On the number of extreme measures with fixed marginals

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Introduction:

In his paper [3], K. R. Parthasarathy gives a bound for the number of extreme points of the convex set of all $G-$ invariant probability measures on $X \times Y$ with given marginals of full support. The purpose of this paper is to improve this bound.

Section 1:

Let X and Y be finite sets with $|X| = m$ and $|Y| = n$. Let G be a group acting on X and Y. Let G act on $X \times Y$ by $g(x, y) = (g(x), g(y))$ for all $g \in G$ and $(x, y) \in X \times Y$. Let X/G be the set of G orbits of X. Write $|X/G| = m_1$, $|Y/G| = n_1$ and $|(X \times Y)/G| = m_{12}$. Let π_1 and π_2 denote the projection maps from $X \times Y$ to X and Y respectively. The sets $G(x), G(y)$ and $G(x, y)$ respectively denote the $G-$orbits of $x \in X, y \in Y$ and $(x, y) \in X \times Y$.

Let $\mu_1$and μ_2 be $G-$ invariant probability measures with full support on X and Y respectively. Then $K\ (\mu_1, \mu_2)$ denotes the convex set of all G -invariant probability measures μ on $X \times Y$ with marginals $\mu_1$and μ_2. Note that for any measure $\mu \in K\ (\mu_1, \mu_2)$, the support $S(\mu)$ of μ is $G-$ invariant. Let $E(\mu_1, \mu_2)$ denote the set of extreme points of $K\ (\mu_1, \mu_2)$. In [3], K.R.Parthasarathy gives an estimate for the number of points in $E(\mu_1, \mu_2)$:

$$|\mathbf{E}(\mu_1, \mu_2)| \leq \sum_{\max(m_1,n_1) \leq r \leq m_1+n_1} \binom{m_{12}}{r}$$

. In this note we prove that

$$|\mathbf{E}(\mu_1, \mu_2)| \leq \binom{m_{12}}{m_1 + n_1 - 1} \tag{1}$$

which considerably improves the above bound. Indeed $\binom{m_{12}}{m_1+n_1-1}$ is one of the terms in the above sum. Moreover, if G acts trivially or if number of G orbits in $G(x) \times G(y)$ is independent of x and y, then

$$|E(\mu_1, \mu_2)| / \binom{m_{12}}{m_1 + n_1 - 1} \longrightarrow 0 \quad \text{as } m_1, n_1 \longrightarrow \infty.$$

In [3], K.R. Parthasarathy has proved the following theorem:

Theorem ([3], Theorem 3.5): *A probability measure $w \in K\ (\mu_1, \mu_2)$ is extreme if and only if there is no nonzero real valued function ζ on $S(w)$ such that*
(i) $\zeta(g(x), g(y)) = \zeta(x, y)$ for all $(x, y) \in S(w), g \in G$;
(ii) $\sum_y \zeta(x, y)\ w(x, y) = 0$ for all x;
(iii) $\sum_x \zeta(x, y)\ w(x, y) = 0$ for all y.

Definition : A $G-$ invariant subset $S \subset X \times Y$ is said to be $G-$*good* if any $G-$ invariant real (or complex) valued function f defined on S can be written as $f(x, y) = u(x) + v(y)$ for all $(x, y) \in S$ for some $G-$invariant functions u and v on X and Y respectively.

Proposition 1: *The support $S = S(\mu)$ of a measure $\mu \in K(\mu_1, \mu_2)$ is $G-$good if and only if $\mu \in E\ (\mu_1, \mu_2)$.*

