))|\Omega_0)=(\Omega|(\langle U(t)\rangle\mathbf{V})|\Omega_0), \tag{13}$$

where the brackets $\langle\cdots\rangle$ denote the average over *only* the statistics of the collisions (not the full average implied in § 1 (cf. eq. (2)). Using (12), the above expression simplifies to

$$(\Omega|(\mathbf{V}(t))|\Omega_0)=v(\Omega|(\langle U(t)\rangle)|\Omega_0)\hat{u}_{\Omega_0}. \tag{14}$$

In the BLM, the collisions are assumed to be Poisson-distributed with a mean rate γ that depends in general on the instantaneous velocity of the particle. The key expression is that of $\langle U(t)\rangle$ in terms of the 'free evolution operator' (i.e., the 'streaming' operator) $U^0(t)$ and the collision operator $\mathbf{J}$. Following ref. [8], this expression is best written in terms of the Laplace transform of $\langle U(t)\rangle$,

$$\langle \tilde{U}(z)\rangle = \sum_{n=0}^{\infty} \tilde{U}^0(z+\gamma(v))[(\gamma\mathbf{J})\tilde{U}^0(z+\gamma)]^n$$
$$= \tilde{U}^0(z+\gamma(v)) + \tilde{U}^0(z+\gamma(v))(\gamma(v)\mathbf{J}).\langle\tilde{U}(z)\rangle. \tag{15}$$

Finally, since the velocity is completely randomized in direction due to collisions, the collision operator $\mathbf{J}$ has the following simple matrix representation

$$(\Omega_0|\mathbf{J}|\Omega) = \frac{1}{4\pi}. \tag{16}$$

Having set up the preliminaries of the BLM, we now make a departure from ref. [8] and consider the expression for the matrix element of the free evolution operator $U^0(t)$, when the particle is charged and is under the influence of a magnetic field. This expression is extracted by first writing down the matrix of the velocity at time t in the collision-free case, from (14)

$$(\Omega|\mathbf{V}(t)|\Omega_0) = v(\Omega|U^0(t)|\Omega_0)\hat{u}_{\Omega_0}. \tag{17}$$

Recalling that the Euler angle Ω_0 in the present case is completely specified by the polar angle θ_0 and the azimuthal angle ϕ_0 (figure 1), the unit vector $\hat{u}_{\Omega_0}$ is

$$\hat{u}_{\Omega_0} = (\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0). \tag{18}$$

Secondly, taking the direction of $\mathbf{B}$ to be the z-axis (cf. figure 1), a direct solution of the equation of motion (i.e., eq. (4) with only the Lorentz term on the right hand side) yields

$$(\Omega|\mathbf{V}(t)|\Omega_0) = v(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\times\, \delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0 + \omega_c t). \tag{19}$$

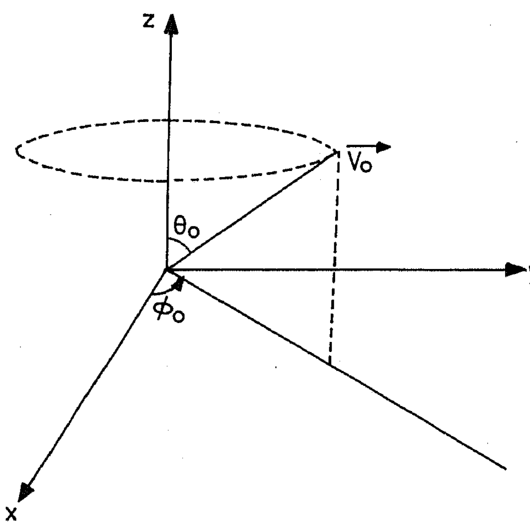

Figure 1. In the absence of collisions, the velocity vector $\mathbf{v}$ precesses around the direction of the magnetic field $\mathbf{B}$, taken along the z-axis. For an initial velocity vector $\mathbf{v}_0 = v\hat{u}_{\Omega_0}$, where $\hat{u}_{\Omega_0} = (\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0)$, the velocity vector after time t is given as $\mathbf{v}(t) = v\hat{u}_{\Omega}(t)$, where $\hat{u}_{\Omega}(t) = (\sin\theta_0\cos\phi(t), \sin\theta_0\sin\phi(t), \cos\theta_0)$ and $\phi(t) = \phi_0 - \omega_c t$.

