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Quantum mechanics of rapidly and periodically driven systems

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Abstract. This review deals with the dynamics of quantum systems that are subject to high frequency external perturbations. Though the problem may look hopelessly time-dependent, and poised on the extreme opposite side of adiabaticity, there exists a 'Kapitza Window' over which the dynamics can be treated in terms of effective time-independent Hamiltonians. The consequent results are important in the context of atomic traps as well as quantum optic properties of atoms in intense and high-frequency electromagnetic fields.

Keywords. Rapid forcing; Kapitza pendulum; effective Hamiltonian; quadrupole and Paul traps; strong-field excitations.

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1. Introduction

Time-dependent perturbations of systematic dynamical evolution are widely studied in a variety of contexts. If the laws of motion of the system are governed by Classical Mechanics, straightforward treatments are available when the amplitude and the frequency of the perturbations are small. When the external perturbations are rapid and intense, however, one would normally think that the analysis would be quite complex and would require numerical treatments. But an early argument due to Kapitza [1], reproduced in the book on *Mechanics* by Landau and Lifshitz [2], showed that a separation of two time-scales, a 'slow' one governing the dynamics of the system at hand, and a 'fast' one, attributed to the external force assumed periodic, yields an effective, 'time-independent' dynamics of the system alone. The Kapitza argument runs as follows.

The Newtonian equation of motion of a mechanical particle of mass m, moving under a systematic potential $V(x)$ but subject to a rapid force $f(x,t)$, is given by

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x} + f(x,t), \tag{1}$$

where the two overhead dots denote double time derivatives. The assumption of one-dimensional motion for the coordinate x does not detract from the general conclusion, discussed below. Further, for the present as well as the subsequent analysis, we shall take the external force to be of the form

$$f(x,t) = a(x)\cos(\omega t), \tag{2}$$

where both the amplitude $a(x)$ and the frequency ω can be arbitrarily large. Again, the assumption of monochromaticity does not cause any loss of generality.

Because ω is assumed much larger than the characteristic frequencies of the system determined by the nature of $V(x)$, it makes sense to split $x(t)$ as

$$x(t) = X(t) + \xi(t), \tag{3}$$

where $\xi(t)$ is a rapidly changing coordinate which averages out to zero over a period $2\pi/\omega$, during which time $X(t)$ changes but very little. Substituting in eq. (1) yields

$$m(\ddot{X}(t) + \ddot{\xi}(t)) = -\left(\frac{\mathrm{d}V}{\mathrm{d}X} + \xi\frac{\mathrm{d}^2V}{\mathrm{d}X^2}\right) + \left(a(X) + \xi\frac{\mathrm{d}a(X)}{\mathrm{d}X}\right)\cos(\omega t), \tag{4}$$

where we have kept terms to linear order in ξ. Since $\xi(t)$ is the fast variable, varying with the periodic force, we immediately read off from eq. (4) that, to a very good approximation,

$$m\ddot{\xi}(t) = a(X)\cos(\omega t), \tag{5}$$

which integrates to, over a time-scale in which X is approximately constant,

$$\xi(t) = -\frac{a(X)}{m\omega^2}\cos(\omega t). \tag{6}$$

Equation (6) provides a retrospective justification of why merely a linear expansion in $\xi(t)$ makes sense: $\xi(t)$ is not only rapidly oscillatory but has an amplitude that has an inverse quadratic dependence on the frequency ω. Furthermore, it is appropriate to view $\xi(t)$ as an expansion parameter – even if a and ω are of the same orders of magnitude (in dimensionless units), $\xi(t)$ is effectively small. The final step is to substitute eq. (6) in eq. (4) and average over a complete cycle of the perturbation (indicated by overhead bar) leading to

$$m\ddot{\bar{X}}(t) = -\frac{\mathrm{d}}{\mathrm{d}\bar{X}}\left[V(\bar{X}) + \frac{a^2(\bar{X})}{4m\omega^2}\right]. \tag{7}$$

Equation (7) is then the desired result which suggests that, once the rapidly oscillating terms die out, the 'smoothly' varying coordinate $\bar{X}(t)$ follows dynamics dictated by an effective *static* potential:

$$V_{\mathrm{eff}}(\bar{X}) = V(\bar{X}) + \frac{a^2(\bar{X})}{4m\omega^2}. \tag{8}$$

An essentially similar result as in eq. (8) was also discussed by Goponov and Miller [3] as well as Weibel and Clark [4] (see also Percival and Richards [5]).