Proof: Let $\mu \in E\ (\mu_1, \mu_2)$ and assume that S is not $G-$ good. Then there exists a $G-$ invariant function f on S which cannot be written as $f = u + v$ where u and v are $G-$ invariant. Let $L^2_G(S, \mu)$ denote the Hilbert space of all $G-$invariant functions defined on S. Let $\Lambda \subset L^2_G(S, \mu)$ denote the set of all $G-$ invariant functions f which have representation $f = u+v$ with u, v $G-$invariant functions on X and Y respectively. Then Λ is a proper subspace of $L^2_G(S, \mu)$. Hence there exists a nonzero $\zeta \in L^2_G(S, \mu)$ which is orthogonal to Λ. Then

$$\sum \zeta(x, y) u(x) \mu(x, y) = 0 \textit{ and } \sum \zeta(x, y) v(y) \mu(x, y) = 0$$

for all $G-$invariant u and v defined on X and Y respectively. In particular

$$\sum_X u(x) \sum_Y \zeta(x, y) \mu(x, y) = 0$$

for all $G-$invariant u on X. For any $x_0 \in X$, taking $u(x) = 1$ on $G(x_0)$ and $u(x) = 0$ on all other orbits in X, we get

$$\sum_Y \zeta(x_0, y) \mu(x_0, y) = 0.$$

Hence

$$\sum_Y \zeta(x,y)\mu(x,y) = 0$$

for all x and similarly,

$$\sum_X \zeta(x,y)\mu(x,y) = 0$$

for all y which contradicts the theorem above. Conversely, suppose S is $G-$good. Then $\Lambda = L^2_G(S,\mu)$. Let $\zeta \in L^2_G(S,\mu)$ satisfy the conditions of the above theorem with respect to the measure μ. Let $f \in L^2_G(S,\mu)$ be any function. Then f can be written as $f = u + v$, for some $G-$invariant functions u and v on X and Y respectively. By condition (ii) of the above theorem,

$$\sum_{X\times Y} u(x)\zeta(x,y)\mu(x,y) = \sum_x u(x) \sum_y \zeta(x,y)\mu(x,y) = 0.$$

Similarly by (iii),

$$\sum \zeta(x,y)v(y)\mu(x,y) \ = 0.$$

Both these equations together imply

$$\sum \zeta(x,y)f(x,y)\mu(x,y) = 0.$$

Since f is arbitrary in $L^2_G(S,\mu)$, $\zeta = 0$. By the above theorem $\mu \in E(\mu_1,\mu_2)$, which proves the proposition.

Remark 1: For any $x \in X$ and $y \in Y$, the $G-$invariant set $G(x) \times G(y)$ can be written as union of $G-$ orbits on $X \times Y$ whose first projection is $G(x)$ and second projection is $G(y)$.

$$G(x)\times G(y) = \cup\{G(z,w)|\, \pi_1(G(z,w)) = G(x) \text{ and } \pi_2(G(z,w)) \ = G(y)\}.$$

This is because the orbit $G(z,w)$ of $(z,w) \in X\times Y$ has $\pi_1(G(z,w)) = G(x)$ and $\pi_2(G(z,w)) = G(y)$ if and only if $G(z,w) \subset G(x)\times G(y)$.

Remark 2: If S is a $G-$ good set then $(G(x)\times G(y))\cap S$ contains atmost one $G-$ orbit. This is because S cannot contain two distinct orbits with the same projections: for, if $G(z,w)$ and $G(a,b)$ are two such orbits with $\pi_1(G(z,w)) = \pi_1(G(a,b)) = G(x)$ and $\pi_2(G(z,w))$ $= \pi_2(G(a,b)) = G(y)$ then for any $G-$ invariant $f = u + v$ defined on S with $f(z,w) \neq f(a,b)$ we will have $f(z,w) = u(z) + v(w) = u(x) + v(y)$ and similarly, $f(a,b) = u(x) + v(y)$, a contradiction.

Section 2:

Let X_1 and Y_1 be two finite sets with $|X_1| = m_1$ and $|Y_1| = n_1$. A subset $S \subset X_1 \times Y_1$ is called *good* (ref. [1]) if every real (or complex) valued function f on S can be expressed in the form

$$f(x) = u(x) + v(x) \text{ for all } (x,y) \in S$$

Let $X_1 = X/G$ and $Y_1 = Y/G$. Then $X_1 \times Y_1$ can be identified with a set whose points are $G(x) \times G(y)$, $x \in X, y \in Y$. The $G-$invariant functions on X and on Y are in one-to-one correspondence with the functions on X_1 and Y_1 respectively.