That is, the velocity vector in the presence of **B** is simply rotated about the z-axis in the clockwise direction by an angle $\omega_c t$, ω_c being the cyclotron frequency. Comparing (19) with (17) and keeping in mind (18), we arrive at

$$(\Omega|U^0(t)|\Omega_0) \equiv (\theta, \phi|(U^0(t))|\theta_0, \phi_0) = \delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0 + \omega_c t). \tag{20}$$

With the machinery of eqs (15), (16) and (20) at hand, we are now ready to evaluate the autocorrelation function $C(t)$. The latter is given by (cf. eq. (11))

$$C(t) = \langle \mathbf{v}(0)\cdot\mathbf{v}(t)\rangle = \langle v^2 \langle \cos\chi(t)\rangle\rangle, \tag{21}$$

where $\chi(t)$ is the angle between the vectors $\mathbf{v}(0)$ and $\mathbf{v}(t)$, and $\langle\cdots\rangle$ denotes the average over the statistics of collisions, for a fixed value of the magnitude v. (The final average over a Maxwellian distribution of the velocity is indicated by double angular brackets.) For calculational convenience, we rewrite (21) with the aid of the spherical harmonics addition theorem [11]

$$C(t) = \langle v^2 S_v(t)\rangle, \tag{22}$$

where

$$S_v(t) = \frac{4\pi}{3}\sum_{m=-1}^{+1} \langle Y_{1m}(\theta(t), \phi(t))\, Y^*_{1m}(\theta_0, \phi_0)\rangle. \tag{23}$$

In terms of the averaged time-evolution operator $\langle U(t)\rangle$, eq. (23) can be further re-expressed as

$$S_v(t) = \frac{4\pi}{3}\sum_{m=-1}^{+1}\frac{1}{4\pi}\int \mathrm{d}\Omega_0\mathrm{d}\Omega\, Y_{1m}(\Omega)(\Omega|\langle U(t)\rangle|\Omega_0)\, Y^*_{1m}(\Omega_0). \tag{24}$$

We turn our attention to the evalution of $S_v(t)$ and in particular its Laplace transform $\tilde{S}_v(z)$ by employing the series solution for $\langle\tilde{U}(z)\rangle$ and the matrix representation of the collision operator **J**, in the BLM (cf. eqs (15) and (16)). Thus

$$\begin{aligned}
\tilde{S}_v(z) &= \frac{1}{3}\sum_{m=-1}^{+1}\Bigg[\int \mathrm{d}\Omega_0\mathrm{d}\Omega\, Y_{1m}(\Omega)(\Omega|\tilde{U}^0(z+\gamma(v))|\Omega_0)\, Y^*_{1m}(\Omega_0) \\
&\quad + \int \mathrm{d}\Omega\mathrm{d}\Omega'\, Y_{1m}(\Omega)(\Omega|\tilde{U}^0(z+\gamma(v))|\Omega') \\
&\quad \times \frac{\gamma(v)}{4\pi}\cdot\sum_{n=0}^{\infty}\left[\frac{\gamma(v)}{4\pi}\int \mathrm{d}\Omega_1\,\mathrm{d}\Omega_2(\Omega_2|\tilde{U}^0(z+\gamma(v))|\Omega_1)\right]^n \\
&\quad \times \int \mathrm{d}\Omega_0\mathrm{d}\Omega_0'(\Omega_0'|\tilde{U}^0(z+\gamma(v))|\Omega_0)\, Y^*_{1m}(\Omega_0)\Bigg] \\
&= \frac{1}{3}\sum_{m=-1}^{+1}\Bigg[\int \mathrm{d}\Omega_0\mathrm{d}\Omega\, Y_{1m}(\Omega)(\Omega|\tilde{U}^0(z+\gamma(v))|\Omega_0)\, Y^*_{1m}(\Omega_0) \\
&\quad + \frac{\gamma(v)}{4\pi}\frac{[\int \mathrm{d}\Omega\mathrm{d}\Omega'\, Y_{1m}(\Omega)(\Omega|\tilde{U}^0(z+\gamma(v))|\Omega')][\int \mathrm{d}\Omega_0\mathrm{d}\Omega_0'\ldots.]}{1-(\gamma(v)/4\pi)\int \mathrm{d}\Omega_1\,\mathrm{d}\Omega_2(\Omega_2|\tilde{U}^0(z+\gamma(v))|\Omega_1)}\Bigg].
\end{aligned} \tag{25}$$

