Lazy Rectangular Hybrid Automata

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Abstract. We introduce the class of *lazy* rectangular hybrid automata. The key feature of this class is that both the observation of the continuous state and the rate changes associated with mode switchings take place with bounded delays. We show that the discrete time dynamics of this class of automata can be effectively analyzed without requiring resetting of the continuous variables during mode changes.

1 Introduction

We introduce here a class of linear rectangular hybrid automata called *lazy hybrid automata* and study its discrete time behavior. An important feature of this class is that the sensors report the current values of the variables and the actuators effect changes in the rates of evolution of the variables with bounded delays. More specifically, the state observed at T_k is a state that held at some time in a bounded interval contained in (T_{k-1}, T_k). Further, if an instantaneous mode change has taken place at T_k, then any necessary change in the rate of a variable will not kick in immediately. Rather, it will do so at some time in a bounded interval contained in (T_k, T_{k+1}). A final -but realistic- restriction is that each variable's allowed range of values is bounded. For convenience, we study the case where there is a single rate vector associated with each control state instead of a bounded rectangular region of vectors as is customary for rectangular hybrid automata [2].

Since both sensors and actuators have delays associated with them, a single symbolic trajectory of the automaton can give rise to uncountably many concrete trajectories; even in discrete time setting with the initial region being a singleton. Hence computing the discrete time behavior of a lazy hybrid automaton is non-trivial. Our main result is that this can be carried out effectively. As a corollary, we also show that the discrete time behavior of a network of lazy hybrid automata that communicate by synchronizing on common actions can be effectively computed.

As is well known, the continuous variables available to an hybrid automaton and the fact that their rates of evolution can change instantaneously during a mode switch endows them with a great deal of expressive power. As a result, in a variety of settings, the control state reachability problem becomes undecidable, as reported for instance, in [3]. A sharp characterization of the boundary between decidable and undecidable features of hybrid automata is provided in [7] as well as [2]. These results, as also the positive results reported elsewhere - for example, [4, 10, 9, 8] - make it clear that except under very restrictive settings, one can not expect to get decidability if the continuous variables don't get reset during mode changes; particularly in case their rates change as a result of the mode change. Viewed as a model of digital control systems that interact with physical plants through sensors and actuators, the resetting requirement severely restricts the modeling power of the automaton. Our results show that by introducing bounded delays into the functioning of the sensors and actuators, we can allow the variables to retain their values during mode changes. Admittedly, our positive results are obtained in the restricted setting of rectangular hybrid automata but the wealth of research concerning this setting (for instance, [5, 7, 4, 6]) suggests that this is a natural and well motivated starting point.

We study the discrete time semantics of lazy hybrid automata. From a technical standpoint, our work is a generalization of [6] where the discrete time behavior of rectangular hybrid automata is studied with the requirement that all instantaneous transitions should take place only at integer-valued instances of time. In our terms, [6] further assumes that the sensors and actuators function with zero delays which simplifies their analysis problem. In our setting, things are more complicated due to the non-zero delays associated with the sensing of values and actuating rate changes. Further, we feel that the approach proposed here is of some independent interest from a modeling point of view. It also has the potential to lead to the tractable analysis of larger classes of hybrid automata. Finally, our focus on discrete time semantics is relevant -as also argued in [6]- in that, as a model of digital controllers for continuous plants, the discrete time semantics of hybrid automata is more natural and useful than the continuous time semantics.

In the next section, we formulate the model of lazy hybrid automata. In section 3 we prove our main result, namely, the language of state sequences and action sequences generated by a lazy hybrid automaton are regular. Moreover, finite state automata representing these languages can be effectively computed. In section 4 we discuss the restrictions placed on lazy automata and point out that many of them can be easily relaxed. We also show that our main result can be easily extended to networks of automata which communicate by performing common actions together. In the concluding section we briefly discuss the prospects for extending the results reported here.

2 Lazy Hybrid Automata

Fix a positive integer n and one function symbol x_i for each i in $\{1, 2, \ldots, n\}$. We will view each x_i as a function $x_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ with $\mathbb{R}$ being the set of reals and $\mathbb{R}_{\geq 0}$, the set of non-negative reals. We let $\mathbb{Q}$ denote the set of rationals and

$\mathcal{I}$ denote the set of closed intervals of the form $[l, r]$ with $l, r \in \mathbb{Q}$ and $l < r$. We view $[l, r]$ as the subset of $\mathbb{R}$ given by $\{z \mid l \leq z \leq r\}$.

A *lazy hybrid automaton* is a structure
$\mathcal{A} = (Q, Act, q_{in}, V_{in}, D, \{\rho_q\}_{q \in Q}, \mathrm{B}, \longrightarrow)$ where:

- Q is a finite set of *control states.*
- Act is a finite set of *actions.*
- $q_{in} \in Q$ is the initial control state.
- $V_{in} \in \mathbb{Q}^n$ is the initial valuation.
- $D = \{g, \delta_g, h, \delta_h\} \subseteq \mathbb{Q}$ is the *set of delay parameters* such that $0 < g < g + \delta_g < h < h + \delta_h < 1$.
- $\rho_q \in \mathbb{Q}^n$ is a rate vector which specifies the rate $\rho_q(i)$ at which each x_i evolves when the system is in the control state q.
- $\mathrm{B} = [B_{min}, B_{max}] \in \mathcal{I}$ is the *allowed range.*
- $\longrightarrow \subseteq Q \times Act \times \mathcal{I}^n \times Q$ is a transition relation such that $q \neq q'$ for every (q, a, I, q') in $\longrightarrow$. Furthermore, if $(q, a, I, q'), (q, a, I', q') \in \longrightarrow$ then $I = I'$.

We shall study the discrete time behavior of our automata. At each time instant T_k, the automaton receives a measurement regarding the current values of the x_i's. However, the value of x_i that is observed at T_k is the value that held at some $t \in [T_{k-1} + h, T_{k-1} + h + \delta_h]$. If at T_k, the automaton is in control state q and observed n-tuple of values $(v_1, v_2, \ldots, v_n)$ is in I with (q, a, I, q') being a transition, then the automaton may perform this transition instantaneously by executing the action a and move to the control state q'. Thus as usual, the x_i's will cease to evolve at the rates ρ_q and instead start evolving at the rates $\rho_{q'}$. However, this change in the rate of evolution will not kick in at T_k but at some time $t \in [T_k + g, T_k + g + \delta_g]$. In this sense, both the sensing of the values of the x_i's and the rate changes associated with mode switching take place in a lazy fashion but with bounded delays.. We expect g to be close to 0 and h to be close to 1 so that in the idealized setting, the change in rates due to mode switching would kick in immediately $(g = 0)$ and the value observed at T_k is the value that holds at exactly T_k $(h = 1)$. Indeed, this is the setting considered in [6].

B specifies the range of values within which the automaton's dynamics are valid. The automaton gets stuck if any of the x_i's ever assume a value outside the allowed range $[B_{min}, B_{max}]$. A number of the restrictions that we have imposed are mainly for ease of presentation. Later, we will discuss how these restrictions can be relaxed. Our main result is that the control state and action sequence languages generated by a lazy hybrid automaton are both regular. Furthermore, these language can be computed effectively.

2.1 The Transition System Semantics

Through the rest of this section we fix a lazy hybrid automaton $\mathcal{A}$ as defined above and assume its associated notations and terminology. The behavior of $\mathcal{A}$ will be defined in terms of an associated transition system.

A *valuation* is just a member of $\mathbb{R}^n$. We let i range over $\{1, 2, \ldots, n\}$. The valuation V will be viewed as prescribing the value $V(i)$ to each variable x_i.

A *configuration* is a triple (q, V, q') where q, q' are control states and V is a valuation. q is the control state holding at the current time instant and q' is the control state that held at the previous time instant. V captures the *actual* values of the variables at the current instance. The configuration (q, V, q') is *feasible* iff $V(i) \in [B_{min}, B_{max}]$ for every i. The initial configuration is, by convention, the triple (q_{in}, V_{in}, q_{in}). We assume without loss of generality that the initial configuration is feasible. We let $C_{\mathcal{A}}$ denote the set of configurations. Since $\mathcal{A}$ will be clear from the context, we will often write C instead of $C_{\mathcal{A}}$.

