ON THE PROBLEM OF EVANESCENT PROCESSES

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1. Let R denote the set of real numbers. Let G be a countable dense subgroup of R. We construct a nontrivial σ-finite measure m on R such that

(i) m is nonatomic, i.e. $\mu(\{x\})=0$ for every real number x,

(ii) m is singular with respect to the Lebesgue measure L on R,

(iii) m is invariant under translation by members of G, i.e., $m(A+g) =m(A)$ for all $g\in G$ and for all Borel subsets $A\in\mathcal{B}$. (Here and in sequel $\mathcal{B}$ denotes the class of Borel subsets of R.)

Such measures are intimately connected with the problem of evanescent processes and analytic functions on compact tori raised by Henry Helson and David Lowdenslager in their paper [**3**]. We establish this connection in §3. In §2 we shall study the group of unitary operators T^g, $g\in G$, defined on $L_2(R, m)$ by $(T^gf)(\lambda)=f(\lambda+g)$, $f\in G$. In the references we list papers connected with the present work. We proceed with the construction of the measure in steps.

Step 1 (*Cantor's decimal set D*). Expand every number x in the unit interval $I=\{x=0\leqq x<1\}$ in the decimal system, i.e. write $x=\sum_{n=1}^{\infty}(\alpha_n/10^n)$, $\alpha_n=0, 1, 2, \cdots 9$; $n=1, 2, \cdots$ and let D be the set of all those numbers x in whose expansion α_n takes values 0 or 9. More accurately D is the set of all numbers x in the unit interval I such that x can be expanded by using 0 and 9 alone. Geometrically D is the Cantor set obtained by deleting the middle 8/10ths.

Step 2. Here we state a known result and indicate its proof. Let A_5 denote the set of all those numbers in the unit interval I whose decimal expansion does not involve the number 5.

LEMMA 1. *A_5 has Lebesgue measure zero.*

PROOF. Let Q_n be the set of numbers $x\in I$ such that 5 is in the nth decimal place of the expansion of x but not in the first $(n-1)$ places. Then each Q_n is measurable and its Lebesgue measure can be shown to be $9^{n-1}/10^n$. Q_n's form a disjoint sequence of sets and the Lebesgue measure of $\bigcup_{n=1}^{\infty} Q_n$ is $\sum_{n=1}^{\infty}(9^{n-1}/10^n)=1$. But $A_5=I-\bigcup_{n=1}^{\infty} Q_n$. So the Lebesgue measure of A_5 is zero, q.e.d

Step 3. Let $Q=\{x-y: x, y\in D\}$.

LEMMA 2. *Q has Lebesgue measure zero.*

PROOF. $Q=Q_+\cup Q_-$ where

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$$Q_+ = \{x - y: x, y \in D, x \geqq y\},$$
$$Q_- = \{x - y: x, y \in D, x \leqq y\} = -Q_+.$$

So it is enough to show that Q_+ has Lebesgue measure zero. We shall show that $Q_+ \subset A_5$ of Lemma 1. Let $x, y \in D$, with $x > y$, have decimal expansions $\cdot\alpha_1\alpha_2 \cdots$ and $\cdot\beta_1\beta_2 \cdots$ respectively. Let x_n and y_n be the numbers obtained from x, y by terminating their decimal expansions at the nth stage. Then

$$x_n - y_n = \cdot\alpha_1\alpha_2 \cdots \alpha_n - \cdot\beta_1\beta_2 \cdots \beta_n.$$

Since α's and β's take values 0 or 9 only (since $x, y \in D$) it follows that $x_n - y_n$ does not involve the number 5 in its decimal expansion. Hence $x - y = \lim_{n\to\infty}(x_n - y_n)$ does not involve the number 5 in its decimal expansion. So the Lebesgue measure of $Q_+ = 0$. Hence $L(Q) = L(Q_+) + L(Q_-) = 0$. q.e.d.

Remark. It is interesting to note that if D were the well-known Cantor ternary set, then the set $Q = \{x - y \cdot x, y \in D\}$ would be the entire interval from -1 up to 1.

Step 4. Let Q be the set of Lebesgue measure zero of Step 3. Write $F = \{(x+m)/n: x \in Q, m, n \text{ arbitrary integers}, n \neq 0\}$.

Since Q has Lebesgue measure zero, F has Lebesgue measure zero. Hence there exists an irrational number $\lambda \notin F$.