Hence, the special property of the collision operator, indicated in (20), has enabled us to write $S_v(t)$ entirely in terms of the matrix element of the streaming operator: $\tilde{U}^0(z+\gamma(v))$.

In evaluating (25), we use the following definition of the spherical harmonics [11]

$$Y_1^m(\Omega) = \sqrt{\frac{(2l+1)}{4\pi}\cdot\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)e^{im\phi}. \tag{26}$$

It is then straightforward to show from (20) that

$$\int d\Omega d\Omega' Y_{1m}(\Omega)(\Omega|\tilde{U}^0(z+\gamma(v))|\Omega')$$
$$= \int d\Omega_0 d\Omega_0'(\Omega_0'|\tilde{U}^0(z+\gamma(v))|\Omega_0)Y_{1m}^*(\Omega_0) = 0. \tag{27}$$

Thus the numerator in the second term on the right of eq. (25) vanishes identically and we are left with simply the first term, which yields

$$\tilde{S}_v(z) = \frac{1}{3}\left[\frac{1}{z+\gamma(v)} + \frac{1}{z+\gamma(v)-i\omega_c t} + \frac{1}{z+\gamma(v)+i\omega_c t}\right]. \tag{28}$$

Taking the inverse Laplace transform of (28) and using (23), we finally obtain

$$C(t) = \tfrac{1}{3}\langle v^2 \exp(-\gamma(v)|t|)\rangle(1+2\cos\omega_c t). \tag{29}$$

If the collision rate were velocity independent, i.e., $\gamma(v)=\gamma$, (29) would be identical in form to the correlation function in the WCM, recalling that the mean squared velocity is given by its 'equipartition' value (cf. eq. (11)). However, in the BLM [8]

$$\gamma(v) = \pi n_p a^2 v, \tag{30}$$

where n_p is the number of scatterers per unit volume and a is an effective scattering radius. Hence, under a Maxwell-Boltzmann distribution $p(v)$, (29) reduces to

$$C(t) = \tfrac{1}{3}(1+2\cos\omega_c t)\cdot\int_0^\infty dv v^2 p(v) e^{-\gamma(v)|t|}, \tag{31}$$

where

$$p(v) = \sqrt{\frac{2}{\pi}}\left(\frac{m}{k_B T}\right)^{3/2}\cdot\exp\left(-\frac{mv^2}{2k_B T}\right). \tag{32}$$

The integral in (31) may be viewed as a continuous superposition of exponential correlations (in time) leading to a correlation function that would, in general, have a non-exponential behaviour. Such behaviour is known to occur in the theory of jump stochastic processes, such as in the case of the "Kangaroo process" [7, 12, 13]. Substituting the explicit form of $\gamma(v)$ from (30) we obtain

$$C(t) = \frac{k_B T}{m}(1+2\cos\omega_c t) \times [1+4(\nu t)^2 + \tfrac{4}{3}(\nu t)^4]\exp(\nu^2 t^2) erfc(\nu|t|)$$
$$-\frac{\nu|t|}{\sqrt{\pi}}(3+\tfrac{4}{3}\nu^2 t^2), \tag{33}$$

where

$$\nu \equiv \pi . \rho a^2 \cdot \sqrt{\frac{k_B T}{2m}} \tag{34}$$

and the complementary error function is defined by

$$erfc(\nu|t|) \equiv \frac{2}{\sqrt{\pi}} \int_{\nu|t|}^{\infty} dx \exp(-x^2). \tag{35}$$