Let $\widetilde{S} \subset (X \times Y)/G$ denote the set of all $G-$orbits in a $G-$invariant subset S of $X \times Y$. Define $\phi : \widetilde{S} \longrightarrow X_1 \times Y_1$ by $\phi(G(x,y)) = G(x) \times G(y)$.

One can show that subsets of good sets are good and every good set $S \subset X_1 \times Y_1$ is contained in a maximal good subset of $X_1 \times Y_1$. Further every maximal good set of $X_1 \times Y_1$ contains $m_1 + n_1 - 1$ elements. (ref [1])

Proposition 2: *A $G-$invariant subset $S \subset X \times Y$ is $G-$good if and only if ϕ is one-to-one on $\widetilde{S}$ and $\phi(\widetilde{S}\ )$ is good in $X_1 \times Y_1$. Further, S is maximal $G-$good set if and only if ϕ is one-to-one on S and $\phi(\widetilde{S}\ )$ is maximal good set in $X_1 \times Y_1$.*

Proof: Assume S is $G-$good. By remark 2, if S is $G-$ good, then ϕ is one-to-one on $\widetilde{S}$. Let f be any real (or complex) valued function defined on $\phi(\widetilde{S}\ )$. Define g on $\widetilde{S}$ by $g = f \circ \phi$. This map g gives rise to a $G-$invariant map on S, again denoted by g. Writing $g = u + v$, where u and v are $G-$invariant functions on X and Y respectively, and noting that u and v are constant on each orbit, we can define $\widetilde{u}$ and $\widetilde{v}$ on X_1 and Y_1 by $\widetilde{u}(G(x)) = u(x)$ and $\widetilde{v}(G(y)) = v(y)$. It is easy to see that $f = \widetilde{u} + \widetilde{v}$. So $\phi(\widetilde{S})$ is good. Conversely, let $S \subset X \times Y$ be such that ϕ is one-to-one on $\widetilde{S}$ and $\phi(\widetilde{S}\ )$ is good. Since ϕ is one-to-one, any $G(x) \times G(y)$ intersects S in atmost one orbit. Given a function g on S we can define f on $\phi(\widetilde{S})$ as $f = g \circ \phi^{-1}$. Since f is defined on the good set $\phi(\widetilde{S})$ we can write f as $f = \widetilde{u} + \widetilde{v}$ where $\widetilde{u}$, $\widetilde{v}$ are defined on $\pi_1(\phi(\widetilde{S}))$ and $\pi_2(\phi(\widetilde{S}))$ respectively. Defining $u(x) = \widetilde{u}(G(x))$ and

$v(y) = \widetilde{v}(G(y))$ we get $G-$ invariant functions u and v with $g = u+v$. Now suppose S is a maximal $G-$good set. We know from the first part of the theorem that ϕ is one-to-one on $\widetilde{S}$. If $\phi(\widetilde{S}\ )$ is not a maximal good set, there exists a point, say $G(a)\times G(b) \notin \phi(\widetilde{S}\ )$, such that $\phi(\widetilde{S})\cup\{G(a)\times G(b)\}$ is good. Then, since $\{G(a)\times G(b)\}\cap S = \emptyset$, the map ϕ is one-to-one on $\widetilde{T}$ where $T = G(a,b)\cup S$. Using the first part of the theorem, T is $G-$good contradicting the maximality of S. The converse can be proved in a similar manner. This completes the proof of the proposition.