Step 5. Choose an irrational number $\lambda \notin F$, where F is as in Step 4. Let G be the group $m + \lambda n$, where m, n are integers. The group G is dense in R. (A nondense subgroup of R is necessarily isomorphic to the group of integers.) Let D be the Cantor decimal set of Step 1.

Lemma 3. *Translates of D by members of G are disjoint.*

Proof. Let $D + m + \lambda n$, $D + p + \lambda q$ be two translates of D. Suppose that $(D + m + \lambda n) \cap (D + p + \lambda q) \neq \emptyset$. Then there exists $x, y \in D$ such that $x + m + \lambda n = y + p + \lambda q$, i.e. $x - y = p - m + \lambda(q - n)$. If $q = n$, then $p = m$ (since $0 \leqq x, y < 1$) so that we do not have distinct translates. If $q \neq n$, then $(x - y + m - p)/(q - n) = \lambda$; but $(x - y + m - p)/(q - n) \in F$ and $\lambda \notin F$, so we again get a contradiction. Hence translates of D by members of G are disjoint.

Step 6 (*The Cantor function* f). Let $x \in I$ have the decimal expansion $x = \cdot\alpha_1\alpha_2\alpha_3, \cdots, \alpha_i = 0, 1, 2, \cdots, 9$. Let $n = n(x)$ be the first index for which $\alpha_n \in \{1, 2, \cdots, 8\}$ and $\alpha_n \notin \{0, 9\}$. If there is no such n, i.e., if $x \in D$, write $n(x) = \infty$. Define the function f by

$$f(x) = \frac{1}{9}\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{2^i}\right) + \frac{1}{2^n}, \qquad n = n(x).$$

The function f is continuous and monotonically nondecreasing with points of increase only in the set D of Lebesgue measure zero.

Step 7 (*Construction of m*). Let μ be the finite measure induced by the monotone function f of Step 6. μ is obviously nonatomic and singular with respect to the Lebesgue measure on I. Extend μ by setting $\mu(A)=0$ for sets A outside I. Let G be the countable dense subgroup of Step 5 and define m by

$$m(A) = \sum_{m,n=-\infty}^{\infty} \mu(A + m + \lambda_n) = \sum_{g\in G} \mu(A + g), \qquad A \in \mathcal{B}.$$

Clearly m is invariant under translation by members of G. Further m is nonatomic. Finally we observe that m is supported on $\bigcup_{g\in G}(D+g)$, the union of countable number of disjoint sets $D+g$, $g\in G$, and that $m(D+g)=m(D)=1$. Hence m is σ-finite. This completes the construction of m.

REMARK. We have constructed the measure m invariant under translation by a dense subgroup with two generators. But this is not a restriction. With little manipulation one can construct a σ-finite nonatomic singular measure invariant under translation by any countable subgroup of the real line.

2. From now on we shall denote by G a fixed countable dense subgroup of R. A measure m on $\mathcal{B}$ is called nonatomic singular G-invariant if

(i) m is nonatomic,

(ii) m is singular with respect to the Lebesgue measure on R,

(iii) $m(A+g)=m(A)$ for all $A\in\mathcal{B}$ and $g\in G$,

(iv) There exists a Borel set D of finite m measure such that the translates $D+g$, $g\in G$, of D by members of G are pair wise disjoint and $\bigcup_{g\in G}(D+g)$ supports m.

A method of constructing such measures was given in §1.

Now fix a continuous singular G-invariant measure m on R. Let $L_2(R, m)$ be the linear space of functions square integrable with respect to m. Let D be the set (the existence of which is guaranteed by (iv)) such that the sets $D+g=D_g$ are pair wise disjoint and $\bigcup_{g\in G} D_g$ supports m. Then clearly $L_2(R, m)=\sum_{g\in G}\oplus L_2(D_g, m)$ where $L_2(D_g, m)$ is the set of functions in $L_2(R, m)$ that vanish outside D_g. The orthogonal projection of $f\in L_2(R, m)$ on $L_2(D_g, m)$ is given by fI_g, where I_g is the characteristic function of D_g.

Now m is G-invariant so we get a group T^g, $g\in G$, of unitary operators defined on $L_2(R, m)$ by $(T^g f)(\lambda)=f(\lambda+g)$, $f\in L_2(R, m)$, $g\in G$.