3. The characteristic function and the spectral lineshape

a. *The weak collision model.* The characteristic function $\phi(t)$ is defined in (1) and (2). If we employ the definition (2), the evaluation of $\phi(t)$ can be carried out, using the Fokker–Planck equation (7). Equivalently, we can calculate $\phi(t)$ from definition (1) by employing the full phase space equation for the probability [10]. However, we follow here a simpler method by taking cognizance of the fact that the underlying stochastic process in the WCM is a Gaussian–Markov process. Thus, all cumulants above the two-point vanish identically [4] and hence, (2) yields

$$\phi(t) = \exp\left[-\frac{1}{2} \sum_{jl} k_j k_l \int_0^t dt' \int_0^t dt'' \langle v_j(t') v_l(t'') \rangle \right]. \tag{36}$$

Further, using stationarity, the above expression simplifies to

$$\phi(t) = \exp\left[-\frac{1}{2} \sum_{jl} k_j k_l \int_0^t d\tau (t - \tau) \langle v_j(0) v_l(t) \rangle \right]. \tag{37}$$

We specialize, now, to the geometry of figure 1 in which the external field is taken along the z-axis, in order to, explicitly, demonstrate the anisotropic nature of the diffusion. The velocity-correlation function is already given in (9) using which we derive

$$\phi(t) = \exp - \tfrac{1}{2}[(k_x^2 + k_y^2) S_\perp(t) + k_z^2 S_\parallel(t)], \tag{38}$$

where $S_\parallel(t)$ and $S_\perp(t)$ are precisely the "variance of the displacement" in the longitudinal and transverse directions respectively [10]

$$S_\parallel(t) = \frac{2k_B T}{m\gamma^2} [\gamma t - 1 + \exp(-\gamma t)], \tag{39}$$

$$\begin{aligned} S_\perp(t) = \frac{2k_B T}{m(\gamma^2 + \omega_c^2)^2} &[\gamma t(\gamma^2 + \omega_c^2) - (\gamma^2 - \omega_c^2) \\ &+ \exp(-\gamma t)[(\gamma^2 - \omega_c^2)\cos(\omega_c t) - 2\gamma\omega_c \sin(\omega_c t)]]. \end{aligned} \tag{40}$$

It is evident that the longitudinal component is associated with the free diffusive behaviour [14], as the motion remains unaffected along the direction of the magnetic field. The transverse component, on the other hand, exhibits the interplay of diffusive motion characterized by the collision rate γ and deterministic orbital motion characterized by the cyclotron frequency ω_c. Of course, in the absence of the magnetic field, $\omega_c = 0$, and (38) reduces to the result in the usual WCM [7].

