The Myth about Einstein

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In common perception, Einstein comes out as a strong mathematical physicist. This is however a myth. The 1905 Einstein was close to real life phenomena. This article presents how he used simple mathematics to understand experiments, especially on Brownian Motion and Photoelectric Effect, employing the underlying concept of thermodynamic fluctuations.

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The popular image of Albert Einstein, mostly drawn from his portraits at a somewhat ripe age, is not merely that he was a dreamy-eyed, long-haired and a bit absent-minded scientist, but that he was very strongly mathematically oriented. This perception of Einstein is bolstered by his work on the General Theory of Relativity (1915) that impacts on our universe and the cosmos, which not only pose challenging issues to trained physicists but also conjure up in the minds of the uninitiated, a sense of distant vastness, almost bordering on mysticism.

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In pursuit of the general theory of relativity Einstein did actually master covariant and contravariant tensor analysis, as propounded by the stalwart mathematicians Gauss, Riemann, and others and did become an expert on many aspects of geometry. This resulted in the departure from Euclidean geometry and heralded the birth of fascinating new ideas of space-time geometry. It is this development which has perhaps led to the commonly perceived vision of Einstein. But the point is, and that point is going to be a recurring theme of this presentation, that Einstein employed Mathematics merely as a tool in developing a new structure of a theory, not so much to make new contributions to the subject of

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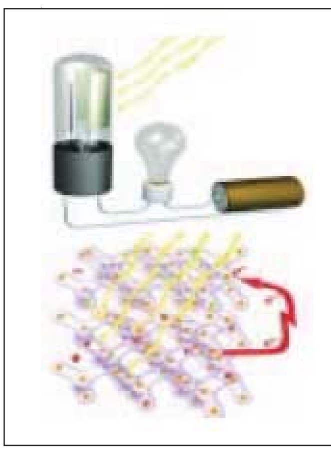
mathematics itself but to understand the phenomenon of gravity and gravitational fields.

The above perception of Einstein in the popular mind that he was a powerful mathematical physicist whose work was beyond comprehension of ordinary mortals, is one myth that I would like to dispel. Indeed as we come to discuss the 1905 Einstein, which is what our task is in this *annus mirabilis*, we will discover below that the mathematics Einstein used was actually simple, within the realm of understanding by present plus-two students. Einstein's primary focus was to elucidate natural phenomena, as evident in everyday life and as seen through experiments.

Given this background, let me turn attention to two of Einstein's stupendous contributions in the year 1905, the photoelectric effect (*Figure* 1), which fetched him the Nobel Prize, and Brownian motion (*Figure* 2). What is presented in *Figure* 1 is a schematic illustration of how by shining light on a metallic plate, an electric current can be generated from the ejected electrons, whereas *Figure* 2 depicts an application of diffusion, emanating from the theory of Brownian motion, in simulating Kanji script on the surface of copper with the aid of diffusing iron atoms.



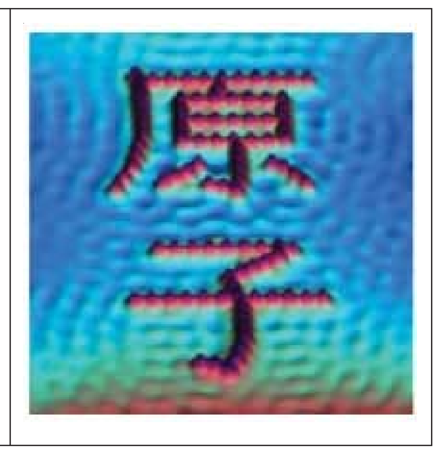


Figure 1 (left). Schematic illustration of photo electric effect.

Figure 2 (right). Kanji script on the surface of copper with the aid of diffusing iron atoms.

What is a remarkable commonality between the Brownian motion and the photo electric effect is Einstein's usage of Boltzmann's idea of statistical fluctuations. In our discussion on these two topics I will try to pinpoint where Einstein makes ingenious application of the concept of thermally-aided fluctuations that give rise to the classical Equipartition Theorem of Kinetic Theory.