As in the case of timed automata [1], a convenient way to understand the dynamics is to break up each move of the automaton into a time-passage move followed by an instantaneous transition. At T_0, the automaton will be in its initial configuration. Suppose the automaton is in the configuration (q_k, V_k, q_{k+1}) at T_k. Then one unit of time will pass[3] and at time instant T_{k+1}, the automaton will make an instantaneous move by performing an action a or the silent action τ and move to a configuration $(q_{k+1}, V_{k+1}, q'_{k+1})$. The silent action will be used to record that no mode change has taken place during this move. Again, as often done in the case of timed automata, we will collapse the two sub-steps of a move (unit-time-passage followed by an instantaneous transition) into one "time-abstract" transition labeled by a member of *Act* or by τ.

With this intuition in mind, we now define the transition relation $\Longrightarrow \subseteq C \times Act \cup \{\tau\} \times C$ as follows.

- Let (q, V, q') and $(q1, V1, q1')$ be configurations and $a \in Act$. Then $(q, V, q') \overset{a}{\Longrightarrow} (q1, V1, q1')$ iff $q1' = q$ and there exists in $\mathcal{A}$ a transition of the form $q \xrightarrow{a,I} q1$ and there exist $\widehat{t1} \in [g, g + \delta_g]^n$ and $\widehat{t2} \in [h,\, h + \delta_h]^n$ such that the following conditions are satisfied for each i.
  (1) $V1(i) = V(i) + \rho_{q'}(i) \cdot \widehat{t1}(i) + \rho_q(i) \cdot (1 - \widehat{t1}(i))$.
  (2) $V(i) + \rho_{q'}(i) \cdot \widehat{t1}(i) + \rho_q(i) \cdot (\widehat{t2}(i) - \widehat{t1}(i)) \in I(i)$ for each i.
- Let (q, V, q') and $(q1, V1, q1')$ be configurations. Then $(q, V, q') \overset{\tau}{\Longrightarrow} (q1, V1, q1')$ iff $q1 = q1' = q$ and there exists $\widehat{t1} \in [g,\, g + \delta_g]^n$ such that $V1(i) = V(i) + \rho_{q'}(i) \cdot \widehat{t1}(i) + \rho_q(i) \cdot (1 - \widehat{t1}(i))$ for each i.

Basically there are four possible transition types as illustrated in Figure 2.1 depending on whether $q = q'$ and $\alpha \in Act$. Suppose $(q, V, q') \overset{a}{\Longrightarrow} (q1, V1, q1')$ with $a \in Act$. Assume that $q \xrightarrow{a,I} q1$ in $\mathcal{A}$ and $\widehat{t1} \in [g,\, g{+}\delta_g]^n$ and $\widehat{t2} \in [h,\, h{+}\delta_h]^n$ are as specified above. We first note that $q1 \neq q$ by the definition of the transition relation of $\mathcal{A}$. The requirement $q1' = q$ captures follows from our convention that $q1'$ is the control state that held in the previous instant and we know this was q.

First consider the case $q \neq q'$ and let us suppose that the configuration (q, V, q') holds at T_k. We take $q \neq q'$ to mean that a change of mode from q' to q has just taken place (instantaneously) at T_k based on the observations that were made available at T_k. However, at T_k, the automaton will continue

[3] We assume that the unit of time has been fixed at some suitable level of granularity and that the rate vectors $\{\rho_q\}$ have been scaled accordingly.

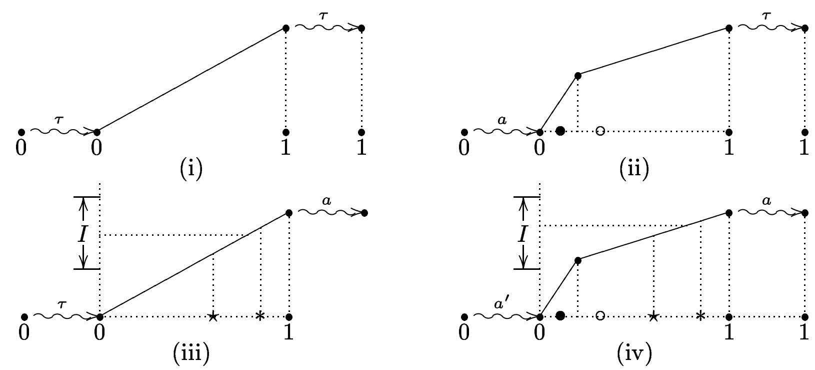

Fig. 2.1. The four transition types

to evolve at the rate dictated by ρ'_q. Indeed, each x_i will, starting from T_k, evolve at rate $\rho'_q(i)$ until some $T_k + t_1$ with $t_1 \in [g,\, g + \delta_g]$. It will then start to evolve at rate $\rho_q(i)$ until T_{k+1}. Consequently, at T_{k+1}, the value of x_i will be $V1(i) = V(i) + \rho'_q(i) \cdot t_1 + \rho_q(i) \cdot (1 - t_1)$. On the other hand, $q1 \neq q$ implies that another instantaneous mode change has taken place at T_{k+1} based on the measurements made in the interval $[T_k{+}h,\, T_k{+}h{+}\delta_h]$. Suppose x_i was measured at $T_k{+}t_2$ with $t_2 \in [T_k{+}h,\, T_k{+}h{+}\delta_h]$. Then in order for the transition $q \xrightarrow{a,I} q1$ of $\mathcal{A}$ to be enabled at T_{k+1}, it must be the case that the observed value of x_i at $T_k + t_2$ falls in $I(i)$. But then this value is $V(i) + \rho'_q(i) \cdot t_1 + \rho_q(i) \cdot (t_2 - t_1)$. This explains the demands placed on the transition $(q, V, q') \overset{a}{\Longrightarrow} (q1, V1, q1')$. It is worth noting that in case $q = q'$ (i.e. no mode change has taken place at T_k) then $V1(i) = V(i) + \rho_q(i) \cdot t_1 + \rho_q(i) \cdot (1 - t_1) = V(i) + \rho_q$ as it should be. Furthermore, $V(i) + \rho_q(i) \cdot t_1 + \rho_q(i) \cdot (t_2 - t_1) = V(i) + \rho_q(i) \cdot t_2$ and this must fall in $I(i)$ as to be expected.

Similar (and simpler) considerations motivate the demands placed on transitions of the form $(q, V, q') \overset{\tau}{\Longrightarrow} (q1, V1, q1')$. Here again, it is worth noting that, in case $q = q'$, $V1(i)$ is determined uniquely, namely, $V1(i) = V(i) + \rho_q(i)$.

We now define the transition system
$TS_{\mathcal{A}} = (RC_{\mathcal{A}}, (q_{in}, V_{in}, q_{in}), Act \cup \{\tau\}, \Longrightarrow_{\mathcal{A}})$ via:

- $RC_{\mathcal{A}}$, the set of *reachable configurations* of $\mathcal{A}$ is the least subset of C that contains the initial configuration (q_{in}, V_{in}, q_{in}) and satisfies:
  Suppose (q, V, q') is in $RC_{\mathcal{A}}$ and is a feasible configuration. Suppose further, $(q, V, q') \overset{\alpha}{\Longrightarrow} (q1, V1, q)$ for some $\alpha \in Act \cup \{\tau\}$. Then $(q1, V1, q) \in RC_{\mathcal{A}}$.
- $\Longrightarrow_{\mathcal{A}}$ is $\Longrightarrow$ restricted to $RC_{\mathcal{A}} \times Act \cup \{\tau\} \times RC_{\mathcal{A}}$.

We will often write RC instead of $RC_{\mathcal{A}}$ and write $\Longrightarrow$ instead of $\Longrightarrow_{\mathcal{A}}$. We note that a reachable configuration can be the source of a transition in $TS_{\mathcal{A}}$ only if it is feasible. Thus infeasible reachable configurations will be deadlocked in $TS_{\mathcal{A}}$.

A *run* of $TS_{\mathcal{A}}$ is a finite sequence of the form
$\sigma = (q_0, V_0, q_0')\, \alpha_0\, (q_1, V_1, q_1')\, \alpha_1\, (q_2, V_2, q_2') \ldots (q_k, V_k, q_k')$ where (q_0, V_0, q_0') is the initial configuration and $(q_m, V_m, q_m') \overset{\alpha_m}{\Longrightarrow} (q_{m+1}, V_{m+1}, q_{m+1}')$ for $0 \leq m < k$. The *st-sequence (state sequence)* induced by the run σ above is denoted as $st(\sigma)$ and it is the the sequence $q_0 q_1 \ldots q_n$. On the other hand, the *act-sequence* induced by σ is denoted as $act(\sigma)$ and it is the sequence $\alpha_0 \alpha_1 \ldots \alpha_n$. We now define the languages $\mathcal{L}_{st}(\mathcal{A})$ and $\mathcal{L}_{act}(\mathcal{A})$ as :

- $\mathcal{L}_{st}(\mathcal{A}) = \{st(\sigma) \mid \sigma \text{ is a run of } \mathcal{A}\}$.
- $\mathcal{L}_{act}(\mathcal{A}) = \{act(\sigma) \mid \sigma \text{ is a run of } \mathcal{A}\}$.