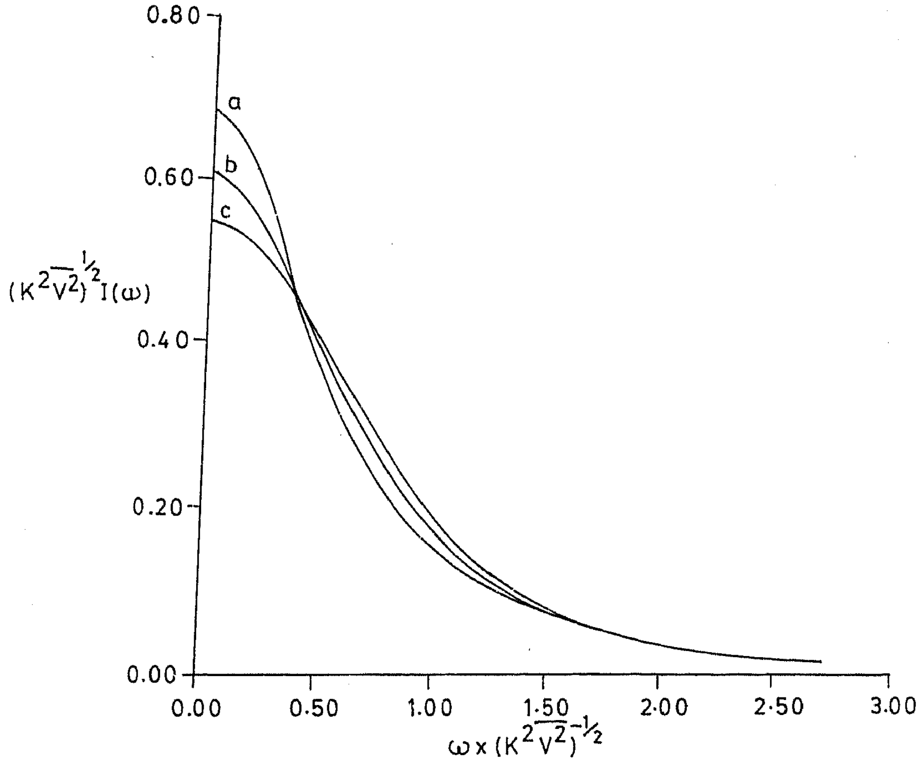

Figure 2. The collision broadened line shape in the weak collision model in the presence of magnetic field. Curves a, b, c correspond to $\bar{\omega}_c = 1$, $\bar{\omega}_c = 0{\cdot}7$ and $\bar{\omega}_c = 0$ respectively, where $\bar{\omega}_c = \omega_c(k^2 \overline{v^2})^{-1/2}$ and $\bar{\gamma} = 1$.

We turn our attention to the issue of what influence the magnetic field has on the lineshape, especially, in the context of collision-broadening. For the sake of definiteness, we choose $k_y = k_z = 0$ and $k_x = k$. As mentioned earlier, the spectral lineshape is obtained from [5]

$$I(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \mathrm{d}t \exp(-i\omega t)\phi(t). \tag{41}$$

We substitute for $S_\perp(t)$ from (40) into (38) and evaluate the integral in (41), numerically. The lineshape is plotted in figure 2 in terms of scaled parameters $\bar{\omega}_c(\bar{\omega}_c \equiv \omega_c(k^2\langle v^2\rangle)^{-1/2})$ and $\bar{\gamma}(\bar{\gamma} \equiv \gamma(k^2\langle v^2\rangle)^{-1/2})$. For a fixed value of $\bar{\gamma}$, which can be ascertained by fixing, say the temperature and pressure, the spectral lines are seen to show narrowing as $\bar{\omega}_c$ is increased. Thus the magnetic field seems to have a similar constraining effect as the collisions do on the velocity of the emitter, leading to a motional narrowing-like phenomenon.

b. *The Boltzmann–Lorentz model.* Our aim in this sub-section is to compute the characteristic function (cf. eq. (2)) and its Fourier transform in the BLM, and compare the respective results with those derived in the WCM (§ 3.a). Before we do this, it is useful to recapitulate the basic premise of the BLM. The time interval 0 to t is divided into $(n+1)$ parts at instants $t_1, t_2, \cdots t_n$, at which time the tagged particle is assumed to undergo collisions with the scatterers. The instants $t_1, t_2 \cdots$ are randomly distributed and assumed to be governed by

Poisson statistics. In-between collisions, the velocity vector of the particle evolves under the influence of the applied magnetic field, and the corresponding evolution operator $G^0(t)$ has matrix elements (cf. eq. (19))

$$(\Omega|G^0(t)|\Omega_0) = \delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0 + \omega_c t)\exp\left[iv\left[tk_z\cos\theta_0 + \frac{\sin\theta_0}{\omega_c}[k_x(\sin\phi_0 - \sin\phi) + k_y(\cos\phi - \cos\phi_0)]\right]\right]. \quad (42)$$