Let us define a spectral measure E on $\mathcal{B}$ by writing $E(\sigma)f=I_\sigma f$, $f\in L_2(R, m)$ where I_σ is the characteristic function of σ. It is easily varified that E and T^g are connected by the relation $T^gE(\sigma)T^{-g} = E(\sigma - g)$ for all $g \in G$. But T^g is not the only commutative group of unitary operators which satisfies this equation with E. The general commutative group U^g, which with E, satisfies $U^gE(\sigma)U^{-g} = E(\sigma - g)$, $g \in G$, has the following form. U^g is defined by

$$(U^gf)(\lambda) = A(g, \lambda)f(\lambda + g), \qquad g \in G, f \in L_2(R, m),$$

where $A(g, \lambda)$ is an m-measurable function of λ for every fixed g such that

(i) $|A(g, \lambda)| = 1$,

(ii) $A(g+h, \lambda) = A(g, \lambda)A(h, \lambda + g)$ for almost all λ with respect to the m-measure.

The set of m-measure zero where (ii) does not hold may vary with the pair (g, h).

Functions satisfying the functional equation (ii) occur very crucially in the study of spectral measures E on $\mathcal{B}$ for which there exists a commutative group U^g, $g \in G$, satisfying the equation $U^gE(\sigma)U^{-g} = E(\sigma + g)$, $g \in G$, $\sigma \in \mathcal{B}$ (cf. §4).

The group U^g has a spectral measure associated with it as follows. (See [6, p. 392].)

Let $B = \hat{G}_d$ be the compact dual of G_d, the group G with the discrete topology. Since U^g is a commutative group of unitary operators, by Godement's extension of Stone's theorem on the representation of unitary operators [1] there exists a Hermitian projection valued spectral measure F on the Borel subsets $\mathfrak{F}$ of B such that $U^g = \int_B \chi_g(\lambda) dF_\lambda$ in the sense that

$$(*) \qquad (U^gf, h) = \int_B \chi_g(\lambda)(dF_\lambda f, h), \qquad f, h \in L_2(R, m).$$

Here χ_g denotes the character on B corresponding to $g \in G_d$. For $f, h \in L_2(R, m)$, $(F(\cdot)f, h)$ defines a complex valued finite measure on $\mathfrak{F}$ so that for $\sigma \in \mathfrak{F}$ the value of this measure is $(F(\sigma)f, h)$.

We show that for every f and h, $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on B.

THEOREM 1. *If $f \in L_2(D_{g_0}, m)$ for some $g_0 \in G$, then the measure $(F(\cdot)f, f)$ is a constant multiple of the Haar measure on B. For any $f, h \in L_2(R, m)$, the measure $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on B.*

PROOF. Let $f \in L_2(D_{g_0}, m)$, then $U^gf \in L_2(D_{g_0} - g, m)$. Hence, the elements $\{U^gf: g \in G\}$ are mutually orthogonal. Now by (*) $(U^gf, f) = \int_B \chi_g(\lambda)(dF_\lambda f, f) = 0$ if $g \neq 0$. Hence $(F(\cdot)f, f)$ is a constant multiple of the Haar measure on B. The constant multiple is, of course, nonzero if and only if $f\neq 0$ in $L_2(D_{g_0}, m)$. Now let $f\in L_2(D_{g_0}, m)$, $h\in L_2(D_{g_1}, m)$, then by the polarization formula it is easy to see that $(F(\cdot)f, h)$ is absolutely continuous with respect to the Haar measure on B. Finally choose any $f,h\in L_2(R, m)$. Let $f=\sum_{g\in G} f_g$, $h=\sum_{g\in G} h_g$, f_g, $h_g\in L_2(D_g, m)$. Then clearly

$$(F(\cdot)f, h) = \sum_{g,g'\in G} (F(\cdot)f_g, h_{g'}).$$

Since each $(F(\cdot)f_g, h_g')$ is absolutely continuous with respect to the Haar measure on B, it follows that $(F(\cdot)f, h)$ has the same property, q.e.d.

The next theorem shows that Wiener closure theorem has no analogue for a nonatomic singular G-invariant measure.

THEOREM 2. *There is no $f\in L_2(R, m)$ such that $\{U^g f \cdot g\in G\}$ spans $L_2(R, m)$.*

To prove this theorem we need a known result which we state here without proof for the sake of completeness.