Our main result is that $\mathcal{L}_{st}(\mathcal{A})$ is a regular subset of $Q^\star$ while $\mathcal{L}_{act}(\mathcal{A})$ is a regular subset of $(Act \cup \{\tau\})^\star$. Moreover, we can effectively construct finite state automata representing these languages. As a consequence, a variety of verification problems for lazy hybrid automata can be effectively solved.

3 Proof of the Main Result

We shall first establish the main result for the one dimensional case. As is often the case with rectangular hybrid automata [4], it will then be easy to lift the proof to the n-dimensional case with the help of a (Cartesian) product operation.

3.1 The One Dimensional Case

Let $\mathcal{A} = (Q, Act, q_{in}, V_{in}, D, \{\rho_q\}_{q \in Q}, \mathrm{B}, \longrightarrow)$ be a lazy automaton. We assume for $\mathcal{A}$, the terminology and notations defined in the previous section. Until further notice , we set $n = 1$ and we will write x instead of x_i and ρ_q instead of $\rho_q(i)$ for $q \in Q$.

The key idea is quantize the unit time interval and correspondingly the phase interval $[B_{min},\, B_{max}]$. We first define Δ to be the largest positive rational number that *integrally* divides every number in the finite set of rational numbers $\{g, \delta_g, h, \delta_h, 1\}$. We next define Γ to be the largest positive rational number that *integrally* divides each number in the finite set of rational numbers $\{\rho_q \cdot \Delta \mid q \in Q\} \cup \{B_{min}, B_{max}\} \cup \{l, r \mid (q, a, [l, r], q') \text{ is a transition in } \mathcal{A}\}$.

Let $\mathbb{Z}$ denote the set of integers. We now define the map
$\|\cdot\| : \mathbb{R} \to \mathbb{Z} \times (\{0,1\} \cup \{\perp\})$ as follows.

- If $v \in (-\infty, B_{min})$, then $\|v\| = (k_{min} - 1, \perp)$ where $k_{min} \cdot \Gamma = B_{min}$. If $v \in (B_{max}, \infty)$ then $\|v\| = (k_{max}, \perp)$ where $k_{max} \cdot \Gamma = B_{max}$.
- Suppose $v \in [B_{min},\, B_{max}]$ and $k \in \mathbb{Z}$ and $\widehat{v} \in [0,\, \Gamma)$ such that $v = k \cdot \Gamma + \widehat{v}$. Then $\|v\| = (k, 0)$ if $\widehat{v} = 0$ and $\|v\| = (k, 1)$ if $\widehat{v} \neq 0$.

The map $\|\cdot\|$ can be extended in a natural way to configurations. Denoting this extension also as $\|\cdot\|$, we define $\|(q, v, q')\|$ to be $(q, \|v\|, q')$. Let $\mathcal{D}_{\mathcal{A}} = \{\|c\| \mid c \in C_{\mathcal{A}}\}$. Clearly $\mathcal{D}_{\mathcal{A}}$ is a finite set and we will often write $\mathcal{D}$ instead of $\mathcal{D}_{\mathcal{A}}$. Our goal is to show that the equivalence relation over the reachable configurations RC of $\mathcal{A}$ induced by the map $\|\cdot\|$ in turn induces a right congruence of finite index over $Q^\star$. The proof will make use of the following simple observation. In stating the observation and elsewhere, we will use the following notations. For $q, q' \in Q$ we let N_q and $N_{qq'}$ be the positive integers such that $|\rho_q \cdot \Delta| = N_q \cdot \Gamma$ and $\left|(\rho_q - \rho_q') \cdot \Delta\right| = N_{qq'} \cdot \Gamma$. Clearly, N_q and $N_{qq'}$ exist because of the choice of Δ and Γ.

Lemma 1. *Let $q, q' \in Q$. Define the functions f_q and $f_{qq'}$ as:*

(1) $f_q : [0,\ \Delta/N_q] \to [0,\ \Gamma]$ and is given by $f_q(\theta) = |\rho_q \cdot \theta|$.
(2) $f_{qq'} : [0,\ \Delta/N_{qq'}] \to [0,\ \Gamma]$ and is given by $f_{qq'}(\theta) = |(\rho_q - \rho_{q'}) \cdot \theta|$.

Then both f_q and $f_{qq'}$ are well-defined, continuous and onto.

Proof. Follows easily from the definitions and the basic property of monotonic real valued functions over bounded domains. □

We are now ready to tackle the main part of the proof.

Theorem 1. *Let $c1$ and $c2$ be two reachable configurations such that $\|c1\| = \|c2\|$. Suppose $\alpha \in Act \cup \{\tau\}$ and $c1'$ is a reachable configuration such that $c1 \overset{\alpha}{\Longrightarrow}_{\mathcal{A}} c1'$. Then there exists a reachable configuration $c2'$ such that $c2 \overset{\alpha}{\Longrightarrow}_{\mathcal{A}} c2'$ and $\|c1'\| = \|c2'\|$.*

Proof. Clearly $c1$ is feasible and since $\|c1\| = \|c2\|$, it follows that $c2$ is also feasible.

Assume that $c1 = (q1, V1, q1')$ and $c2 = (q2, V2, q2')$ and that $\|V1\| = (K1, z1)$ and $\|V2\| = (K2, z2)$. Since $\|c1\| = \|c2\|$, we can set $q = q1 = q2$, $q' = q1' = q2'$ and $(K, z) = (K1, z1) = (K2, z2)$. If $z = 0$ then $V1 = V2$ and hence $c1 = c2$ and the result follows.

So assume that $z = 1$ and $V1 \neq V2$. Hence $V1, V2 \in (K.\Gamma, (K+1).\Gamma)$ and hence $\|(q, V1, q')\| = \|(q, V2, q')\| = (q, (K, 1), q')$. Furthermore, there exist $v1, v2 \in (0, \Gamma)$ such that $v1 \neq v2$ and $V1 = K \cdot \Gamma + v1$ and $V2 = K \cdot \Gamma + v2$. In what follows, for the sake of convenience, we will assume that $0 \leq \rho_{q'} \leq \rho_q$ and that $v2 < v1$. From the structure of the proof it will be obvious that this involves no loss of generality.

Let $c1' = (s, V1', q)$. Then we have $(q, V1, q') \overset{\alpha}{\Longrightarrow} (s, V1', q)$. We are required to show that there exists $V2'$ such that $(q, V2, q') \overset{\alpha}{\Longrightarrow} (s, V2', q)$ with $\|V1'\| = \|V2'\|$. We shall do this by considering four cases.

Case 1: $q = q'$ **and** $\alpha = \tau$.

Since $q = q'$, no mode change has taken place in the previous time interval. Hence the automaton will evolve at rate ρ_q during the current unit interval. On the other hand, $\alpha = \tau$ implies that $s = q$ and hence no mode change takes place at the end of this unit interval either. Consequently, we must have $V1' = V1 + \rho_q$. We now set $V2' = V2 + \rho_q$. Then it follows that $(q, V2, q') \overset{\alpha}{\Longrightarrow} (q, V2', q)$. We need to argue that $\|V1'\| = \|V2'\|$.

In what follows, we define for $\zeta \in \{g, \delta_g, h, \delta_h, 1\}$, N_ζ to be the positive integer satisfying $\zeta = N_\zeta \cdot \Delta$. These positive integers must exist by the choice of Δ.

Now $\rho_q = \rho_q \cdot 1 = \rho_q \cdot N_1 \cdot \Delta = N_q \cdot N_1 \cdot \Gamma$. (Recall that $\rho_q \cdot \Delta = N_q \cdot \Gamma$). But then $V1, V2 \in (K \cdot \Gamma, (K+1) \cdot \Gamma)$ and hence $V1', V2' \in ((K + N_q \cdot N_1) \cdot \Gamma,\ (K + 1 + N_q \cdot N_1) \cdot \Gamma)$. This at once leads to $\|V1'\| = \|V2'\|$.

Case 2: $q = q'$ **and** $\alpha \in Act$.

Since $q = q'$ we again have that no mode change has taken place in the previous interval and hence the automaton will evolve at rate ρ_q in the current interval. Hence, as in the previous case, we must have $V1' = V1 + \rho_q$. Again, we set $V2' = V2 + \rho_q$. Consequently as shown in the previous case, $\|V1'\| = \|V2'\|$. So if we show that $(q, V2, q') \overset{\alpha}{\Longrightarrow} (s, V2', q)$, then we are done.

