MAGNETOTELLURIC SOUNDING OVER MODELS OF CONTINUOUSLY VARYING CONDUCTIVITY

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(*Received 26 July 1975*)

In this paper, the analytical expressions for cagniard impedance are presented for a half space in which, as a whole or part thereof, the conductivity is assumed to vary continuously with the vertical depth from the free surface (air-earth boundary). Mathematically, the assumed variations are taken as

$$(i) \quad \sigma(z) = \sigma_0 \, e^{\alpha' z}$$

$$(ii) \quad \sigma(z) = \sigma_0 \left[1 + \frac{z}{\alpha} \right]^n$$

where σ_0 is the conductivity just at the surface and α' and α are constants. The impedance, thus obtained, can be utilized for preparing apparent resistivity master curves for the proposed models.

INTRODUCTION

As in other electrical methods, the assumption of isotropic and homogeneous layered earth is most common, so is in the magnetotelluric methods as well. This assumption, however, does not always represent the actual earth conditions in the field. So the existence of the situation whose closest approximation may be considered as the continuous (gradual) variation of conductivity in the whole or part of a medium should not be deemed unrealistic. In some recent investigations, transition in the electrical properties has been studied with D.C. (Mallick & Roy 1968; Paul & Banerjee 1970; Jain 1972; and Sri Niwas & Upadhyay 1972, 1974) and with telluric fields (Mallick 1970; and Paul & Banerjee 1970) have considered a more general type of models of continuously varying conductivity and developed the expressions for potentials due to a point current source placed at the surface of the model. However, Mallick (1971) has shown that there is no ready way of detecting transition zone from the resistivity field sounding curve and for this a pre-knowledge of the geological history of the area of investigations is a necessity. He also gave a procedure by which the presence of transition can be detected directly from the field curves obtained from magnetotelluric observations. Keeping this detectibility of presence of transition, the models of Paul and Banerjee (1970) are being studied using telluric fields in the present study. The expressions obtained for cagniard impedance can be utilized for preparing master curves for respective models.

Note: The paper was accepted on 3-1-1976.

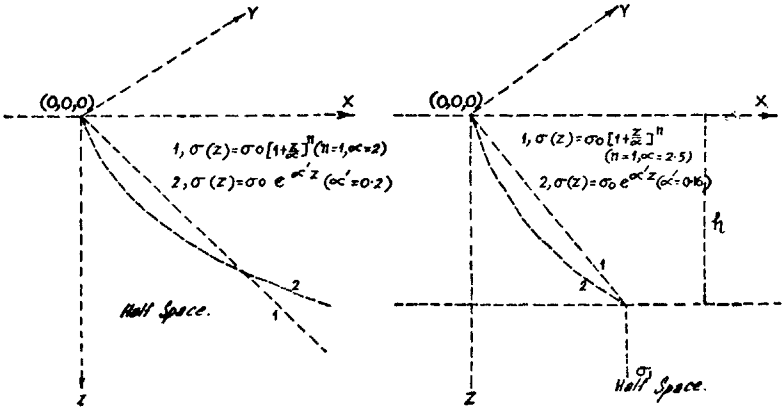

FIG. 1. Geometry of the problem.

STATEMENT OF THE PROBLEM

The proposed models and the nature of conductivity variations are presented in the Fig. 1 (*a*, *b*). The four models are:

(*i*) a half space in which the conductivity varies exponentially along the direction of inward normal to its free surface;

(*ii*) a half space in which the conductivity is proportional to $(1 + \frac{z}{\alpha})^n$, z being measured along the direction of the inward normal to its free-surface;

(*iii*) a uniformly thick layer over a homogeneous half space in which the conductivity varies exponentially along the direction of inward normal to its free surface; and

(*iv*) a uniformly thick layer over a homogeneous half space in which the conductivity is proportional to $(1 + \frac{z}{\alpha})^n$, z being measured veritically downward.

In terms of radiation constants in electromagnetic c.g.s. units the relation for exponential and generalized power law variation can be represented as

$$k(z) = k_0\, e^{\frac{\alpha' z}{2}} \qquad \ldots \quad (1)$$

and

$$k(z) = k_0 \left[1 + \frac{z}{\alpha}\right]^{n/2} \qquad \ldots \quad (2)$$

where $k(z) = \sqrt{4\pi i \omega \sigma(z)},$

$k_0 = \sqrt{4\pi i \omega \sigma_0}$

and $i = \sqrt{-1}.$

Proceeding parallel to Mallick (1971) we express the field equations.