With the operator $G^0(t)$ at hand, the full time evolution operator $\langle U(t)\rangle$ has the Laplace transform given by the series solution in (15) except, $\tilde{U}^0(z+\gamma(v))$ is replaced by $\tilde{G}^0(z+\gamma(v))$. The most simplifying result of the BLM that emerges from the structure of the collision operator **J** in (16) is that the matrix of $\langle \tilde{U}(z)\rangle$ is expressible entirely in terms of $\tilde{G}^0(z+\gamma(v))$. Thus [8]

$$\frac{1}{4\pi}\int d\Omega_0 d\Omega(\Omega|(\langle\tilde{U}(z)\rangle)|\Omega_0) = \frac{\tilde{C}_0(z+\gamma(v))}{1-\gamma(v)\tilde{C}_0(z+\gamma(v))}, \quad (43)$$

where

$$\tilde{C}_0(z+\gamma(v)) \equiv \frac{1}{4\pi}\int d\Omega_0 d\Omega(\Omega|(\tilde{G}^0(z+\gamma(v))|\Omega_0). \quad (44)$$

Using (42), we easily obtain

$$\tilde{C}_0(z+\gamma(v)) = \int_0^\infty dt\exp[-(z+\gamma(v))t] \times \frac{1}{4\pi}\int \sin\theta_0 d\theta_0 \int d\phi_0 \exp\left[iv\left(k_z t\cos\theta_0 + \frac{\sin\theta_0}{\omega_c}[k_x(\sin\phi_0 - \sin(\phi_0 - \omega_c t)) + k_y(\cos(\phi_0 - \omega_c t) - \cos\phi_0)]\right)\right]. \quad (45)$$

We are now ready to write down the expression for the spectral lineshape in the BLM. Recall from (41) that

$$I(\omega) = \frac{1}{\pi}\operatorname{Re}\lim_{\delta\to 0}\phi(z), \quad z = -i\omega + \delta, \quad (46)$$

where

$$\tilde{\phi}(z) = \int_0^\infty dv p(v)\frac{\tilde{C}_0(z+\gamma(v))}{1-\gamma(v)\tilde{C}_0(z+\gamma(v))}, \quad (47)$$

$p(v)$ being given by (32). As before, we specialize to the geometry for which $k_y = k_z = 0$ and $k_x = k$ and compute $\tilde{C}_0(z+\gamma(v))$ from (44) by numerical integrations. The results are then substituted in (46) and one additional numerical integration is performed in order to derive $\tilde{\phi}(z)$. The resultant $I(\omega)$ is again plotted (figure 3) in terms of the scaled parameters $\bar{\gamma}$ and $\bar{\omega}_c$, defined earlier, Once again we notice a narrowing of the spectral line with increasing magnetic field although the lines are more intense in this case than in the WCM.