We are given that $(q, V1, q') \overset{\alpha}{\Longrightarrow} (s, V1', q)$. Hence there exists a transition of the form (q, α, I, s) in $\mathcal{A}$ and there exists $t1 \in [h,\ h+\delta_h]$ such that $V1 + \rho_q \cdot t1 \in I$. We just need to show that there exists $t2 \in [h,\ h+\delta_h]$ such that $V2 + \rho_q \cdot t2 \in I$.

In order to fix $t2$, recall that $h = N_h \cdot \Delta$ and $\delta_h = N_{\delta_h} \cdot \Delta$. We first note that $t1 \in [N_h \cdot \Delta,\ (N_h + N_{\delta_h}) \cdot \Delta]$. Noticing that $\rho_q \cdot \Delta = N_q \cdot \Gamma$ and hence $\rho_q \cdot (\Delta / N_q) = \Gamma$ we set $\Delta_q = \Delta / N_q$, and observe that $t1 \in [N_h \cdot N_q \cdot \Delta_q,\ (N_h + N_{\delta_h}) \cdot N_q \cdot \Delta_q]$. Let N be the least integer in the interval $[N_h \cdot N_q,\ (N_h + N_{\delta_h}) \cdot N_q]$ such that $t1 \in [N \cdot \Delta_q,\ (N+1) \cdot \Delta_q]$. Let $\theta 1 = t1 - N \cdot \Delta_q$. Clearly $\theta 1 \in [0,\ \Delta_q]$.

Suppose $\theta 1 = 0$. Then $\rho_q \cdot t1 = \rho_q \cdot N \cdot \Delta_q = N \cdot \Gamma$ and hence $\widehat{V1} = V1 + \rho_q \cdot t1 \in ((K+N) \cdot \Gamma,\ (K+1+N) \cdot \Gamma)$. Set $t2 = t1$. Then $\widehat{V2} = V2 + \rho_q \cdot t1 \in ((K+N) \cdot \Gamma,\ (K+1+N) \cdot \Gamma)$ too. Now assume that $I = [l,\ r]$. Then there exist integers N_l and and N_r such that $l = N_l \cdot \Gamma$ and $r = N_r \cdot \Gamma$ with $N_l < N_r$. Since $\widehat{V1} \in [l,\ r]$, we must have $N_l \leq (K+N) < (K+N+1) \leq N_r$. But this implies that $\widehat{V2} = V2 + \rho_q \cdot t1 \in [l, r]$ too. Hence $(q, V2, q') \overset{\alpha}{\Longrightarrow} (s, V2', q)$.

The case $\theta 1 = \Delta_q$ can be dealt with in a similar manner by again setting $t2 = t1$.

So now assume that $\theta 1 \in (0,\ \Delta / N_q)$. Then clearly $\widehat{V1} = V1 + \rho_q \cdot t1 \in [v1 + (K+N) \cdot \Gamma,\ v1 + (K+N+1) \cdot \Gamma]$. (Recall that $v1 = V1 - K \cdot \Gamma$ and $v2 = V2 - K \cdot \Gamma$.) There are three possibilities to consider.

Firstly, suppose $\widehat{V1} \in [v1 + (K+N) \cdot \Gamma,\ (K+N+1) \cdot \Gamma)$. Then we set $t2 = N \cdot \Delta_q$. Clearly $\widehat{V2} = V2 + \rho_q \cdot N \cdot \Delta_q \in ((K+N) \cdot \Gamma, (K+N+1) \cdot \Gamma)$. But then $\widehat{V1} \in [v1 + (K+N) \cdot \Gamma,\ (K+N+1) \cdot \Gamma)$ implies $\widehat{V1} \in ((K+N) \cdot \Gamma,\ (K+N+1) \cdot \Gamma)$. Consequently $\widehat{V1} \in [l,\ r]$ implies $N_l \leq (K+N) < (K+N+1) \leq N_r$ as before and this in turn implies $\widehat{V2} \in [l,\ r]$. This leads to $(q, V2, q') \overset{\alpha}{\Longrightarrow} (s, V2', q)$.

Secondly, suppose $v1 = (K+N+1) \cdot \Gamma$. Then, $(K+N+1) \cdot \Gamma \in (v2 + (K+N) \cdot \Gamma,\ v2 + (K+N+1) \cdot \Gamma)$. From Lemma 1, it follows that there exists θ_2 in $[0,\ \Delta_q]$ such that $v2 + (K+N) \cdot \Gamma + \rho_q \cdot \theta_2 = (K+N+1) \cdot \Gamma$. Set $t2 = N \cdot \Delta_q + \theta_2$. Clearly, $\widehat{V2} = V2 + \rho_q \cdot t2 = \widehat{V1} = (K+N+1) \cdot \Gamma$. Again, $\widehat{V1} \in [l,\ r]$ implies $\widehat{V2} \in [l,\ r]$ as required.

Thirdly, suppose $\widehat{V1} \in ((K+N+1) \cdot \Gamma,\ v1 + (K+N+1) \cdot \Gamma]$. Then we set $t2 = (N+1) \cdot \Delta_q$. Clearly $\widehat{V2} = V2 + \rho_q \cdot (N+1) \cdot \Delta_q \in ((K+N+1) \cdot \Gamma,\ (K+N+2) \cdot \Gamma)$. But then $\widehat{V1} \in ((K+N+1) \cdot \Gamma,\ v1 + (K+N+1) \cdot \Gamma]$ implies $v1 \in ((K+N+1) \cdot \Gamma,\ (K+N+2) \cdot \Gamma)$. Thus again, $\widehat{V1} \in [l,\ r]$ implies $\widehat{V2} \in [l,\ r]$.

Case 3: $q \neq q'$ **and** $\alpha = \tau$.

Since $q \neq q'$, an instantaneous transition has taken place at the end of the time-passage move leading to $(q, V1, q')$. Hence the automaton will continue to evolve at rate $\rho_{q'}$ until some $t1 \in [g,\ g+\delta_g]$ and then will evolve at the rate ρ_q for the rest of the period $1-t1$. Moreover $t1$ is such that $V1' = V1+\rho_q' \cdot t1+\rho_q \cdot (1-t1)$. We need to find $t2 \in [g,\ g+\delta_g]$ such that $V2' = V2 + \rho_{q'} \cdot t2 + \rho_q \cdot (1-t2)$ and $\|V1'\| = \|V2'\|$. In order to fix $t2$, let $g = N_g \cdot \Delta$ and $\delta_g = N_{\delta_g} \cdot \Delta$.

Noticing that $(\rho_q - \rho_{q'}) \cdot \Delta = N_{qq'} \cdot \Gamma$ and hence $(\rho_q - \rho_{q'}) \cdot (\Delta / N_{qq'}) = \Gamma$ we set $\Delta_{qq'} = \Delta / N_{qq'}$, and observe that $t1 \in [N_g \cdot N_{qq'} \cdot \Delta_{qq'},\ (N_g + N_{\delta_g}) \cdot N_{qq'} \cdot \Delta_{qq'}]$. Let N be the least integer in the interval $[N_g \cdot N_{qq'},\ (N_g + N_{\delta_g}) \cdot N_{qq'}]$ such that $t1 \in [N \cdot \Delta_{qq'},\ (N+1) \cdot \Delta_{qq'}]$. Let $\theta 1 = t1 - N \cdot \Delta_{qq'}$. Clearly $\theta 1 \in [0, \Delta_{qq'}]$.

We now have $V1' = V1 + \rho_{q'} \cdot N \cdot \Delta_{qq'} + \rho_{q'} \cdot \theta 1 + \rho_q \cdot (\Delta_{qq'} - \theta 1) + \rho_q \cdot (N_1 \cdot N_{qq'} - N - 1) \cdot \Delta_{qq'}$. (Recall that $N_1 \cdot \Delta = 1$.) Expanding this expression and simplifying using the definitions of N_q, $N_{q'}$, $N_{qq'}$ and $\Delta_{qq'}$, we get:
$V1' = V1 + (N_1 \cdot N_q - N) \cdot \Gamma - (\rho_q - \rho_{q'}) \cdot \theta 1$. We recall that $v1 = V1 - K \cdot \Gamma$ and $v2 = V2 - K \cdot \Gamma$. Since $\theta 1$ ranges over $[0,\ \Delta_{qq'}]$, we have that $(\rho_q - \rho_{q'}) \cdot \theta 1$ ranges over $[0,\ \Gamma]$. Hence $V1' \in [v1 + (K + N_1 \cdot N_q - N)\Gamma,\ v1 + (K + N_1 \cdot N_q - N + 1) \cdot \Gamma]$. Again there are three situations to consider. For convenience, let $K' = N_1 \cdot N_q - N$.