$$\frac{\partial^2 E_x}{\partial z^2} + k_j^2\ E_x = 0 \quad (j = 1, 2) \qquad \ldots \quad (3)$$

and

$$i\omega H_y = \frac{\partial E_x}{\partial z} \qquad \ldots \quad (4)$$

in all the four models considered:

Case (i):

Substituting the value of $k(z)$ from Eq. (1) to Eq. (3) the electric field equation in transition half space is

$$\frac{\partial^2 E_x}{\partial z^2} + k_0^2\ e^{\alpha' z}\ E_x = 0 \qquad \ldots \quad (5)$$

choosing a new variable η such as

$$\eta = e^{\alpha' z}$$

the Eq. (5) is reduced to

$$\eta^2\ \frac{d^2 E_x}{d\eta^2} + \eta\ \frac{dE_x}{d\eta} + \frac{k_0^2}{\alpha'^2}\eta E_x = 0 \qquad \ldots \quad (6)$$

The solution of equation can be written as (Wylie 1960):

$$E_x = A I_0\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right) + B K_0\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right) \qquad \ldots \quad (7)$$

Hence the corresponding magnetic field equation is,

$$H_y = \frac{k_0}{i\omega}\sqrt{\eta}\left[A I_1\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right) - B K_1\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right)\right] \qquad \ldots \quad (8)$$

The boundary condition in this medium is that the field should vanish at $z \to \infty$. Applying this, the Eqs. (7) and (8) can be written as

$$E_x = B K_0\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right) \qquad \ldots \quad (9)$$

and

$$H_y = -\frac{k_0\sqrt{\eta}}{i\omega}\ B K_1\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right) \qquad \ldots \quad (10)$$

Hence the cagniard impedance at the surface of the model is

$$z = \frac{E_x}{H_y}\left(z = 0\right) = \frac{1}{\sqrt{2\sigma_0 T}}\ \frac{K_0\left(\frac{2k_0}{\alpha'}\right)}{K_1\left(\frac{2k_0}{\alpha'}\right)} e^{-\frac{3\pi}{4}i} \qquad \ldots \quad (11)$$

where

$$\omega = \frac{2\pi}{T} \text{ and } i^{-3/2} = e^{-\frac{3\pi}{4}i}$$

Case (ii) :

Substituting the value of $k(z)$ from (2) in Eq. (3), the electric field equation in this case would be

$$\frac{d^2E_x}{dz^2} + k_0^2\left[1 + \frac{z}{\alpha}\right]^n E_x = 0 \qquad \ldots \quad (12)$$

letting $t = \left[1 + \frac{z}{\alpha}\right]$ the Eq. (12) can be written as

$$t^2 \frac{d^2E_x}{dE^2} + k_0^2\,\alpha^2\, t^{n+2}\, E_x = 0 \qquad \ldots \quad (13)$$

Again the solution of Eq. (13) can be derived from Wylie (1960) as

$$E_x = \sqrt{t}\left[AJ_{\frac{1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right) + BJ_{-\frac{1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right)\right] \qquad \ldots \quad (14)$$

and the corresponding magnetic field is

$$H_y = \frac{k_0}{i\omega}\left(\sqrt{t}\right)^{n+1}\left[AJ_{-\frac{n+1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right) - BJ_{\frac{n+1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right)\right] \qquad \ldots \quad (15)$$

where $\lambda = \frac{2k_0\alpha}{n+2}$

Since the field vanishes at $z \rightarrow \infty$ the field equations can be rewritten as

$$E_x = \sqrt{t}\; BJ_{-\frac{1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right) \qquad \ldots \quad (16)$$

and

$$H_y = \frac{ik_0}{\omega}\left(\sqrt{t}\right)^{n+1} BJ_{\frac{n+1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right) \qquad \ldots \quad (17)$$

Hence the cagniard impedance at the surface, $(z = 0)$,

$$Z = \frac{1}{\sqrt{2\sigma_0 T}}\,\frac{J_{-\frac{1}{n+2}}(\lambda)}{J_{\frac{n+1}{n+2}}(\lambda)}\, e^{-\frac{3\pi}{4}i} \qquad \ldots \quad (18)$$

The general Eq. (18) can be simplified for particular cases of linear variations $(n = 1)$ and quadratic variations $(n = 2)$.