By corollary 3.6 of [3], different extreme points of $K(\mu_1, \mu_2)$ have distinct supports. As pointed out by the referee, this fact is also a consequence of proposition 1: Assume that $\mu, \nu \in E\ (\mu_1, \mu_2)$, with $\mu \neq \nu$, having the same support S. By proposition 1 S is $G-$good. But this is a contradiction since S is also the support of $(\mu + \nu)/2$, which is not extreme. Further, for μ and $\nu \in E\ (\mu_1, \mu_2)$ the measure $(\mu + \nu)/2 \in K\ (\mu_1, \mu_2)$ is not extreme, and so by Proposition 1 its support $S(\mu) \cup S(\nu)$ is not a $G-$ good set. Further, for μ and $\nu \in E\ (\mu_1, \mu_2)$ the measure $(\mu + \nu)/2 \in K\ (\mu_1, \mu_2)$ is not extreme, and so by Proposition 1 its support $S(\mu) \cup S(\nu)$ is not a $G-$ good set. This shows that supports of different measures in $E(\mu_1, \mu_2)$ are contained in different maximal $G-$good sets of $X \times Y$: Because, if $\mu \neq \nu \in E\ (\mu_1, \mu_2)$ such that $S(\mu) \subset S$ and $S(\nu) \subset S$ for some maximal $G-$good set S then the measure $(\mu + \nu)/2 \in K\ (\mu_1, \mu_2)$ has its support $S(\mu) \cup S(\nu)$ contained in S. Since S is $G-$good, $S(\mu) \cup S(\nu)$ is also $G-$good a contradiction to proposition 1 as $(\mu + \nu)/2$ is not extreme. Therefore, $|E(\mu_1, \mu_2)|$ is bounded by the number of maximal $G-$good sets of $X \times Y$.

Let S be a maximal $G-$good set in $X \times Y$. By Proposition 2, $\phi(\widetilde{S})$ is a maximal good set in $X_1 \times Y_1$. Since ϕ is one-to-one on $\widetilde{S}$, $\widetilde{S}$ contains $m_1 + n_1 - 1$ orbits of G. Since the number of orbits in $X \times Y$ is m_{12}, and any maximal $G-$good set in $X \times Y$ is of the form $\phi(\widetilde{S}\ )$, the total number of maximal $G-$good sets in $X \times Y$ is less than or equal to $\binom{m_{12}}{m_1+n_1-1}$. This proves (1).

We give an example to show that the above bound is sharp. Let G be the group S_n, the permutation group on n elements. Let $X = \{1, 2, ..., n\}$ and Y be the set S_n . Here $|X| = n$ and $|Y| = n!$. Then G acts on X in the obvious manner and on Y by $g(h) = g \circ h$. The only $G-$ invariant subset of X is X itself and the only $G-$ invariant subset of Y is Y itself. Then G also acts on $X \times Y$ diagonally. That is, $g(x, y) = (g(x), g(y))$. For any $(x, y) \in X \times Y$, the set $G(x, y) = \{(g(x), g(y))|g \in G\}$ is a $G-$ invariant subset of $X \times Y$ with $n!$ number of elements and

$G(x)\times\ G(y)$ is the whole set $X\times Y$. In this case, $|X/G| = m_1 = 1$ and $|Y/G| = n_1 = 1$ and $|(X\times Y)/G| = m_{12} = n$. Therefore, $\binom{m_{12}}{m_1+n_1-1} = n$.

The only $G-$ invariant probability measures on X and Y are uniform measures. That is, $\mu_1(x) = \frac{1}{n}$ for all $x\in X$ and $\mu_2(y) = \frac{1}{n!}$ for all $y\in Y$. So the only $G-$ invariant functions on X and Y are constant functions. If $\mu\in E(\mu_1,\mu_2)$, then the support S of μ should be $G-$ good. Any $G-$ invariant function f defined on S, can be written as $f = u+v$ where u, v are $G-$ invariant functions on X and Y respectively. This shows that f must be constant, which means S consists of a single orbit, say $S = G(x,y)$. Then $\mu((g(x),g(y)) = \frac{1}{n!}$ for all $g\in G$. Observe that the collection $\{g(y)|g\in G\}$ has all $n!$ different elements whereas in the collection $\{g(x)|g\in G\}$ every value of $g(x)$ is repeated $(n-1)!$ times. This shows that every such uniform measure μ supported on any single orbit $G(x,y)$ has marginals μ_1 and μ_2. Since there are n orbits in $X\times Y$, we get $|E(\mu_1,\mu_2)| = n$.

Now we state some results about good subsets of $X_1\times Y_1$ not necessarily G-good sets (ref. [1], [2]).