LEMMMA 4. *Let μ be a finite positive regular measure on the Borel subsets $\mathfrak{F}$ of B. Let h, $f\in L_2(B, \mu)$. Then $\int_B \chi_g(\lambda)h(\lambda)\bar{f}(\lambda)d\mu=0$ for all $g\in G_d$ if and only if h vanishes almost everywhere with respect to μ on the set where $|f|>0$.*

This lemma is an easy consequence of the fact that a finite regular Borel measure on a locally compact abelian group is uniquely determined by its Fourier-Stieltjes transform [7, p. 17].

Consider the measures on $\mathfrak{F}$ defined by $\int_\sigma |h(\lambda)|^2 d\mu$, $\int_\sigma |f(\lambda)|^2 d\mu$. Then the lemma is equivalent to the following fact:

$$\int_B \chi_g(\lambda)h(\lambda)\overline{f(\lambda)}d\mu = 0$$

for all $g\in G_d$ if and only if the measures $\int_{(\cdot)} |h(\lambda)|^2 d\mu$ and $\int_{(\cdot)} |f(\lambda)|^2 d\mu$ are mutually singular.

PROOF OF THEOREM 2. Suppose that there exists $f\in L_2(R, m)$ such that $\{U^g f\cdot g\in G\}$ spans $L_2(R, m)$. By (*), $(U^g f, f)=\int\chi_g(\lambda)(dF_\lambda f, f)$ $=\int\chi_g(\lambda)d\mu$, where μ is the measure defined by $\mu(\sigma)=(F(\sigma)f, f)$, $\sigma\in\mathfrak{F}$. By Theorem 1, μ is absolutely continuous with respect to the Haar measure on B. The mapping S: $SU^g f=\chi_g$ extends by linearity to an invertible isometry from the space spanned by $\{U^g f$: $g\in G\}$ to $L_2(B, \mu)$. Now let D_1, D_2 be two disjoint measureable subsets of D

such that $D=D_1\cup D_2$ and $m(D_1)$, $m(D_2)>0$. Let h_1 and h_2 denote the characteristic functions of D_1 and D_2. Write $f_1=Sh_1$, $f_2=Sh_2$. It is clear that

(i) $U^g h_1$ are all mutually orthogonal in $L_2(R, m)$,

(ii) $U^g h_2$ are all mutually orthogonal in $L_2(R, m)$,

(iii) $U^g h_1 \perp U^{g'} h_2$ for all g, $g'\in G$.

This is because the translates of D by members of G are disjoint.

Now for all $g\in G$, $(U^g h_1, h_1)=\int \chi_g(\lambda)|f_1(\lambda)|^2 d\mu=\int_B \chi_g(\lambda)(dF_\lambda h_1, h_1)=0$ if $g\neq 0$. Similarly $(U^g h_2, h_2)=\int \chi_g(\lambda)|f_2(\lambda)|^2 d\mu=\int_B \chi_g(\lambda)(dF_\lambda h_2, h_2)=0$.

Hence the measures $(F(\cdot)h_1, h_1)$ and $(F(\cdot)h_2, h_2)$ are nonzero constant multiples of the Haar measure on B. Hence the measures $\int_{(\cdot)}|f_1(\lambda)|^2 d\mu$ and $\int_{(\cdot)}|f_2(\lambda)|^2 d\mu$ are nonzero constant multiples of the Haar measure on B. But $U^g h_1 \perp U^{g'} h_2$ for all g, g' by (iii). Hence $(U^g h_1, h_2)=\int_B \chi_g f_1 \bar{f}_2 d\mu=0$ for all g. Hence by Lemma 4 the measures $\int_{(\cdot)}|f_1(\lambda)|^2 d\mu$ and $\int_{(\cdot)}|f_2(\lambda)|^2 d\mu$ are mutually singular. This is a contradiction, q.e.d.

3. In this section we show how the measures of the type discussed in §2 are excluded in the problem of evanescent processes. First we must explain this problem.

Let G and B be as in §2. Let f be a nonzero positive function on B summable with respect to the Haar measure on B. Let $L_2(B, f) = \{\psi: |\psi|^2 f$ is summable with respect to the Haar measure on $B\}$. Let H_t be the subspace of $L_2(B, f)$ spanned by $\{\chi_g: g<t\}$, where χ_g denotes the character on B corresponding to the real number $g\in G$. It is clear that $H_t\subseteq H_{t'}$ whenever $t<t'$. It can be shown that either $H_t=H_{t'}$ for all t, t' or $\bigcap_t H_t=\{0\}$ and $H_t \subsetneqq H_{t'}$ whenever $t<t'$. This has been shown by Helson and Lowdenslager in their paper [3]. The problem of evanescent processes can be stated as follows: Assume that $H_t\neq H_{t'}$ for $t<t'$, then is it always true that $(\bigcap_{t>0} H_t)\ominus H_0\neq\{0\}$?