(*a*) linear variation $(n=1)$.

$$Z = \frac{1}{\sqrt{2\sigma_0 T}}\,\frac{J_{-\frac{1}{3}}\left(\frac{2k_0\alpha}{3}\right)}{J_{\frac{2}{3}}\left(\frac{2k_0\alpha}{3}\right)}\, e^{-\frac{3\pi}{4}i} \qquad \ldots \quad (19)$$

(*b*) quadratic variation ($n = 2$)

$$Z = \frac{1}{\sqrt{2\sigma_0 T}} \frac{J_{-\frac{1}{4}}\left(\frac{k_0\alpha}{2}\right)}{J_{\frac{3}{4}}\left(\frac{k_0\alpha}{2}\right)} e^{-\frac{3\pi}{4} i} \qquad \ldots \quad (20)$$

Case (iii) :

In this case the value of α' would be given by $\frac{1}{h} e^{n}\left(\frac{\sigma_1}{\sigma_1}\right)$ (σ_1 being conductivity of lower half space and h is the thickness of the upper layer) and hence the electric field and magnetic field equations in the upper layer can be written from (7) and (8) as

$$E_{xt} = AI_0\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right) + BK_0\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right) \qquad \ldots \quad (21)$$

and

$$H_{yt} = \frac{k_0}{i\omega}\sqrt{\eta}\left[AI_1\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right) - BK_1\left(\frac{2k_0}{\alpha'}\sqrt{\eta}\right)\right] \qquad \ldots \quad (22)$$

The corresponding field equations for the lower half space are

$$E_{x1} = B_1 e^{ik_1 z} \qquad \ldots \quad (23)$$

and

$$H_{y1} = \frac{k_1}{\omega} B_1 e^{ik_1 z} \qquad \ldots \quad (24)$$

The boundary conditions of continuity of electric and magnetic field at the interface ($z = h$) are given by

$$\left.\begin{aligned} E_{xt} &= E_{x_1} \\ H_{yt} &= H_{y_1} \end{aligned}\right| \text{ at } z = h \qquad \ldots \quad (25)$$

The application of (25) to (21), (22), (23) and (24) yields

$$B_1 e^{ik_1 h} = AI_0\left(\frac{2k_1}{\alpha'}\right) + BK_0\left(\frac{2k_1}{\alpha'}\right) \qquad \ldots \quad (26)$$

$$-B_1 e^{ik_1 h} = i\left[AI_1\left(\frac{2k_1}{\alpha'}\right) - BK_1\left(\frac{2k_1}{\alpha'}\right)\right] \qquad \ldots \quad (27)$$

Solving equations (26) and (27) we get

$$\frac{B}{A} = \frac{\nu_1}{\nu_2} \qquad \ldots \quad (28)$$

where

$$\nu_1 = iK_1\left(\frac{2k_1}{\alpha'}\right) - k_0\left(\frac{2k_1}{\alpha'}\right)$$

and

$$\nu_2 = iI_1\left(\frac{2k_1}{\alpha'}\right) + I_0\left(\frac{2k_1}{\alpha'}\right)$$

Hence Eqs. (21) and (22) can be written for ($Z = 0$) as

$$\frac{E_{zt}}{A} = I_0\left(\frac{2k_0}{\alpha'}\right) + \frac{\nu_1}{\nu_2} K_0\left(\frac{2k_0}{\alpha'}\right) \quad \ldots \quad (29)$$

and

$$\frac{H_{yt}}{A} = \frac{k_0}{i\omega}\left[I_1\left(\frac{2k_0}{\alpha'}\right) - \frac{\nu_1}{\nu_2} K_1\left(\frac{2k_0}{\alpha'}\right)\right] \quad \ldots \quad (30)$$

Hence cagniard impedance is

$$Z = \frac{1}{\sqrt{2\sigma_0 T}} \frac{\nu_2 I_0\left(\frac{2k_0}{\alpha'}\right) + \nu_1 K_0\left(\frac{2k_0}{\alpha'}\right)}{\nu_2 I_1\left(\frac{2k_0}{\alpha'}\right) - \nu_1 K_1\left(\frac{2k_0}{\alpha'}\right)} e^{-\frac{\pi}{4} i} \quad \ldots \quad (31)$$

Case (iv):

In this case $\alpha = h \Big/ \left(\frac{\sigma_1}{\sigma_0} - 1\right)$

σ_1 being the conductivity of the lower half space and h is the thickness of the upper layer. The field equations. for upper transition layer can be written from (14) and (15) as

$$E_{xt} = \sqrt{t}\left[AJ_{\frac{1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right) + BJ_{-\frac{1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right)\right] \quad \ldots \quad (32)$$

and

$$H_{yt} = \frac{k_0}{i\omega}\left(\sqrt{t}\right)^{n+1}\left[AJ_{-\frac{n+1}{n+2}}\left(\lambda t^{\frac{n+2}{2}}\right) - BJ_{\frac{n+1}{n+2}}\left(\pi t^{\frac{n+2}{2}}\right)\right] \quad \ldots \quad (33)$$