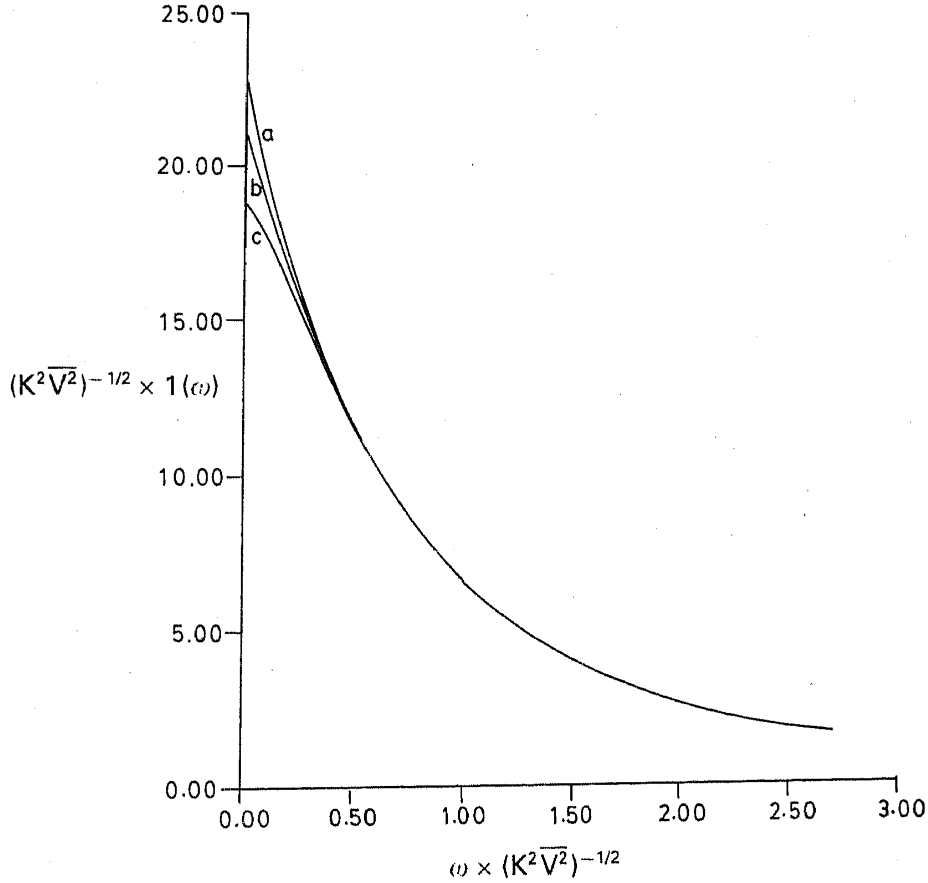

Figure 3. The collision broadened line shape in the Boltzmann Lorentz model in the presence of magnetic field. Curve a corresponds to $\bar{\omega}_c = 1$, and curves b and c to $\bar{\omega}_c = 0{\cdot}7$ and 0 respectively, where $\bar{\omega}_c = \omega_c(k^2\overline{v^2})^{-1/2}$. All curves correspond to $(n_p a^2/k) = 1$ and $\bar{\gamma} = 1$.

4. Summary and conclusions

In this paper, we have employed stochastic modelling for studying classical motion of a charged particle in the presence of a magnetic field. Two distinct stochastic models have been used—one based on diffusion processes, the other on jump processes. These are known in the literature on collision broadening of spectral lines as the weak collision model (WCM) and the Boltzmann–Lorentz model (BLM) respectively. We have made a comparative investigation of the influence of the magnetic field on the spectral line shape, in the WCM and BLM. One of the common features of both these models is *narrowing* of spectral lines, as the magnetic field is increased. Normally, narrowing results from an enhanced rate of collisions, which can be effected by raising the temperature or pressure of the gas. Thus one of our findings in this paper is that the magnetic field can be used as a tuning parameter, in addition to the temperature and pressure, in modulating the width of the spectral lines, in gas-phase spectroscopy.

Apart from the application to spectroscopy, the present study also has a bearing on plasma physics. The magnetic field is, of course a ubiquitous feature in plasma physics and therefore, the results derived here are expected to be of some relevance in magneto-hydrodynamics. One of the sensitive tools for measuring the temperature inside a hot plasma (such as the one present in a tokomak) is to analyze the width of a spectral line emitted by an ion. This analysis would therefore have to take into account the alteration of the width by the magnetic field, as has been demonstrated in this paper.

Our treatment has been entirely classical. It is important to extend the method to quantum mechanics, especially in the context of magneto-transport of an electron, which is of interest in the measurement of Hall coefficient and magneto resistance, in solid state physics, Another intriguing question is what effect does dissipation (induced by collisions in the present context) have on the Landau diamagnetism, which is inherently a quantum phenomenon. These issues will be discussed in a forthcoming publication.

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