Consider any two points $(x,y),(z,w)\in S\subset X_1\times Y_1$ where S is any (not necessarily good) subset of $X_1\times Y_1$. We say that $(x,y),(z,w)$ are *linked* if there exists a sequence of points $(x_1,y_1) = (x,y),(x_2,y_2)...(x_n,y_n) = (z,w)$ of points of S such that

(i) for any $1\le i\le n-1$ exactly one of the following equalities hold:

$x_i = x_{i+1}$ or $y_i = y_{i+1}$;

(ii) if $x_i = x_{i+1}$ then $y_{i+1} = y_{i+2}$, and if $y_i = y_{i+1}$ then $x_{i+1} = x_{i+2}, 1\le i\le n-2$.

We also call this a *link* joining (x,y) to (z,w). A nontrivial link joining (x,y) to itself is called a *loop*.

Theorem (ref. [1], cor. 4.11): A subset $S\subset X_1\times Y_1$ is good if and only if S contains no loops.

Remark 3: Let the orbits in $\widetilde{S}$ be

$$G(x_1,y_1),G(x_2,y_2),...,G(x_{m_1+n_1-1},y_{m_1+n_1-1}).$$

Then $S\cap(G(x_i)\times G(y_i)) = G(x_i,y_i)$ for $1\le i\le m_1+n_1-1$. Let $G(z,w)$ be any other orbit in $G(x_i)\times G(y_i)$ and let $S' = (S\setminus G(x_i,y_i))\cup G(z,w)$. It is clear that S' is maximal $G-$good set with $\phi(\widetilde{S}) = \phi(\widetilde{S}')$. If α_i denote the number of orbits in $G(x_i)\times G(y_i)$, then there are $\alpha_1\alpha_2...\ \alpha_{m_1+n_1-1}$ many maximal $G-$good sets in $X\times Y$ with image $\phi(\widetilde{S})$ under ϕ.

It seems likely that $|E(\mu_1, \mu_2)|/\binom{m_{12}}{m_1+n_1-1} \longrightarrow 0 \quad$ as $m_1, n_1 \longrightarrow \infty$. We show this in the case $G = (e)$ and more generally when number of G orbits in $G(x) \times G(y)$ is independent of x and y. For that we first prove the following theorem.

Theorem: *Let $X = \{x_1, x_2, ...x_m\}$ and $Y = \{y_1, y_2, ...y_n\}$ be two finite sets. Then*

(i) *the number of maximal good sets contained in $X \times Y$ is $m^{n-1}n^{m-1}$.*

(ii) *the number of maximal good sets among them with exactly k fixed points, (say $(x_i, y_{j_1}), ...(x_i, y_{j_k})$) having a fixed first coordinate say x_i is : $kn^{m-2}(m-1)^{n-k}$, $1 \leq k \leq n$.*

(iii) *the number of maximal good sets with exactly k fixed points having a fixed second coordinate say y_j is: $km^{n-2}(n-1)^{m-k}$, $1 \leq k \leq m$.*

Proof: We use induction on $m+n$. The result is true for $m = 1$ and $n = 1$. Assume the result for all values of $|X| \leq m$ and $|Y| \leq n$. We prove the result for $|X| = m$ and $|Y| = n+1$. Let $X = \{x_1, x_2, ...x_m\}$ and $Y = \{y_1, y_2, ...y_n, y_{n+1}\}$. Consider a $m \times (n+1)$ grid of $m(n+1)$ cells with m rows corresponding to $\{x_1, x_2, ..., x_m\}$ and $n+1$ columns correspondiong to $\{y_1, y_2, ..., y_{n+1}\}$ Associate (i, j)th cell with the point $(x_i, y_j) \in X \times Y$. We say that $(x_i, y_j) \in (i, j)$th cell.

To prove (iii) let S be a maximal good set in $X \times Y$. Then $|S| = m+n$. Suppose S contains exactly k points with fixed second coordinate, say y_{n+1}. Without loss of generality we assume them to be (x_1, y_{n+1}), (x_2, y_{n+1}), ... (x_k, y_{n+1}). Denote

$$K = \{(x_1, y_{n+1}), (x_2, y_{n+1}), ...(x_k, y_{n+1})\}.$$

(i) Atleast one of these first k rows contain atleast two points of S, i.e., there exist a point (x_i, y_j) of S with $1 \leq i \leq k$ and $1 \leq j \leq n$.