and for lower half space,

$$E_{x1} = B_1 e^{ik_1 z} \quad \ldots \quad (34)$$

and

$$H_{y1} = \frac{k_1}{\omega} B_1 e^{ik_1 z} \quad \ldots \quad (35)$$

Applying the boundary condition (25) to (32), (33), (34) and (35) and solving them for $\frac{B}{A}$ we get

$$\frac{B}{A} = \frac{S_1}{S_2} \quad \ldots \quad (36)$$

where

$$S_1 = iJ_{\frac{n+1}{n+2}}\left(\frac{2k_1\alpha}{n+2}\right) - J_{-\frac{1}{n+2}}\left(\frac{2k_1\alpha}{n+2}\right)$$

and

$$S_2 = iJ_{-\frac{n+1}{n+2}}\left(\frac{2k_1\alpha}{n+2}\right) + J_{\frac{1}{n+2}}\left(\frac{2k_1\alpha}{n+2}\right)$$

Hence the expression for cagniard impedance at the surface $z = 0$ is

$$Z = \frac{1}{\sqrt{2\sigma_0 T}} \frac{S_2 J_{\frac{1}{n+2}}\left(\frac{2k_0\alpha}{n+2}\right) + S_1 J_{-\frac{1}{n+2}}\left(\frac{2k_0\alpha}{n+2}\right)}{S_2 J_{-\frac{n+1}{n+2}}\left(\frac{2k_0\alpha}{n+2}\right) - S_1 J_{\frac{n+1}{n+2}}\left(\frac{2k_0\alpha}{n+2}\right)} e^{-\frac{\pi}{4} i} \quad \ldots \quad (37)$$

Particular Cases

(*a*) for linear variation ($n = 1$)

$$S_1 = iJ_{\frac{2}{3}}\left(\frac{2k_1\alpha}{3}\right) - J_{-\frac{1}{3}}\left(\frac{2k_1\alpha}{3}\right)$$

$$S_2 = iJ_{-\frac{2}{3}}\left(\frac{2k_1\alpha}{3}\right) + J_{\frac{1}{3}}\left(\frac{2k_1\alpha}{3}\right)$$

Thus

$$Z = \frac{1}{\sqrt{2\sigma_0 T}} \frac{S_2 J_{\frac{1}{3}}\left(\frac{2k_0\alpha}{3}\right) + S_1 J_{-\frac{1}{3}}\left(\frac{2k_0\alpha}{3}\right)}{S_1 J_{-\frac{2}{3}}\left(\frac{2k_0\alpha}{3}\right) - S_1 J_{\frac{2}{3}}\left(\frac{2k_0\alpha}{3}\right)} e^{-\frac{\pi}{4} i} \quad \ldots \quad (38)$$

(*b*) for quadratic variation ($n = 2$)

$$S_1 = iJ_{\frac{3}{4}}\left(\frac{k_1\alpha}{2}\right) - J_{-\frac{1}{4}}\left(\frac{k_1\alpha}{2}\right)$$

$$S_2 = iJ_{-\frac{3}{4}}\left(\frac{k_1\alpha}{2}\right) + J_{\frac{1}{4}}\left(\frac{k_1\alpha}{2}\right)$$

and

$$Z = \frac{1}{\sqrt{2\sigma_0 T}} \frac{S_2 J_{\frac{1}{4}}\left(\frac{k_0\alpha}{2}\right) + S_1 J_{-\frac{1}{4}}\left(\frac{k_0\alpha}{2}\right)}{S_2 J_{-\frac{3}{4}}\left(\frac{k_0\alpha}{2}\right) - S_1 J_{\frac{3}{4}}\left(\frac{k_0\alpha}{2}\right)} e^{-\frac{\pi}{4} i} \quad \ldots \quad (39)$$

Eqs. (11), (18), (31) and (37) represent the analytical expressions for cagniard expressions in the four models considered for this study.

Concluding Remarks

The values for cagniard impedance for the proposed four models of continuously varying conductivity have been calculated [expressions (18) and (37) for generalized power law variation and (11) and (31) for exponential variation]. The expressions (19), (38) and (20), (39) are the values of impedance for particular cases of linear ($n = 1$) and quadratic ($n = 2$) variations of generalized power law variation.

Using these expressions for cagniard impedance, the apparent resistivity values can be calculated by using the expression:

$$\rho_a = -\frac{i}{\omega\mu} \mid Z \mid^2 \quad \ldots \quad (40)$$

The numerical values can be calculated on an electronic computer for different models using various combinations of different parameters and sets of master-curves can be prepared. These numerical calculations are, however, proposed to be presented in another paper.

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