Einstein's doctoral thesis is a remarkable exposition of his contact with real life phenomena – it is a description of a new method for determining the Avogadro number (*N*) and the size (*a*) of a molecule.

A. Brownian Motion

In this section I describe Einstein's work on Brownian motion, preceded by his doctoral thesis which was submitted on 30 April 1905 but published later in 1906. The thesis is a remarkable exposition of Einstein's contact with real life phenomena – it is a description of a new method for determining the Avogadro number (N) and the size (a) of a molecule. It is incredible today to imagine that something like 'Molecular Reality' which we take for granted as a concept, found no consensus amongst physicists and chemists in as late as the 19th century! Einstein's PhD thesis deals with bulk rheological properties of particle suspensions and contains results with an extraordinary range of applications, relevant to:

(i) Construction industry, based on what has now emerged as a novel interdisciplinary subject of granular matter;
(ii) Dairy industry, through the colloidal suspension properties of, for instance, casein micelles in cow's milk; and
(iii) Ecology, involving the Brownian movement of aerosol particles in clouds.

On this, Abraham Pais writes in his most readable biography on Einstein: *Subtle is the Lord*: "... Einstein might have enjoyed hearing this, since he was quite fond of applying physics to practical situations".

I now turn attention to Einstein's thesis in which he

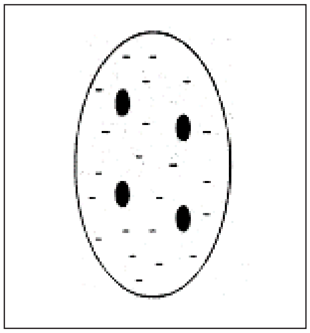


Figure 3. Model: Region of solvent containing dilute amounts of much bigger sized solute molecules.

produced two separate ingeniously developed ideas that led to two separate relations between N, the Avogadro number and a, the radius of a molecule, assumed spherical.

1. *'Renormalized' Viscosity*

Like all practitioners of the art Einstein considers a simple model, a region of solvent, containing dilute amounts of solute whose molecules are much bigger in size than the solvent molecules (*Figure* 3). How big are the solute molecules? Well, their radii are smaller than 10^{-3} mm, which in today's parlance, translates into 1000 nm, and hence it is relevant to mention how critically important Brownian motion is, in the contemporarily significant topic of Nanoscience. The system at hand could be a dilute solution of sugar molecules as solutes, dissolved in water as solvent.

Einstein makes the reasonable assumption that the (bare) viscosity η of the solvent, upon mixing with solute, must be enhanced to a renormalized value η^* where the fractional increase must be proportional to the volume of the solute per unit volume of the solvent:

$$\frac{\eta^*}{\eta} = 1 + \frac{4}{3}\pi a^3 \frac{N}{M}\rho. \tag{1}$$

In (1), $\frac{4}{3}\pi a^3$ is the volume of a solute molecule. The quantity $\frac{N}{M}\rho$ is the number of solute molecules in a unit volume of the solvent, wherein ρ is the density i.e. mass of the solute per unit volume of the solvent, N is the Avogadro number and M is the molecular weight of the solute. Einstein then makes use of the experimental data on η^* and η for sugar solutions, and knowing M and η, determines the product (Na^3).

2. *Thermodynamic plus Dynamic Argument*

In order to evaluate N and a separately Einstein, of course, needed another relation between N and a, and in striving to arrive at that, produced a brilliant combination of arguments from Boltzmann's thermodynamics and fluid dynamics *à la* Fick and Stokes.