A well-known result of Helson and Lowdenslager [2] answers the question in the affirmative under the assumption that $\log f$ is summable with respect to the Haar measure on B. In what follows we give further evidence in favor of the affirmative answer to the question.

The increasing subspaces H_t give rise to a spectral measure E on the Borel subsets of R. For intervals $(a, b]$, E is given by $E(a, b]=$ orthogonal projection on $H_b\ominus H_a$. In $L_2(B, f)$ there is a commutative group U^g of unitary operators defined by $U^g\psi=\chi_g\psi$, $\psi\in L_2(B, f)$, $g\in G$. Further the following two identities are easily verified

(A) $U^g(H_b\ominus H_a)=H_{b+g}\ominus H_{a+g}$ where a, b $(a<b)$ are any two real numbers.

(B) For any $\psi\in L_2(B, f)$, $\|E(a, b]\psi-\psi\|^2=\|U^g E(a, b]\psi-U^g\psi\|^2$.

(A) and (B) together imply that U^g and E are connected by the relation $U^gE(\sigma)U^{-g}=E(\sigma+g)$ for all $\sigma\in\mathcal{B}$ and $g\in G$. Helson and Lowdenslager have shown that if $E\{x\}\neq 0$ for some x, then the spectral measure E is purely discrete and E has no continuous component. Now it can be shown that E cannot have a component absolutely continuous with respect to the Lebesgue measure on R, i.e., there does not exist a nonzero $\psi\in L_2(B, f)$ such that $(E(\cdot)\psi, \psi)$ is absolutely continuous with respect to the Lebesgue measure on R. In what follows we show that E has no component absolutely continuous with respect to a nonatomic singular G-invariant measure on R.

THEOREM 3. *Assume that $E\{x\}=0$ for all x. There does not exist a Borel set D such that:*
(i) *the sets $D+g$, $g\in G$ are mutually disjoint,*
(ii) $E(D)\neq 0$.

PROOF. Suppose not. Then there exists a set D such that the sets $D+g$, $g\in G$, are mutually disjoint and $E(D)\neq 0$. Since E has no discrete spectrum, we can find two nonzero vectors Φ, ψ in $E(D)$ such that Φ and ψ are mutually orthogonal. Now $U^g\Phi=U^gE(D)\Phi=E(D+g)U^g\Phi\in E(D+g)$ and similarly $U^g\psi\in E(D+g)$. Since the sets $D+g$, $g\in G$, are mutually disjoint, we see that $U^g\Phi\perp\Phi$, $U^g\psi\perp\psi$ for all $g\neq 0$ and $U^g\Phi\perp U^{g'}\psi$ for all g, g'. So

(i) $(U^g\Phi, \Phi)=\int_B\chi_g(\lambda)|\Phi(\lambda)|^2f(\lambda)d\sigma=0$ for $g\neq 0$.
(ii) $(U^g\psi, \psi)=\int_B\chi_g(\lambda)|\psi(\lambda)|^2f(\lambda)d\sigma=0$ for $g\neq 0$.
(iii) $(U^g\Phi, \psi)=\int_B\chi_g(\lambda)\Phi(\lambda)\overline{\psi(\lambda)}f(\lambda)d\sigma=0$ for all g.
(Here $d\sigma$ is the normalized Haar measure on B.)

The first two equations above say that $|\Phi|^2fd\sigma$ and $|\psi|^2fd\sigma$ are nonzero constant multiples of the Haar measure on B and the third equation says that $\Phi\bar{\psi}f$ is equal to zero almost everywhere with respect to the Haar measure on B. This is impossible, q.e.d.

4. Let E be a spectral measure on the Borel subsets of R and let G be a countable dense subgroup of R. We call a spectral measure E G-stationary if there exists a commutative group U^g of unitary operators such that $U^gE(\sigma)U^{-g}=E(\sigma+g)$ for all $\sigma\in\mathcal{B}$ and $g\in G$. If one tries to obtain the canonical representation of G-stationary spectral measures like the one there is for a pair of commutative groups of unitary operators satisfying Weyl's commutativity relation one at once faces the following question.