Suppose $V1' \in [v1 + (K+K') \cdot \Gamma, (K+K'+1) \cdot \Gamma)$. Then we set $t2 = N \cdot \Delta_{qq'}$. Then it is easy to see that $t2 \in [g, g+\delta_g]$. Now let $V2' = V2 + \rho_{q'} \cdot t2 + \rho_q \cdot (1-t2)$. Then by our choice of $t2$, we have, $V2' = V2 + \rho_{q'} \cdot N \cdot \Delta_{qq'} + \rho_q \cdot (N_1 \cdot N_{qq'} - N) \cdot \Delta_{qq'}$. Simplifying this expression, we get $V2' = V2 + K' \cdot \Gamma$. Since $V2 = v2 + K \cdot \Gamma$, we then get $V2' \in ((K+K') \cdot \Gamma, (K+K'+1) \cdot \Gamma)$. As a result, $\|V1'\| = \|V2'\|$. By the choice of $t2$, it is also clear that $(q, V2, q') \overset{\tau}{\Longrightarrow} (s, V2', q)$.

The case $V1' \in ((K+K'+1) \cdot \Gamma,\ v1 + (k+K+1) \cdot \Gamma]$ is handled in a similar manner by setting $t2 = (N+1) \cdot \Delta_{qq'}$.

So suppose that $V1' = (K+K'+1) \cdot \Gamma$. Then by Lemma 1 we can find $\theta 2 \in (0,\ \Delta_{qq'})$ such that with $t2 = N \cdot \Delta_{qq'} + \theta 2$, and $V2' = V2 + \rho_{q'} \cdot t2 + \rho_q \cdot (1-t2)$, we can obtain $V2' = (K+K'+1) \cdot \Gamma$. This follows from the fact that as $\theta 2$ ranges over $[0,\ \Delta_{qq'}]$, we will have $V2'$ ranging continuously over $[v2 + (K+K') \cdot \Gamma,\ v2 + (K+K'+1) \cdot \Gamma]$ and surely $(K+K'+1) \cdot \Gamma$ lies within this range. Clearly by the choice of $t2$ and $V2'$, we have $(q, V2, q') \overset{\tau}{\Longrightarrow} (s, V2', q)$. It also follows at once that $\|V1'\| = \|V2'\|$.

Case 4: $q \neq q'$ **and** $\alpha \in \mathit{Act}$.

This is the most general case where the rate will change during the current period *and* the time-pass move will be followed by an instantaneous execution of a transition of $\mathcal{A}$.

Since $(q, V1, q') \overset{\alpha}{\Longrightarrow} (s, V1', q)$, there exist $t1 \in [g,\ g+\delta_g]$ and $t1' \in [h,\ h+\delta_h]$ and a transition $q \xrightarrow{(\alpha, I)} s$ in $\mathcal{A}$ such that $V1' = V1 + \rho_{q'} \cdot t1 + (1-t1) \cdot \rho_q$ and $V1 + \rho_{q'} \cdot t1 + \rho_q \cdot (t1' - t1) \in I$. We need to find $t2 \in [g,\ g+\delta_g]$ and $t2' \in [h,\ h+\delta_h]$ such that $V2 + \rho_{q'} \cdot t2 + \rho_q \cdot (t2' - t2) \in I$ and $\|V1'\| = \|V2'\|$ where $V2' = V2 + \rho_{q'} \cdot t2 + (1-t2) \cdot \rho_q$.

As before, we set $\Delta_{qq'} = \Delta / N_{qq'}$ and let N be the least integer in the interval $[N_g \cdot N_{qq'},\ (N_g + N_{\delta_g}) \cdot N_{qq'}]$ such that $t1 \in [N \cdot \Delta_{qq'},\ (N+1) \cdot \Delta_{qq'}]$. Let $\theta 1 = t1 - N \cdot \Delta_{qq'}$. Clearly $\theta 1 \in [0,\ \Delta_{qq'}]$. Using the argument developed to settle the previous case, we can conclude that $V1' = V1+(N_1 \cdot N_q - N)\cdot \Gamma - (\rho_q - \rho_{q'})\cdot \theta 1$. As before, we set $K' = N_1 \cdot N_q - N$. We need to examine two cases. (It is worth recalling here that we are operating under the assumptions $0 \leq \rho'_q < \rho_q$ and $v2 < v1$).

Suppose $V1' \in [v1+(K+K')\cdot \Gamma,\ v2+(K+K'+1)\cdot \Gamma]$. Consider $t2 = (N+1)\cdot \Delta_{qq'} + \theta 2$ for some $\theta 2 \in [0, \Delta_{qq'}]$. Define $V2' = v2+(K'+K+1)\cdot \Gamma - \theta 2\cdot(\rho_q - \rho_{q'})$. As $\theta 2$ ranges over $[0,\ \Delta_{qq'}]$, $V2'$ will range over $[v2 + (K' + K) \cdot \Gamma,\ v2 + (K + K' + 1) \cdot \Gamma]$. Hence, by Lemma 1, we can fix a $\theta 2$ such that $V2' = V1'$.

Suppose on the other hand, $V1' \in (v2+(K+K'+1)\cdot \Gamma,\ v1+(K+K'+1)\cdot \Gamma]$. Then we set $\theta 2 = 0$ so that $t2 = (N+1)\cdot \Delta_{qq'}$ and hence $V2' = v2+(K+K'+1)\cdot \Gamma$. Clearly both $V1'$ and $V2'$ lie in $((K + K' + 1) \cdot \Gamma,\ (K + K' + 2) \cdot \Gamma)$. Hence $\|V1'\| = \|V2'\|$.

We note that in either case, our choice of $\theta 2$ guarantees that $V1' = V2'$ or $V2' < V1'$ with $V1' - V2' \leq v1 - v2$.

Turning to the choice of $t2'$, we define as before, $\Delta_q = \Delta / N_q$. Let J be the least integer in the interval $[N_h \cdot N_q,\ (N_h + N_{\delta_h} - 1) \cdot N_q]$ such that $t1' \in [J \cdot \Delta_q,\ (J + 1) \cdot \Delta_q]$. Let $\theta 1' = t1' - (J \cdot \Delta_q)$. Clearly $\theta 1' \in [0,\ \Delta_q]$.

Let $V1'' = V1+\rho_{q'}.t1+\rho_q \cdot (t1'-t1)$. Then $V1'' = V1+\rho_{q'}\cdot N\cdot \Delta_{qq'} + \rho_{q'}\cdot \theta 1 + \rho_q \cdot (\Delta_{qq'} - \theta 1) + \rho_q \cdot (J \cdot N_q \cdot \Delta_q - (N+1)\cdot \Delta_{qq'}) + \rho_q \cdot \theta 1'$. Again expanding and simplifying this expression, we get $V1'' = V1 + (N_q \cdot J - N) \cdot \Gamma - (\rho_q - \rho_{q'}) \cdot \theta 1 + \rho_q \cdot \theta 1'$. Let $L = N_q \cdot J - N$. Then $V1'' = v1 + (K + L) \cdot \Gamma - (\rho_q - \rho_{q'}) \cdot \theta 1 + \rho_q \cdot \theta 1'$.

Now $V1''' = v1+(K+L)\cdot \Gamma - (\rho_q - \rho_{q'})\cdot \theta 1$ must lie in $[v1+(K+L)\cdot \Gamma,\ v1+(K+L+1)\cdot \Gamma]$. Suppose $V1'''$ lies in $[v1+(k+L)\cdot \Gamma,\ v2+(K+L+1)\cdot \Gamma]$. Then our choice of $\theta 2$ ensures that $v2+(K+L)\cdot \Gamma - (\rho_q - \rho_{q'})\cdot \theta 2 = v1+(K+L)\cdot \Gamma - (\rho_q - \rho_{q'})\cdot \theta 1$. We now set $\theta 2' = \theta 1'$ and $t2' = J \cdot \Delta_q + \theta 2'$. Clearly $t2' \in [h,\ h + \delta_h]$ and $V2'' = V2 + \rho_{q'} \cdot t2 + \rho_q \cdot (t2' - t2) = V1'' \in I$ and hence we have, as required, $(q, V2, q') \xRightarrow{\alpha} (s, V2', q)$ with $\|V1'\| = \|V2'\|$.