Consider again the region of *Figure* 3 and the fluid motion, never meant to stop at any finite temperature, taken to be in one direction, say x, for the sake of simplicity. In arriving at this picture Einstein was most definitely influenced by the original observation of Robert Brown that the particles in the liquid, in this case the solute molecules, are in incessant, random motion due to the temperature of the solvent. Thus the solvent is expected to provide a push to the solutes, and if F is the concomitant force on each solute molecule, the force per unit volume of the solvent will be F multiplied by our earlier encountered factor of $\frac{N}{M}\rho$, which measures the number of solute molecules in a unit volume of the solvent (see *Figure* 4). This must equal the pressure gradient, hence

$$\frac{\partial P}{\partial x} = F\frac{N}{M}\rho. \tag{2}$$

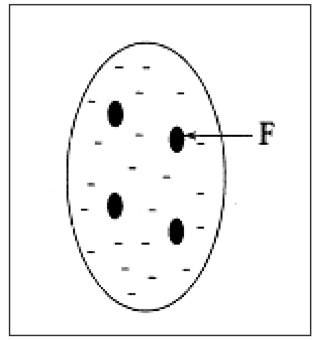


Figure 4. Model: 'Internal' viscous force F on each solute molecule.

Einstein now takes recourse to Boltzmann's concept of thermal fluctuations and making the bold assumption that the dilute solute particles should behave as the molecules of an ideal gas, writes for the pressure

$$P = \frac{RT}{M}\rho, \tag{3}$$

where R is the gas constant ($= Nk_B, k_B$ being the Boltzmann constant, as we know now), and T is the temperature. Therefore,

$$\frac{\partial \rho}{\partial x} = \frac{FN}{RT}\rho. \tag{4}$$

Einstein then moves onto fluid dynamical arguments and proposes that the force F must be of viscous origin, given by the Stokes law:

$$F = -6\pi\eta a v, \tag{5}$$

where, on the other hand, the velocity v must be given by Fick's law of diffusion:

$$\rho\nu = -D\frac{\partial\rho}{\partial x}, \tag{6}$$

D being the diffusion coefficient. Miraculously then, F drops out from the analysis (which underscores the depth of Einstein's analysis) and so do ρ and $\frac{\partial\rho}{\partial x}$, resulting in:

$$D = \frac{RT}{6\pi N\eta a}. \tag{7}$$

Equation (7), through the measurement of D at a given temperature T, leads to another relation determining now the product (Na) which, when combined with the result based on equation (1), yields a and N separately. Einstein thus computed N as 6.6×10^{23}, very close indeed to the presently accepted value.

3. *Diffusion*

Eleven days after submitting the thesis appears the paper by Einstein on the Brownian motion in which Einstein provides a mathematical formulation of the physical ideas already embodied in the thesis. Consider again one dimension x in which $n(x,t)$ measures the number of suspended solute particles per unit volume around x at time t. Because the total number of particles is conserved $n(x,t)$ must obey an equation of continuity:

$$\frac{\partial n(x,t)}{\partial t} = -\frac{\partial(nv)}{\partial x}. \tag{8}$$

Since n is proportional to the density ρ, the right hand side of (8) obeys the same equation as in (6), yielding the diffusion equation:

$$\frac{\partial n(x,t)}{\partial t} = D\frac{\partial^2 n(x,t)}{\partial x^2}. \tag{9}$$

This equation, for free boundary conditions, has the solution

$$n(x,t) = \frac{n}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right), \tag{10}$$

with the initial condition:

$$n(x,0) = n\delta(x), \tag{11}$$

and the normalization:

$$\int n(x,t)dx = n. \tag{12}$$

From equation (10), we can immediately write down an expression for the mean square displacement (from the origin):

$$\langle x^2 \rangle = \frac{1}{n} \int x^2 n(x,t)dx, \tag{13}$$

which, upon substitution of equation (10), yields

$$\langle x^2 \rangle = 2Dt. \tag{14}$$

The remarkable aspect of this analysis is that while the thesis dealt with the macroscopic aspects of diffusion, equation (14) puts it in the proper perspective of fluctuation phenomena: $\langle x^2 \rangle$ is related to the mean square fluctuation of $x(\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2)$. Furthermore, inserting the expression for D in equation (7) yields

$$\langle x^2 \rangle = \frac{RT}{3\pi N\eta a} t. \tag{15}$$

This term is the first clear statement of the Fluctuation-Dissipation theorem. While the left hand side of (15) measures fluctuations, the right hand side, because of its dependence on the viscosity and hence, friction, embodies dissipation! Apart from this theoretical nicety, (15) provides a remarkable operational aid to computing the

Avogadro number N: prepare a set of small spheres, be it of sugar molecules or of polystyrene, or whatever,... use a stopwatch and a microscope and keep measuring $\langle x^2 \rangle$ – the result yields N!