Proof: Otherwise leaving these k rows and the last column, the remaining points of S will be a good set with $m+n-k$ points using $m+n-k$ coordinates which is not possible.

(ii) If $(x_i, y_j) \in S$ with $1 \leq i \leq k$ and $1 \leq j \leq n$. Then the $j$the column (which contains the point (x_i, y_j)) has no other point (x_l, y_j) of S with $1 \leq l \neq i \leq k$ because the four points $\{(x_i, y_j), (x_i, y_{n+1}), (x_l, y_{n+1}), (x_l, y_j)\}$ form a loop.

(iii) Suppose $(x_i, y_j) \in S$ for some $1 \leq i \leq k$ and $1 \leq j \leq n$. Then the set got by dropping the point (x_i, y_j) and adding $(x_l, y_j), 1 \leq l \neq i \leq k$ to S clearly contain no loop and so is maximal good.

Let S' be the maximal good set obtained in this way by replacing all the points (x_i, y_j), $1 \leq i \leq k$ and $1 \leq j \leq n$ of S by $(x_1, y_j), 1 \leq j \leq n$.

Then each of the rows corresponding to $x_2, ..., x_k$ contains exactly one point of S'. The set S'' got from S' by dropping these rows and the last column will be a maximal good set in $\{x_1, x_{k+1}, ..., x_m\} \times \{y_1, y_2, ...y_n\}$ and contains $m + n - k$ elements.

By induction hypothesis, the number of maximal good sets in $\{x_1, x_{k+1}, ..., x_m\} \times \{y_1, y_2, ...y_n\}$ having exactly r points in r fixed positions in the first row, is: $rn^{m-k-1}(m-k)^{n-r}$, for $1 \leq r \leq n$. Consider any such maximal good set, say A. Further add the dropped rows and the last column. Enlarge A by adding the first k points of the $(n+1)$th column, call this set B. It is a maximal good set in $\{x_1, x_2, ..., x_m\} \times \{y_1, y_2, ...y_{n+1}\}$. Any point $(x_1, y_j), 1 \leq j \leq n$ in B can be replaced by (x_l, y_j), for any $1 \leq l \leq k$ and the resulting set will continue to remain maximal good in $\{x_1, x_2, ..., x_m\} \times \{y_1, y_2, ...y_{n+1}\}$. In this way each one of $rn^{m-k-1}(m-k)^{n-r}$ maximal good set A gives rise to k^r maximal good sets in the original $m \times (n+1)$ matrix. . Further, we can choose the r points in the first row in $\binom{n}{r}$ ways . Adding over r, the total number of maximal good sets with exactly k cells in k fixed positions of the last column is:

$$\sum_{r=1}^{n} \binom{n}{r} k^r r n^{m-k-1}(m-k)^{n-r} = kn^{m-k-1} n \sum_{r=0}^{n-1} \binom{n-1}{r} k^{r-1}(m-k)^{n-r}$$

$$= kn^{m-k}(m-k+k)^{n-1} = kn^{m-k}m^{n-1}$$

which is (iii) for $m \times (n+1)$ matrix.

To prove (i), since we can choose the k points in the last column in $\binom{m}{k}$ ways, the total number of maximal good sets with exactly k points from the last column is: $\binom{m}{k}\, kn^{m-k}m^{n-1}$. The total number of maximal good sets in $X \times Y$ is got by adding these numbers as k varies from 1 to m :

$$\sum_{k=1}^{m} \binom{m}{k} kn^{m-k}m^{n-1} = \sum_{k=0}^{m-1} \binom{m-1}{k} n^{m-k-1}m^n = m^n(n+1)^{m-1}.$$

(ii) can be proved in a similar way as (iii). This completes the proof of the theorem.