Given this background on how to treat a classical, dynamical system under rapid forcing, we turn to the corresponding issue in quantum mechanics. Indeed the rest of the review, much of which is described in §2, is devoted to a comparative study of different treatments, extant in the literature, for analyzing quantum evolution under the influence of rapidly varying periodic forces. With the advent of intense and coherent laser beams as well as high-frequency magnetic fields, such issues have become pertinent within the realm of experiments. A few selective examples are to be found in the context of motion in atom traps and dynamics of cold atoms [6–8], atomic billiards [9,10], quantum spin dynamics in the presence of high frequency magnetic fields, a wide class of light–atom interactions [11–14] and so on. One simplifying assumption that we shall make at the outset is that while the system variables are to be treated as quantum operators, the external perturbations can be viewed in classical, parametric terms. Such an assumption is eminently reasonable in the context of coherent laser fields, not to mention laboratory magnetic fields.

The analysis of quantum evolution in the presence of time-dependent perturbations is of course not new. The entire field of resonance spectroscopy, in the context of nuclear magnetic resonance, electron spin resonance, fluorescence, etc., is replete with such phenomena. But in all these applications, the frequency of the external fields matches with the 'intrinsic' frequency of the system and hence first-order time-dependent perturbation theory is quite adequate, especially when the field amplitudes are small [15]. We are however interested in situations in which the external frequencies are so large that the concept of transitions between stationary eigenstates of the unperturbed Hamiltonian is no longer valid. The question therefore is: is there a Kapitza-like analysis for quantum systems; further can one go beyond the $\frac{1}{\omega^2}$-expansion implied in eq. (6)?

The direct quantum analogue of the classical equation (1) is provided by the Heisenberg picture although the most physical and graphic formulation of quantum mechanics is in the Schrödinger picture [16]. In §2.1, we discuss a method due to Cook *et al* [17], which is the simplest and closest quantum version of the Kapitza ansatz, through the formulation of the Schrödinger equation. The result, like in the Kapitza method, is independent of the specific forms of the unperturbed potential $V(x)$ and the perturbing force $f(x)$, but is restricted to $O(1/\omega^2)$. To the best of our knowledge the first serious attempt to generalize the elegantly facile treatment of Cook *et al* is by Grozdanov and Raković [18]. The latter authors employ a powerful approach using a time-dependent canonical transformation but in actual implementation, their results are restricted to uniform, space-independent forcing. Further, even though the results can be extended to $O(1/\omega^4)$ and beyond, the higher order terms lose their validity if the unperturbed potential is not differentiable as discussed in §2.2. On the other hand, the Cook *et al* method does not suffer from this limitation, though the results are valid up to $O(1/\omega^2)$ only. In §2.3 we discuss an analysis due to Gilary and Moiseyev [19] who employ the so-called Kramers–Henneberger representation [20]. This analysis, however, assumes from the very beginning the constancy (in space) of the external force, and in addition, is shown to be equivalent to the approach of Cook *et al.* The Grozdanov–Raković treatment is taken to its logical and rightful extension by Rahav *et al* [21] who base their

calculation on the extensive use of the Floquet theory. By ensuring at every step the classical limit of the theory, Rahav *et al* were able to remove the restrictive assumptions of Grozdanov and Raković. Indeed as discussed in §2.4, the results are a true generalization of those derived by Cook *et al*, except once again, the terms of $O(1/\omega^4)$ and higher depend on the differentiability of the unperturbed potential. Section 3 is devoted to a comparative assessment of the various approaches. The first case, discussed in §3.1, is that of the exactly treatable model of a linearly driven quantum oscillator.

2. Effective quantum Hamiltonian: Various approaches

With the preceding introductory discussion we turn in this section to different approaches to the quantum problem when there is a high frequency external perturbation. Our goal, as has indeed been spelt out in §1, is to derive effective static Hamiltonians through which standard quantum procedures can be applied. In the process we will see that the underlying averaging of the external forces leading up to an effective Hamiltonian can in fact introduce new terms which can qualitatively alter the dynamics.