Einstein did not just stop at the diffusion equation (9), he laid the foundation of the Stochastic Theory of Brownian Motion by giving a new interpretation to (9) in terms of what is now called a Markov process. This was based on a simple ansatz that the suspended particles move independently of one another, again justified because of the dilution involved. This means that if the density of particles within the region x and $x + dx$ grows from $n(x,t)$ at time t to $n(x,t+\tau)$ at time $t+\tau$, where τ is an incremental change ($\tau \ll t$), the growth must be at the expense of the density at the preceding step $(x-\Delta)(|\Delta| \ll |x|)$ measured by $n(x-\Delta,t)$. Thus

$$n(x,t+\tau) = \int_{-\infty}^{\infty} d\Delta n(x-\Delta,t)\Phi(\Delta), \ t \gg \tau, \qquad (16)$$

where $\Phi(\Delta)d\Delta$ is the probability that a particle is displaced, in an interval τ, between Δ and $\Delta + d\Delta$. Note that the independence of particle motion is tacitly incorporated in writing probabilities in a multiplicative form, under the integral in (16). The probability $\Phi(\Delta)$ is expected to have the plausible properties:

$$\int_{-\infty}^{\infty} \Phi(\Delta)d\Delta = 1, \qquad (17)$$

and

$$\Phi(\Delta) = \Phi(-\Delta). \qquad (18)$$

Equation (17) means probability is conserved while (18) stipulates that 'jumps' are symmetric, reminiscent of unbiased random walks in one dimension! We may then

Taylor-expand both sides of (16) in space as well as time, to yield

$$n(x,t) + \tau \frac{\partial n(x,t)}{\partial t} = \int_{-\infty}^{\infty} d\Delta \Phi(\Delta) \left[n(x,t) + \frac{\Delta^2}{2} \frac{\partial^2 n(x,t)}{\partial x^2} \right], \tag{19}$$

where the first order term, proportional to Δ, under the integral vanishes because of the symmetry condition (18). Employing then equation (17), we arrive at the diffusion equation (9), but now the diffusion constant has a new interpretation:

$$D = \frac{1}{2\tau} \int_{-\infty}^{\infty} \Delta^2 \Phi(\Delta) d\Delta. \tag{20}$$

This 'microscopic' formulation of diffusion in which the coefficient of diffusion is obtained from the 'random walk' of a tagged particle provides a nice link, through (9), to 'macroscopic' diffusion of a region (between x and $x+dx$) containing a cluster of particles.

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It is interesting to reflect on what Einstein himself thought of his paper on Brownian motion. In 1915, the year in which he worked out the general theory of relativity, Einstein writes "... the theory of Brownian motion is of great importance since it permits an exact computation of N ... The great significance as a matter of principle is, however, ... that one sees directly under the microscope part of the heat energy in the form of mechanical energy ...," and in 1917, Einstein further muses "... because of the understanding of the essence of Brownian motion, suddenly all doubts vanished about the correctness of Boltzmann's interpretation of the thermodynamic laws." In the words of Pais: "Had I to compose a one-sentence scientific biography of Einstein, I would write, better than anyone before or after him, he knew how to invent invariance principles and make use of statistical fluctuations."

The ideas of Brownian motion are still vigorously alive today in myriad branches of science dealing with the condensed phase of matter, biology and materials science – in particular, nanomaterials science. They have also opened new vistas to basic issues, concerning for example, nonequilibrium statistical physics and quantum Brownian motion.

B. Black Body Radiation

Having discussed Brownian motion I now indicate how thermodynamics and fluctuation ideas also shaped Einstein's work on the Black Body Radiation. For this, we turn to the year 1900 when Max Planck proposed his celebrated formula for the energy density of black body radiation. Indeed, the Planck formula is the essential starting point in our effort to appreciate Einstein's path-breaking contribution to the photon and the photoelectric effect.