Let μ be a finite positive measure on $\mathcal{B}$. Call μ G-quasi invariant if μ and μ_g are mutually absolutely continuous for all $g\in G$. Here μ_g is defined by $\mu_g(A)=\mu(A+g)$, $A\in\mathcal{B}$, $g\in G$.

QUESTION 1. μ is G-quasi invariant. Does there exist a σ-definite measure m on $\mathcal{B}$ such that (i) $m(\sigma+g)=m(\sigma)$ for all $\sigma\in\mathcal{B}$, $g\in G$, (ii) m and μ are mutually absolutely continuous?

Now suppose that $\mu=\mu^d+\mu^a+\mu^s$ where μ^d is the atomic part of μ, μ^a = part of μ absolutely continuous with respect to the L, the Lebesgue measure, and μ^s = nonatomic singular part of μ. It is easy to see that each component μ^d, μ^a and μ^s is separately G-quasi invariant.

Further μ^a and the Lebesgue measure are mutually absolutely continuous. Thus for μ^a the question raised above has a solution. One can also show easily that the question raised above has a solution for μ^d. Hence in the question raised above one can assume that μ is nonatomic singular measure.

We give a reformulation of our question in terms of the functions $A(g,\lambda)=(d\mu_g/d\mu)(\lambda)$. One verifies very easily that $A(g,\lambda)$ satisfy the relation $A(g+h,\lambda)=A(g,\lambda)A(h,\lambda+g)$ a.e. $[\mu]$.

THEOREM 4. *Question 1 has a solution if and only if there exists a measurable function B such that $A(g,\lambda)=B(\lambda+g)/B(\lambda)$.*

PROOF. Suppose there exists an m as in Question 1. Write $B(\lambda)=(d\mu/dm)(\lambda)$. Then clearly $A(g,\lambda)=(d\mu_g/d\mu)(\lambda)=(d\mu_g/dm)(\lambda)\cdot(dm/d\mu)(\lambda)=(d\mu_g/dm)(\lambda)\cdot 1/B(\lambda)$. Now by the invariance of m under translation by g it is easy to see that $(d\mu_g/dm)(\lambda)=B(\lambda+g)$; thus $A(g,\lambda)=B(\lambda+g)/B(\lambda)$. Conversely suppose that $A(g,\lambda)=B(\lambda+g)/B(\lambda)$ where B is measurable. Define m by $m(\sigma)=\int_\sigma[B(\lambda)]^{-1}d\mu$. It is clear that m and μ are mutually absolutely continuous. Next to see the G-invariance of m we note that

$$\begin{aligned} m(\sigma+g) &= \int_{\sigma+g}[B(\lambda)]^{-1}d\mu = \int_\sigma [B(\lambda+g)]^{-1}d\mu_g(\lambda) \\ &= \int_\sigma [B(\lambda+g)]^{-1}\frac{d\mu_g}{d\mu}(\lambda)d\mu = \int_\sigma \frac{[B(\lambda+g)]^{-1}B(\lambda+g)}{B(\lambda)}d\mu \\ &= \int [B(\lambda)]^{-1}d\mu = m(\sigma), \qquad \text{q.e.d.} \end{aligned}$$

We conclude by making the following remarks.

Assume that μ of Question 1 is singular. In order that Question 1 have an affirmative solution it is enough that there is a μ-measurable set D such that $D+g$, $g\in G$ are disjoint and $\bigcup_{g\in G}(D+g)$ supports μ. However, there exist singular G-quasi invariant measures for which no such D exists. We illustrate this by the following example. Let C

be the Cantor ternary set and ψ the Cantor function from C onto $[0, 1]$. ψ is strictly increasing and continuous on C with range $[0, 1]$. Let P be the singular measure on the real line induced by ψ. Let G be the group of real members having finitely many terms in their ternary expansions. Let $g_1, g_2, g_3, \cdots$ be a denumeration of G. Write $\mu(A)=\sum_{n=1}^{\infty}(1/2^n)P(A+g_n)$, $A\in\mathfrak{B}$. Clearly μ is G-quasi invariant. Call two members of C equivalent if their difference belongs to G. This equivalence relation partitions C. Choose a member from each equivalence class and call the new set D. Translates $D+g$, $g\in G$ are disjoint and $\bigcup_{g\in G}(D+g)$ supports μ. But D can never be chosen to be μ-measurable, for the difference of two members of $\psi(D)$ has always finite binary expansions, so that $\psi(D)$ is nonmeasurable. Hence $D=\psi^{-1}(\psi(D))$ is non-μ-measurable.

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