2.1 *The method of Cook et al*

The time-dependent, one-dimensional Schrödinger equation can be written as

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \Big[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + v(x)\cos(\omega t)\Big]\psi(x,t), \tag{9}$$

where, in the notation of §1,

$$a(x) = -\frac{\mathrm{d}v(x)}{\mathrm{d}x}. \tag{10}$$

The ansatz of Cook *et al* is to write the wave function $\psi(x,t)$ as the product of an unknown wave function $\phi(x,t)$ and a function that is a solution of eq. (9), were the external perturbation is the only term present. Thus

$$\psi(x,t) = \phi(x,t)\exp\left[-i\frac{v(x)\sin(\omega t)}{\hbar\omega}\right]. \tag{11}$$

Note that the term within the square parentheses is just the solution of eq. (9) (apart from a multiplicative initial wave function) if the oscillatory force was the only dominant presence. The wave function, split off in the above prescribed manner, serves an analogous purpose as does the displacement $x(t)$ in the Newtonian picture.

Substituting eq. (11) in eq. (9) yields

$$i\hbar\frac{\partial}{\partial t}\phi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\phi(x,t)$$

$$+\frac{i\hbar}{2m}\sin(\omega t)\left[\frac{\mathrm{d}v(x)}{\mathrm{d}x}\frac{\partial}{\partial x}+\frac{1}{2}\frac{\mathrm{d}^2v(x)}{\mathrm{d}x^2}\right]\phi(x,t)$$
$$+\frac{1}{2m\omega^2}\left[\frac{\mathrm{d}v(x)}{\mathrm{d}x}\right]^2\sin^2(\omega t)\phi(x,t). \tag{12}$$

Like in the classical treatment, we average over the period $2\pi/\omega$, and derive for the 'smooth' function $\bar{\phi}(x,t)$, an effective Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\bar{\phi}(x,t)=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)+\frac{(v'(x))^2}{4m\omega^2}\right]\bar{\phi}(x,t). \tag{13}$$

Keeping eq. (10) in mind, eq. (13) provides a surprisingly simple derivation of the form of the effective potential, as given in eq. (8). Although the treatment here is restricted to $O(1/\omega^2)$ there is no further condition imposed on the form of either $V(x)$ or $v(x)$. Indeed, as we see in the sequel, attempts to go beyond $O(1/\omega^2)$ are disproportionately harder and are restricted to special structures of $V(x)$ and $v(x)$.

2.2 *Time-dependent canonical transformation*

Unlike the method of Cook *et al*, the starting point of the treatment of Grozdanov and Raković [18] is the Hamiltonian in the Heisenberg picture, written as

$$\mathcal{H}(p,x,t)=\mathcal{H}_0(p,x)+v(x)\cos(\omega t). \tag{14}$$

Actually, in ref. [18], an extra phase factor has been considered in the oscillatory term, which has been ignored in the present discussion without sacrificing any essential content. A quantum version of an explicitly time-dependent canonical transformation (in its classical Lie algebraic formulation [16]) is a unitary transformation effected by a Hermitian operator that is time periodic:

$$S(p,x,t)=S\left(p,x,t+\frac{2\pi}{\omega}\right). \tag{15}$$

The unitary transformation takes the old set $\{p,x\}$ to a new one $\{P,X\}$ such that

$$P(p,x,t)=\exp[-iS(p,x,t)]\,p\exp[iS(p,x,t)]\,,$$
$$X(p,x,t)=\exp[-iS(p,x,t)]\,x\exp[iS(p,x,t)]\,. \tag{16}$$

The idea is to find an S in such a way that the new Hamiltonian $K(P,X)$ which obeys the following equation

$$K(P,X)=\exp[iS(p,x,t)]\,\mathcal{H}\exp[iS(p,x,t)]$$
$$-i\exp[iS(p,x,t)]\,\frac{\partial}{\partial t}\exp[-iS(p,x,t)]\,, \tag{17}$$

is actually time-independent! Some of the details are given in §2.4 below, and are not repeated here.

The underlying strategy is to make an asymptotic expansion of the generator S in inverse powers of ω, exploit the periodicity of the coefficients S_n of the expansion and thereby inductively determine S_n. Once that is achieved, eq. (17) is employed to obtain an effective static Hamiltonian that is also expressed as a power series in ω^{-2}. It is to be noted that only even powers in ω^{-1} are relevant, as is evident also in the classical argument. However, it turns out that the results beyond $O(1/\omega^2)$ terms are not useful unless one assumes the external potential $v(x)$ to be linear in x, *i.e.*, the corresponding force is uniform, in accordance with

$$v(x) = -ax. \tag{18}$$

It should be emphasized that a form such as given by eq. (18) is often employed in the dipole approximation to electromagnetic fields.