In *Figure* 5 I have sketched what may be called a black body. The energy density $\varepsilon(\nu)$ or the energy per unit volume as a function of the frequency ν of radiation and temperature T is shown in *Figure* 6, and can be written as:

$$\varepsilon(\nu) = \frac{\alpha\nu^3}{e^{\frac{\gamma\nu}{T}} - 1}, \tag{21}$$

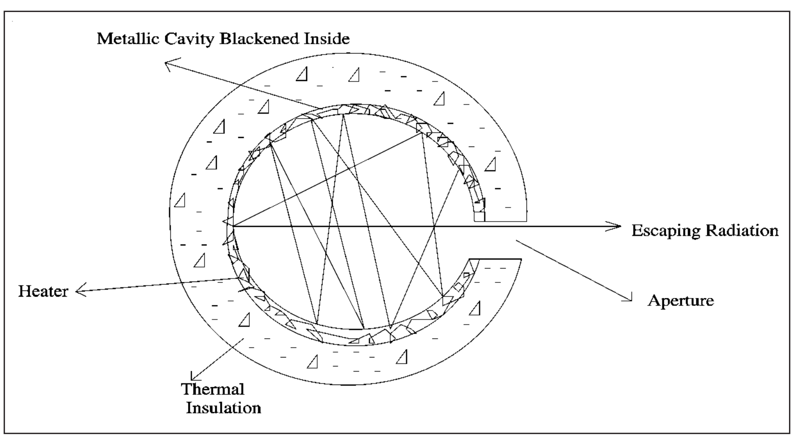

Figure 5. A hollow spherical black body.

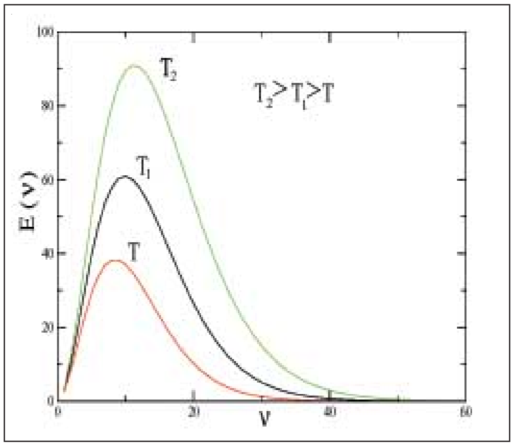

Figure 6. Black body spectrum for various temperatures.

where α and γ are constants, the determination of which constitutes one of the important aims of the 1905 paper of Einstein. The basic point of Planck was that the formula (21) interpolates between the low frequency ($\propto \nu^2$) quadratic behaviour (cf. *Figure* 6) attributed to Rayleigh and Jeans and the high frequency ($\propto \exp(-\frac{\gamma\nu}{T})$) exponential falling-off seen in *Figure* 6, and discovered earlier by Wien.

1. *High T Limit*

Einstein first considered the high temperature (i.e. $\frac{\gamma\nu}{T} \ll 1$) limit of equation (21) to derive

$$\varepsilon(\nu) = \frac{\alpha}{\gamma}\nu^2 T. \qquad (22)$$

He then went on to make the brilliant argument that the black body radiation must be viewed as an ideal photon gas, the energy of which had to be governed by the equipartition theorem of kinetic theory. Therefore, each mode of the electromagnetic field in the black body cavity must contribute an energy $k_B T$ and since the number of modes of frequency ν per unit volume is $\frac{8\pi\nu^2}{c^3}$, c being the speed of light, Einstein concluded:

$$\varepsilon(\nu) = \frac{8\pi\nu^2}{c^3} k_{\mathrm{B}} T. \qquad (23)$$

The fluctuation idea of statistical mechanics has been a recurring theme that shaped Einstein's mind not just in the context of black body radiation but Brownian motion as well.