Finally, assume that $V1''' = v1 + (K + L) \cdot \Gamma - (\rho_q - \rho_{q'}) \cdot \theta 1$ lies in $(v2 + (K + L + 1) \cdot \Gamma,\ v1 + (K + L + 1) \cdot \Gamma]$ Then our choice of $\theta 2$ ensures that $V2''' = v2 + (K + L).\Gamma - (\rho_q - \rho_{q'}) \cdot \theta 2 = v2 + (K + L + 1) \cdot \Gamma$ and thus $V1''' - V2''' \leq v1 - v2$. Now depending on $\theta 1'$, the value of $V1''$ must lie in $(v2 + (K + L + 1) \cdot \Gamma,\ v1 + (K + L + 2) \cdot \Gamma]$. If $V1''$ lies in $(v2 + (k + L + 1) \cdot \Gamma,\ v2 + (k + L + 2) \cdot \Gamma]$ then we can, by lemma 1, pick $\theta 2' \in [0, \Delta_q]$ so that $V2'' = V1''$ where $V2'' = V2 + \rho_{q'} \cdot t2 + \rho_q \cdot (t2' - t2)$ with $t2' = J \cdot \Delta_q + \theta 2'$. If on the other hand, $V1''$ lies in $(v2 + (K + L + 2) \cdot \Gamma,\ v1 + (k + L + 2) \cdot \Gamma]$ we can set $\theta 2' = \Delta_q$ and $t2' = J \cdot \Delta + \theta 2'$ so that $V2'' = v2 + (K + L + 2) \cdot \Gamma$. In either case, we have $t2' \in [h,\ h + \delta_h]$ and $V2'' \in I$ so that $(q, V2, q') \xRightarrow{\alpha} (s, V2', q)$ with $\|V1'\| = \|V2'\|$. □

We now define the finite state automaton $\mathcal{Z}_{\mathcal{A}} = (\mathcal{D}, (q_{in}, (k_0, 0), q_{in}), Act \cup \{\tau\}, \rightsquigarrow, \mathcal{D})$ where $k_0 \cdot \Gamma = V_{in}$ and the transition relation $\rightsquigarrow \subseteq \mathcal{D} \times (Act \cup \{\tau\}) \times \mathcal{D}$ is given by: $(q, (k, d), q') \overset{\alpha}{\rightsquigarrow} (q1, (k1, d1), q1')$ iff there exist configurations (q, V, q') and $(q1, V1, q1')$ such that $(q, V, q') \xRightarrow{\alpha} (q1, V1, q1')$ and $\|V\| = (k, d)$ and $\|V1\| = (k1, d1)$. In what follows, we will often write $\mathcal{Z}_{\mathcal{A}}$ as just $\mathcal{Z}$. Note that,we are setting all the states of $\mathcal{Z}$ to be its final states.

We define $\mathcal{L}_{st}(\mathcal{Z})$ to be the subset of $Q^\star$ as follows. A *run* of $\mathcal{Z}$ is a sequence of the form $(q_0,(l_0,d_0),q'_0)\ \alpha_0\ (q_1,(l_1,d_1),q'_1)\ \alpha_1\ \ldots\ (q_m,(l_m,d_m),q'_m)$ where $(q_0,(l_0,d_0),q'_0)=(q_{in},(k_0,0),q_{in})$ and $(q_j,(l_j,d_j),q'_j) \overset{\alpha_j}{\leadsto} (q_{j+1},(l_{j+1},d_{j+1}),q'_{j+1})$ for $0 \le j < m$. Next we define $q_0q_1\ldots q_m \in \mathcal{L}_{st}(\mathcal{Z})$ iff there exists a run of $\mathcal{Z}$ of the form $(q_0,(l_0,d_0),q'_0)\ \alpha_0\ (q_1,(l_1,d_1),q'_1)\ \alpha_1\ \ldots\ (q_m,(l_m,d_m),q'_m)$. Clearly $\mathcal{L}_{st}(\mathcal{Z})$ is a regular subset of $Q^\star$ and it does not involve any loss of generality to view $\mathcal{Z}_\mathcal{A}$ itself as a representation of this regular language.

Theorem 2. *The automaton $\mathcal{Z}_\mathcal{A}$ can be computed effectively. Moreover $\mathcal{L}_{st}(\mathcal{A})=\mathcal{L}_{st}(\mathcal{Z}_\mathcal{A})$ and $\mathcal{L}_{act}(\mathcal{A})=\mathcal{L}(\mathcal{Z}_\mathcal{A})$ where $\mathcal{L}(\mathcal{Z}_\mathcal{A})$ is the regular subset of $(Act\cup\{\tau\})^\star$ accepted by $\mathcal{Z}_\mathcal{A}$ in the usual sense. (Note that all the states of $\mathcal{Z}_\mathcal{A}$ are final states.)*

Proof. Clearly the finite set of states $\mathcal{D}$ and the initial state $(q_{in},(k_0,0),q_{in})$ can be computed easily. The transition relation $\leadsto$ is expressible in the first order theory of the real ordered field which is a decidable theory [11] . For instance, to determine if $(q,(k,1),q') \overset{a}{\leadsto} (q1,(k1,1),q)$, with $a \in Act$, we first check if there is a transition tr in $\mathcal{A}$ of the form $(q,a,I,q1)$. If there is no such transition then we conclude that $(q,(k,1),q') \overset{a}{\leadsto} (q1,(k1,d1),q)$ is *not* a transition in $\mathcal{Z}_\mathcal{A}$. If there is such a transition then for each such transition tr we construct the formula φ_{tr}, take the disjunction of all such formulas and check for its satisfiability.

Suppose $tr=(q,a,I,q1)$. Then φ_{tr} will conjunctively assert the following:

- There exists V such that $k\cdot\Gamma < V < (k+1)\cdot\Gamma$.
- There exists $t1$ such that $g \le t1 \le g+\delta_g$ and $k1\cdot\Gamma < V+\rho_{q'}\cdot t1+\rho_q\cdot(1-t1) < (k1+1)\cdot\Gamma$.
- There exists $t2$ such that $h \le t2 \le h+\delta_h$ and $l \le V+\rho_{q'}\cdot t1+\rho_q\cdot(t2-t1) \le r$ (where $I=[l,\ r]$).

To see that $\mathcal{L}_{st}(\mathcal{A})=\mathcal{L}_{st}(\mathcal{Z})$ we first note that $\mathcal{L}_{st}(\mathcal{A})\subseteq\mathcal{L}_{st}(\mathcal{Z})$ follows from the definition of $\mathcal{Z}_\mathcal{A}$. To conclude inclusion in the other direction, we will argue that for each run $(q_0,(l_0,d_0),q'_0)\,\alpha_0\,(q_1,(l_1,d_1),q'_1)\,\alpha_1\ldots(q_m,(l_m,d_m),q'_m)$ of $\mathcal{Z}$ there exist $V_0,V_1\ldots V_m\in\mathbb{R}$ such that $(q_0,V_0,q'_0)\ \alpha_0\ (q_1,V_1,q'_1)\ \alpha_1\ldots(q_m,V_m,q'_m)$ is a run of $TS_\mathcal{A}$. And furthermore, $\|V_j\|=(l_j,d_j)$ for $0\le j\le m$. The required inclusion will then follow at once. For $m=1$, it is clear from the definitions and so suppose that $(q_0,(l_0,d_0),q'_0)\ \alpha_0\ (q_1,(l_1,d_1),q'_1)\ \alpha_1\ldots(q_m,(l_m,d_m),q'_m)$ $\alpha_m\,(q_{m+1},(l_{m+1},d_{m+1}),q'_{m+1})$ is a run of $\mathcal{Z}$. By the induction hypothesis, there exists a run $(q_0,V_0,q'_0)\ \alpha_0\ (q_1,V_1,q'_1)\ \alpha_1\ldots(q_m,V_m,q'_m)$ of $TS_\mathcal{A}$ with the property, $\|V_j\|=(l_j,d_j)$ for $0\le j\le m$.

Now $(q_m,(l_m,d_m),q'_m) \overset{\alpha_m}{\leadsto} (q_{m+1},(l_{m+1},d_{m+1}),q'_{m+1})$ implies that there exist V'_m and V'_{m+1} such that $(q_m,V'_m,q'_m) \overset{\alpha_m}{\leadsto} (q_{m+1},V'_{m+1},q'_{m+1})$ and $\|V'_m\|=(l_m,d_m)$ and $\left\|V'_{m+1}\right\|=(l_{m+1},d_{m+1})$. But this implies that $\|V'_m\|=\|V_m\|$. Hence by Theorem 1, there exists V_{m+1} such that $(q_m,V_m,q'_m) \overset{\alpha_m}{\leadsto} (q_{m+1},V_{m+1},q'_{m+1})$ and moreover $\left\|V'_{m+1}\right\|=\|V_{m+1}\|$. Thus $\mathcal{L}_{st}(\mathcal{A})=\mathcal{L}_{st}(\mathcal{Z}_\mathcal{A})$. It now also follows easily that $\mathcal{L}_{act}(\mathcal{A})=\mathcal{L}(\mathcal{Z}_\mathcal{A})$. □

In what follows, we will refer to $\mathcal{Z}$ as the *zone version* of $\mathcal{A}$.