Next, we prove that as $m, n \to \infty$ the ratio $m^{n-1}n^{m-1}/\binom{mn}{m+n-1} \to 0$ as $m, n \to \infty$.

$$\lim_{m,n\to\infty} m^{n-1}n^{m-1}/\binom{mn}{m+n-1} =$$

$$\lim_{m,n\to\infty} m^{n-1}n^{m-1}(m+n-1)!((mn-m-n+1)!/(mn)!$$

By Sterling's formula, we know that $n! \sim \frac{\sqrt{2\pi}n^{n+\frac{1}{2}}}{e^n}$ for large n.

Using this expression one can show that

$$m^{n-1}n^{m-1}/\binom{mn}{m+n-1} \leq C\frac{(1-\frac{1}{m})^{mn}(1+\frac{n}{m})^m(1-\frac{1}{n})^{mn}(1+\frac{m}{n})^n}{(1-\frac{1}{m})^n(1-\frac{1}{n})^m(m+n)^{\frac{1}{2}}} \tag{2}$$

for some constant C. If $\frac{m}{n} \geq 1$, since $(1-\frac{1}{m})^m$ increases to e^{-1}, the right hand side of (2) tends to 0 as $m, n \longrightarrow \infty$.

The case where $\frac{m}{n} \leq 1$ is similar because the the expression on the right hand side of (2) is symmetric with respect to m and n.

If $G = (e)$, the maximal G-good sets in $X \times Y$ are just the maximal good sets and the number of maximal good sets, by the previous theorem is, $m^{n-1}n^{m-1}$. In this case $m_{12} = m_1 n_1$. Therefore

$$|E(\mu_1, \mu_2)|/\binom{m_{12}}{m_1+n_1-1} \leq m^{n-1}n^{m-1}/\binom{m_1 n_1}{m_1+n_1-1} \longrightarrow 0$$

as m and $n \longrightarrow \infty$.

Now suppose that the number of G-orbits in $G(x) \times G(y)$ is a constant, say a, for all x and y. Then by remark 3, the number of maximal G-good sets in $X \times Y$ is $a^{m_1+n_1-1}m_1^{n_1-1}n_1^{m_1-1}$ and $m_{12} = am_1 n_1$. Therefore,

$$|E(\mu_1, \mu_2)|/\binom{m_{12}}{m_1+n_1-1} \leq \left(a^{m_1+n_1-1}m_1^{n_1-1}n_1^{m_1-1}\right)/\binom{am_1 n_1}{m_1+n_1-1}$$

$$\leq \left(a^{m_1+n_1-1}m_1^{n_1-1}n_1^{m_1-1}\right)/a^{m_1+n_1-1}\binom{m_1 n_1}{m_1+n_1-1}$$

$$= \left(m_1^{n_1-1} n_1^{m_1-1}\right) / \binom{m_1 n_1}{m_1 + n_1 - 1} \longrightarrow 0$$

as m and $n \longrightarrow \infty$.

Note: The maximal good sets in $X \times Y$ can be associated in a one-to-one manner with the spanning trees of a complete bipartite graph. Consider the complete bipartite graph $K_{m,n}$ where $|X| = m$ and $|Y| = n$. A subset $S \subset X \times Y$ is maximal good if and only if $|S| = m + n - 1$ and in the grid corresponding to $X \times Y$, S contains no loops. Construct an $m \times n$ matrix corresponding to any spanning tree T in $K_{m,n}$ as follows: Identifying the elements of X and Y with the veritces of $K_{m,n}$, let $V = (X, Y)$ denote the vertices of $K_{m,n}$. Whenever the edge $(x_i, y_j) \in T$, put (i, j)th entry in the matrix equal to one; otherwise (i, j)th entry is zero. Since T is a spanning tree, there are exactly $m + n - 1$ nonzero entries in the matrix. As T contains no cycles, the nonzero entries in the matrix donot form a loop. Therefore the nonzero entries of the matrix correspond to a maximal good set in the grid corresponding to $X \times Y$. This correspondence is one-to-one. In [5], it is proved that the number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$. But the proof makes use of the determinant of the matrix and is different from the one given here.

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