(Note that in writing (23) Einstein was using the concept of thermodynamic fluctuation of energy; indeed the fluctuation idea of statistical mechanics has been a recurring theme that shaped Einstein's mind not just in the context of black body radiation but Brownian motion as well, as discussed earlier.) This alternate form of $\varepsilon(\nu)$ in (23) allowed Einstein to immediately compute $\frac{\alpha}{\gamma}$ in terms of the fundamental constants k_B and c:

$$\frac{\alpha}{\gamma} = \frac{8\pi k_{\mathrm{B}}}{c^3}, \tag{24}$$

the value of which turns out to be 1.26×10^{-46} in c.g.s. units.

2. *Low T Limit*

Einstein next proceeded to study the low temperature limit of (21), in which

$$\varepsilon(\nu) \simeq \alpha\nu^3 \exp\left(-\frac{\gamma\nu}{T}\right) \ . \tag{25}$$

This expression then provides an alternate definition of the inverse temperature T^{-1} (by taking logarithm of (25)) which he equated to the thermodynamic definition of T^{-1} from the second law

$$\frac{1}{T} = \frac{1}{\gamma\nu} \ln\left(\frac{\varepsilon}{\alpha\nu^3}\right) = \frac{\partial s}{\partial \varepsilon}, \tag{26}$$

where $s = \frac{S}{V}$ is the specific entropy, S being the extensive entropy for a volume V. (Note once again how much did thermodynamics influence Einstein!)

The last two expressions in equation (26) yield a differential equation for s in terms of ε which, when integrated, yields

$$s = -\frac{1}{\gamma\nu}\left[\varepsilon \ln\left(\frac{\varepsilon}{\alpha\nu^3}\right) - \varepsilon\right] + \text{constant}. \tag{27}$$

Einstein then determined the integration constant by first converting the left hand side of (27) into an equation for the extensive entropy S and equating the reference entropy for a volume V_0 to S_0, thus

$$S - S_0 = \frac{E}{\gamma\nu} \ln\left(\frac{V}{V_0}\right) \ , \qquad (28)$$

where $E(=\varepsilon V)$ is the total energy of the radiation, kept fixed.

Finally, Einstein took recourse to Boltzmann who, as already mentioned, had a profound impact on Einstein's work, and had given the celebrated entropy formula:

$$S - S_0 = k_{\mathrm{B}} \ln\left(\frac{W}{W_0}\right) \ , \qquad (29)$$

W being the probability of configuration of n photons in a volume V. For a photon gas,

$$W = V^n, \qquad (30)$$

and therefore,

$$S - S_0 = nk_{\mathrm{B}} \ln\left(\frac{V}{V_0}\right). \qquad (31)$$

Comparing (31) with (28) Einstein surmised that

$$\frac{E}{\gamma\nu} = nk_{\mathrm{B}}, \qquad (32)$$

or

$$E = n(\gamma k_B)\nu. \qquad (33)$$

Equation (33) then yields the remarkable energy quantization formula:

$$E = nh\nu, \qquad (34)$$

where the constant appearing in the Planck formula (21) can be identified as $\frac{h}{k_{\mathrm{B}}}$. It must however be emphasized that the energy E appearing in (34) is the

The remarkable feature of Einstein's work on the photon is that in one shot he broke away from the shackles of Maxwell's concept of light – light is wave that exhibits interference, diffraction and polarization.

thermodynamic energy which, in modern statistical mechanics, has the interpretation of $\langle H \rangle$, i.e., the average of the Hamiltonian operator. Consequently, n in (34) must be regarded as the average photon number that is given by $\langle \hat{N} \rangle$, $\hat{N}$ being the number operator. It is the unraveling of the angular brackets $\langle \ldots \rangle$ that takes us from thermodynamics to statistical mechanics, i.e. from the macroworld to the microworld. And therein lies the transition from Albert Einstein to Satyendra Nath Bose. While Einstein looked at the Planck formula from the reference frame of the macroworld, Bose's work made possible the further conceptual jump into the microworld and the discovery of a new statistics. But that is a different story.