3.2 The n-dimensional Case

We now consider an n-dimensional hybrid automaton $\mathcal{A}$ defined as in the previous section with the associated terminology and notations. Our goal is to show that $\mathcal{L}_{st}(\mathcal{A})$ is a regular subset of $Q^\star$ while $\mathcal{L}_{act}(\mathcal{A})$ is a regular subset of $(Act \cup \{\tau\})^\star$.

To do so, we first define the family of one dimensional automata $\{\mathcal{A}^i\} = (Q, Act, q^i_{in}, V^i_{in}, D, \{\rho^i_q\}_{q\in Q}, \mathrm{B}, \longrightarrow_i)$ where:

- $V^i_{in}(i)$ is $V_{in}(i)$, the i-th component of V_{in}.
- $\rho^i_q = \rho_q(i)$
- $q \stackrel{(a,I^i)}{\longrightarrow}_i q'$ iff there exists $q \stackrel{(a,I)}{\longrightarrow} q'$ in $\mathcal{A}$ with $I^i = I(i)$. Again, $I(i)$ denotes the i-th component of I.

Let $\mathcal{Z}^i$ be the zone version of $\mathcal{A}^i$ with $\mathcal{Z}^i = (\mathcal{D}^i, (q_{in}, (k^i_0, 0), q_{in}), Act \cup \{\tau\}, \rightsquigarrow_i)$. We now define the finite state automaton $\mathcal{Z}_\mathcal{A} = (\mathcal{D}, (q_{in}, \kappa_0, q_{in}), Act \cup \{\tau\}, \rightsquigarrow, \mathcal{D})$ which will constitute the zone version of the n-dimensional automaton $\mathcal{A}$ as follows.

- $\mathcal{D}$, the states of this automaton, will be of the form (q, κ, q') with $q, q' \in Q$ and $\kappa \in ((\mathbb{Z} \times \{0,1\})^n$. Let $\kappa = ((k_1, d_1), (k_2, d_2) \dots, (k_n, d_n))$. Then $(q, \kappa, q') \in \mathcal{D}$ iff there $(q, (k_i, d_i), q') \in \mathcal{D}^i$ for each i in $\{1, 2, \dots, n\}$.
- $\kappa_0 = ((k^1_0, 0), (k^2_0, 0), \dots, (k^n_0, 0)$
- $\rightsquigarrow \subseteq \mathcal{D} \times (Act \cup \{\tau\}) \times \mathcal{D}$ is given by: Let $(q, \kappa, q'), (q1, \kappa 1, q1') \in \mathcal{D}$ with $\kappa = ((k_1, d_1), (k_2, d_2) \dots, (k_n, d_n))$ and $\kappa 1 = ((k1_1, d1_1), (k1_2, d1_2) \dots, (k1_n, d1_n))$. Then $(q, \kappa, q') \stackrel{\alpha}{\rightsquigarrow} ((q1, \kappa 1, q1')$ iff $(q, (k_i, d_i), q') \stackrel{\alpha}{\rightsquigarrow}_i (q1, (k1_i, d1_i), q)$ for each $i \in \{1, 2, \dots, n\}$.

As before, we will often write $\mathcal{Z}$ instead of $\mathcal{Z}_\mathcal{A}$ and refer to it as the zone version of $\mathcal{A}$. We denote by $\mathcal{L}_{st}(\mathcal{Z})$ the state sequence language of $\mathcal{Z}$ and define it in the obvious way. We also define $\mathcal{L}(\mathcal{Z})$ to be the subset of $(Act \cup \{\tau\})^\star$ accepted by the finite state automaton $\mathcal{Z}$.

Theorem 3. *The automaton $\mathcal{Z}_\mathcal{A}$ can be computed effectively. Moreover $\mathcal{L}_{st}(\mathcal{A}) = \mathcal{L}_{st}(\mathcal{Z}_\mathcal{A})$ and $\mathcal{L}_{act}(\mathcal{A}) = \mathcal{L}(\mathcal{Z}_\mathcal{A})$.*

Proof. Since, by Theorem 2, each of the finite state automata $\mathcal{Z}^i$ can be computed effectively, so can $\mathcal{Z}$ be.

We define the runs of $\mathcal{Z}$ in the obvious way. Now suppose $\sigma = (q_0, V_0, q'_0)\, \alpha_0\, (q_1, V_1, q'_1)\, \alpha_1\, (q_2, V_2, q'_2) \dots (q_n, V_n, q'_n)$ is a run of $TS_\mathcal{A}$. Then by the definition of $\mathcal{A}^i$ we have that $\sigma^i = (q_0, V_0(i), q'_0)\, \alpha_0\, (q_1, V_1(i), q'_1)\, \alpha_1\, (q_2, V_2(i), q'_2) \dots (q_n, V_n(i), q'_n)$ is a run of $TS_{\mathcal{A}^i}$ for each i. Hence by the definition of the transition relations $\rightsquigarrow_i$ and $\rightsquigarrow$, it follows that $(q_0, \kappa 0, q'_0)\, \alpha_0\, (q_1, \kappa 1, q'_1)\, \alpha_1\, (q_2, \kappa 2, q'_2) \dots (q_m, \kappa m, q'_m)$ is a run of $\mathcal{Z}$ where $\kappa j = \|V_j(1)\|, \|V_j(2)\|, \dots, \|V_j(n)\|))$ for $0 \leq j \leq m$. Hence $\mathcal{L}_{st}(\mathcal{A}) \subseteq \mathcal{L}_{st}(\mathcal{Z}_\mathcal{A})$.

To show inclusion in the other direction, consider the configurations (q, V, q') and $(q1, V1, q)$ in $\mathcal{C}_\mathcal{A}$ and suppose α is such that $(q, V(i), q') \stackrel{\alpha}{\Longrightarrow}_i (q1, V1(i), q)$

in $TS_{\mathcal{A}_i}$ for each i. Then, for each i, there exists $t1_i \in [g,\, g+\delta_g]$ such that $V1(i) = (V(i) + \rho^i_{q'}) \cdot t1_i + (\rho^i_q) \cdot (1 - t1_i)$. If $\alpha = act$, then there also exists a (unique!) transition $(q, a, I, q1)$ in $\mathcal{A}$ and for each i there exists $t2_i \in [h,\, h+\delta_h]$ such that $V(i) + \rho^i_{q'} \cdot t1_i + \rho^i_q \cdot (t2_i - t1_i) \in I(i)$. Hence we can conclude that $(q, V, q') \overset{\alpha}{\Longrightarrow} (q1, V1, q1')$ in $TS_{\mathcal{A}}$ by defining $\widehat{t1} \in [g,\, g+\delta_g]^n$ to be $\widehat{t1}(i) = t1_i$ and $\widehat{t2} \in [h,\, h+\delta_h]^n$ to be $\widehat{t2}(i) = t1_i$ for each i.

So now suppose $(q_0, \kappa 0, q'_0)\ \alpha_0\ (q_1, \kappa 1, q'_1)\ \alpha_1\ (q_2, \kappa 2, q'_2) \ldots (q_m, \kappa m, q'_m)$ is a run of $\mathcal{Z}$. Then for each i,
$(q_0, (k^i_0, d^i_0), q'_0)\ \alpha_0\ (q_1, (k^i_1, d^i_1), q'_1)\ \alpha_1\ (q_2, (k^i_2, d^i_2), q'_2) \ldots (q_m, (k^i_m, d^i_m), q'_m)$ is a run of $\mathcal{Z}^i$, the zone version of $\mathcal{A}^i$ with the assumption that $\kappa j(i) = (k^i_j, d^i_j)$ for $0 \leq j \leq m$ and $1 \leq i \leq n$. Hence from the proof of Theorem 2, it follows that for each i, there exist $V^i_0, V^i_1 \ldots V^i_m$ such that
$(q_0, V^i_0, q'_0)\alpha_0(q_1, V^i_1, q'_1)\alpha_1(q_2, V^i_2, q'_2) \ldots (q_m, V^i_m, q'_m)$ is a run of $TS_{\mathcal{A}^i}$. Now define $V_j \in \mathbb{R}^n$ to be $V_j(i) = V^i_j$ for $0 \leq j \leq m$. Then by the argument above we must also have that $(q_0, V_0, q'_0)\ \alpha_0\ (q_1, V_1, q'_1)\ \alpha_1\ (q_2, V_2, q'_2) \ldots (q_m, V_m, q'_m)$ is a run of $TS_{\mathcal{A}}$. Clearly, this implies $\mathcal{L}_{st}(\mathcal{A}) = \mathcal{L}_{st}(\mathcal{Z}_{\mathcal{A}})$ and this in turn implies $\mathcal{L}_{act}(\mathcal{A}) = \mathcal{L}(\mathcal{Z}_{\mathcal{A}})$. □

4 Some Extensions

In order to simplify the initial presentation, we placed a number of restrictions on our automata. Here we first examine which of these can be relaxed so that, with minor overhead, our main results go through smoothly. We then formulate a composition operation for lazy hybrid automata in a standard way using which large automata can be presented in a succinct fashion. These networks of lazy hybrid automata can also be analyzed effectively.