The remarkable feature of Einstein's work on the photon is that in one shot he broke away from the shackles of Maxwell's concept of light – light is wave that exhibits interference, diffraction and polarization. Einstein discards this notion of light as wave and in a sense, resurrects the corpuscle idea of Newton in treating light! In making this bold step Einstein was of course guided (as he inevitably was, in the year 1905) by experiments – the experiments of Lenard (1902) which revealed that the maximum electron energy ejected from a metal due to impinging light showed "not the slightest dependence on the light intensity". That led to the famous equation $\varepsilon_{\max} = h\nu - \phi$, ϕ being the work function of the metal, which forms the cornerstone of the photoelectric effect. Speaking of this equation Millikan (1915) states: "I spent 10 years testing that 1905 equation of Einstein and contrary to all my expectations, I was compelled to assert its unambiguous verification in spite of its unreasonableness since it seems to violate everything we know about the interference of light ". Millikan further deduced from his experiments the value of the Planck constant h as 6.4×10^{-27} erg-sec.

Taking the two papers of Einstein, on Brownian motion

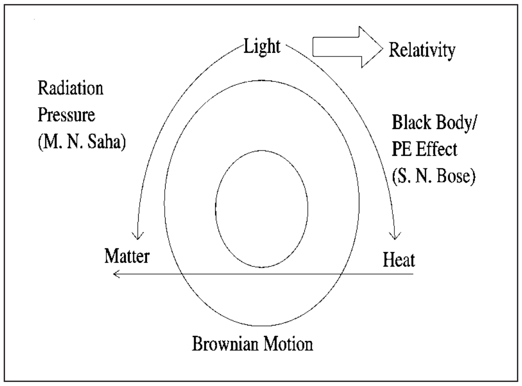

Figure 7. Trinity in Einstein's World.

and the black body radiation, in juxtaposition, we obtain a glimpse of the *Trinity* in Einstein's world: *Light, Matter and Heat* (see *Figure* 7). Light consists of particles called photons which are responsible for the photo electric effect.

At the same time photons constitute the black body radiation within a cavity which is at thermal equilibrium, determining spectrum of energy levels associated with atoms and molecules of the cavity. Thus, light gets related to heat, as can be seen more clearly through the work of Bose and the Bose–Einstein statistics of photons, in which the temperature plays an essential role. On the other hand, light is also electromagnetic waves, described by Maxwell's equations which, in turn, are invariant under the Lorentz transformation of the Special Theory of Relativity. From light to heat, we connect to Brownian motion, which is the result of random motion in a fluid, generated by 'spontaneous' thermal fluctuations due to heat. Thus, heat leads to damping through viscous forces which impinge on matter. Matter is something that we study in mechanics but material motion is not only influenced by heat, as in Brownian movement, but also by light, due to a phenomenon called Radiation Pressure, thus completing the circle of Trinity. Radiation pressure is a subject to which another great savant of Indian Science, Meghnad Saha, made invaluable contributions.

C. Concluding Remarks

I would like to conclude by emphasizing once again that young Einstein was very close to experiments – he was deeply influenced by Lenard's measurements on the photoelectric effect, Brown's observations of random motion of suspended pollens, the viscosity data in sugar solutions, and so on; and he was interested in computing numbers, such as the Avogadro number and the Planck constant. This brings us to point out one very significant attribute, not often widely appreciated, of Einstein's 1905 contributions – they were all motivated by phenomena, as we begin to recognize more in analyzing his treatment of the Brownian motion, discussed in Section A. The second lesson is about Einstein's open-mindedness in embracing ideas and concepts from apparently distant fields. Thus even though he was concerned with the elementary particle of photon in his analysis of the photo electric effect, Einstein was constantly borrowing methods and concepts from Boltzmann's thermodynamics, as elaborated in Section B. Similarly, he adapted the ideas of hydrodynamics, then in the domain of chemical engineers, to his treatment of the Brownian motion. Finally, we should learn from what turns out to be the hallmark of Einstein's approach – his unflinching boldness. Even though Maxwell's electromagnetic theory was well entrenched in terms of the observed wave phenomena of interference and diffraction Einstein shows rare courage in its complete repudiation. Interestingly enough, the same Maxwellian theory is resuscitated by Einstein when he turns later, in the same year of 1905, to the special theory of relativity!

Einstein was not just a *Gyana Yogi* but a *Karma Yogi* as well!

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