Let $\mathcal{A} = (Q, Act, q_{in}, V_{in}, D, \{\rho_q\}_{q \in Q}, \mathrm{B}, \longrightarrow)$ be a lazy hybrid automaton. We could permit a set of initial control states and a set of initial valuations for each initial control state, provided they can be specified using rectangular constraints. Our results will go through with minor modifications. It is also clear that our demand $0 < g < g + \delta_g < h < h + \delta_h < 1$ is only for convenience. We could have different delay parameters for different variables and these delays could spill over more than one time unit.

The restriction that there is at most one a-labeled transition between a pair of control states is mainly for convenience. If this condition is violated we could use renaming to enforce this property, construct the zone automaton and then restore the old names.

We have also avoided the use of state invariants only for convenience. They can be introduced in the expected manner. A similar remark applies to allowing for resets of the variables during a mode switch. Finally, we have avoided the customary use of differential inclusions to specify the rates mainly for convenience. Our results will still go through, with some additional notational overhead if we permit this extension.

The boundedness restriction on the allowed range $B = [B_{min}, B_{max}]$ is crucial. From a modeling point of view however, this is not a crippling limitation.

The fact that we have linear rates is again crucial. Our proof idea breaks down for non-linear rates. The fact that non-empty closed intervals are used for specifying the transitions of $\mathcal{A}$ is not important. However the fact that we have rectangular constraints is important.

4.1 Product Automata

We now show that we can easily cope with networks of lazy hybrid automata in which the component automata communicate by synchronizing on common actions.

Let $\mathcal{P}$ be a finite set of agent names with u, v ranging over $\mathcal{P}$. We define a *product* lazy hybrid automaton to be a structure $\mathcal{A}_{\mathcal{P}} = \prod_{u \in \mathcal{P}} \mathcal{A}_u$ where $\mathcal{A}_u = (Q_u, Act_u, q^u_{in}, V^u_{in}, D, \{\rho^u_q\}_{q \in Q_u}, \mathrm{B}, \longrightarrow_u)$ for each u in $\mathcal{P}$. For convenience, we will write TS_u instead of $TS_{\mathcal{A}_u}$ to denote the transition system over the reachable configurations of $\mathcal{A}_u$ as defined in section 2. The operational behavior of $\prod_{u \in \mathcal{P}} \mathcal{A}_u$ is given by the transition system denoted as $TS_{\mathcal{P}}$ and defined as follows.

Let RC_u be the set of reachable configurations of $\mathcal{A}_u$ with c^u_{in} as the initial configuration for each u. Then $RC_{\mathcal{P}}$, the set of reachable configurations of $\prod_{u \in \mathcal{P}} \mathcal{A}_u$ and the transition relation $\Longrightarrow_{\mathcal{P}}$ are the least sets satisfying:

- $c_{in} = \prod_{u \in \mathcal{P}} c^u_{in} \in RC_{\mathcal{P}}$ and is the initial configuration.
- Suppose $c = \prod_{u \in \mathcal{P}} c^u \in RC_{\mathcal{P}}$ and $c^u \overset{\tau}{\Longrightarrow}_u d^u$ for each u. Then $d = \prod_{u \in \mathcal{P}} d^u \in RC_{\mathcal{P}}$ and $c \overset{\tau}{\Longrightarrow}_{\mathcal{P}} d$.
- Suppose $c = \prod_{u \in \mathcal{P}} c^u \in RC_{\mathcal{P}}$ and $a \in Act$ and $c^u \overset{a}{\Longrightarrow}_u d^u$ for each u satisfying $a \in Act_u$. (Act_u is the set of actions that $\mathcal{A}_u$ participates in.) Furthermore, suppose $c^u \overset{\tau}{\Longrightarrow}_u d^u$ for every u such that $a \notin Act_u$. Then $d = \prod_{u \in \mathcal{P}} d^u \in RC_{\mathcal{P}}$ and $c \overset{a}{\Longrightarrow}_{\mathcal{P}} d$.

We now define $TS_{\mathcal{P}}$ to be the transition system $TS_{\mathcal{P}} = (RC_{\mathcal{P}roc}, Act \cup \{\tau\}, c_{in}, \Longrightarrow_{\mathcal{P}})$. The notion of runs of $TS_{\mathcal{P}}$ and the language of state sequences, $\mathcal{L}_{st}(\prod_{u \in \mathcal{P}} \mathcal{A}_u)$ and $\mathcal{L}_{act}(\prod_{u \in \mathcal{P}} \mathcal{A}_u)$ are defined in the expected manner. These languages are also regular and they can be effectively represented as follows.

Let $\mathcal{Z}^u = (\mathcal{D}^u, (q^u_{in}, \kappa^u_0, q^u_{in}), Act_u \cup \{\tau\}, \rightsquigarrow_u, \mathcal{D}^u)$ be the zone version of $\mathcal{A}_u$ for each u. We now define the automaton $\mathcal{Z}_{\mathcal{P}} = (\mathcal{D}_{\mathcal{P}}, \prod_{u \in \mathcal{P}} \kappa^u_0, Act \cup \{\tau\}, \rightsquigarrow_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ via:

- $\mathcal{Z}_{\mathcal{P}} = \prod_{u \in \mathcal{P}} \mathcal{D}^u$.
- Suppose $X = \prod_{u \in \mathcal{P}} (q1^u, \kappa^u, q2^u)$ and $Y = \prod_{u \in \mathcal{P}} (q3^u, \kappa^u, q4^u)$ are in $\mathcal{D}_{\mathcal{P}}$ and $(q1^u, \kappa^u, q2^u) \overset{\tau}{\rightsquigarrow}_u (q3^u, \kappa^u, q4^u)$ for each u. Then $X \overset{\tau}{\rightsquigarrow}_{\mathcal{P}} Y$.
- Suppose X and Y are as above and $a \in Act$ and $(q1^u, \kappa^u, q2^u) \overset{a}{\rightsquigarrow}_u (q3^u, \kappa^u, q4^u)$ for each u such that $a \in Act_u$. Suppose also $(q1^u, \kappa^u, q2^u) \overset{\tau}{\rightsquigarrow}_u (q3^u, \kappa^u, q4^u)$ for each u such that $a \notin Act_u$. Then $X \overset{\tau}{\rightsquigarrow}_{\mathcal{P}} Y$.

It is now easy to prove that $\mathcal{Z}_{\mathcal{P}}$ represents both $\mathcal{L}_{st}(Aut_{\mathcal{P}})$ and $\mathcal{L}_{act}(\mathcal{A}_{\mathcal{P}})$ in the expected manner.

5 Conclusion

We have formulated here the class of lazy hybrid rectangular automata. These are basically linear rectangular hybrid automata but where each automaton is accompanied by the delay parameters $\{g, \delta_g, h, \delta_h\}$. Our main result is that the discrete time behavior of these automata can be effectively computed if the allowed ranges of values for the variables are bounded.

We have not outlined the verification problems for lazy rectangular hybrid automata that can be settled effectively. It should be clear however, that due to Theorem 1 and Theorem 2, we can model-check the discrete time behavior of our automata against a variety of linear time and branching time temporal logic specifications. We have also not established the precise complexity bounds for the procedure constructing the zone version of our automata.

We believe that associating non-zero bounded delays with the sensors and actuators is a natural assumption. It also cuts down on the expressive power of hybrid automata. We also wish to argue that it is useful to focus on the discrete time behavior of hybrid automata. Finally, it is our hope that the ideas presented here may have a larger scope of application. In particular, we conjecture that the discrete time behavior of lazy linear hybrid automata (with bounded allowed ranges of values) can be effectively computed.

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