With the proviso of eq. (18) the Grozdanov–Raković expression for the effective (and canonically transformed) Hamiltonian is given, upto $O(1/\omega^4)$, by

$$\begin{aligned}\mathcal{H}_{\text{eff}}(P,X) = \mathcal{H}_0(P,X) &- \frac{1}{4(\hbar\omega)^2}\Big[v,[v,\mathcal{H}_0]\Big] \\ &- \frac{1}{4(\hbar\omega)^4}\Big[v,[\mathcal{H}_0,[\mathcal{H}_0[v,\mathcal{H}_0]]]\Big] + \cdots .\end{aligned} \tag{19}$$

With the specific form chosen in eq. (18), eq. (19) reads

$$\mathcal{H}_{\text{eff}}(P,X) = \mathcal{H}_0(P,X) + \frac{a^2}{4m\omega^2} + \frac{a^2}{4m^2\omega^4}\frac{\mathrm{d}^2V(x)}{\mathrm{d}X^2} + O\Big(\frac{1}{\omega^6}\Big). \tag{20}$$

We reiterate that while eq. (20) is an improvement over the result (13) of Cook *et al*, it suffers from the restriction of eq. (18). Further, the method presupposes that the unperturbed potential energy $V(X)$ has non-singular derivatives.

2.3 *Kramers–Henneberger representation*

Our next item of discussion is a method due to Gillary and Moiseyev [19] who introduce the concept of quasi-stationary states, based on what is called the Kramers–Henneberger (KH) representation. This method is however restricted at the outset to the case of a uniform external force (see eq. (18)). Before we analyze the treatment of Gillary and Moiseyev we wish to show how the ansatz of Cook *et al* (see eq. (11)) naturally leads to the KH representation. Recall that the Schrödinger equation (12) for the case of eq. (18) reads as

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\phi(x,t) = \Big(\frac{\hat{p}^2}{2m} + V(x)\Big)\phi(x,t) &- \frac{a}{m\omega}\sin(\omega t)\hat{p}\phi(x,t) \\ &+ \frac{a^2}{2m\omega^2}\sin^2(\omega t)\phi(x,t),\end{aligned} \tag{21}$$

where the momentum operator $\hat{p}$ has its usual representation:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}. \tag{22}$$

In the present case of a uniform force, the appropriate KH representation transforms $\phi(x,t)$ to

$$\chi(x,t) = \mathrm{e}^{-ig_2(t)}\mathrm{e}^{-ig_1(t)\hat{p}}\phi(x,t), \tag{23}$$

where

$$g_1(t) = -\frac{a}{m\omega^2\hbar}\cos(\omega t), \tag{24}$$

$$g_2(t) = -\frac{a^2}{4m\omega^2\hbar}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]. \tag{25}$$

It is easy to see that

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\chi(x,t) &= \mathrm{e}^{-ig_2(t)-ig_1(t)\hat{p}}\Big[i\hbar\frac{\partial\phi(x,t)}{\partial t} + \frac{a}{m\omega}\sin(\omega t)\hat{p}\phi(x,t) \\ &\quad -\frac{a^2}{2m\omega^2}\sin^2(\omega t)\phi(x,t)\Big] \\ &= \mathrm{e}^{-ig_2(t)-ig_1(t)\hat{p}}\Big(\frac{\hat{p}^2}{2m} + V(x)\Big)\phi(x,t), \end{aligned} \tag{26}$$

where we have employed eq. (21). We now use the identities

$$\mathrm{e}^{-ig_1(t)\hat{p}}V(x)\mathrm{e}^{ig_1(t)\hat{p}} = V(x) + \sum_{n=1}^{\infty}\frac{(-ig_1(t))^n}{n!}[\hat{p},[\hat{p},...[\hat{p},V(x)]...]], \tag{27}$$

and

$$[\hat{p},V(x)] = -i\hbar\frac{\partial V(x)}{\partial x}, \tag{28}$$

to reduce the Schrödinger equation for $\chi(x,t)$ to the KH form:

$$i\hbar\frac{\partial}{\partial t}\chi(x,t) = \Big(\frac{\hat{p}^2}{2m} + V(x - \hbar g_1(t))\Big)\chi(x,t). \tag{29}$$

Thus the KH trick is to derive a Schrödinger equation in an 'accelerated frame' in which the displacement coordinate x is transformed into

$$\tilde{x}(t) = x + \frac{a}{m\omega^2}\cos(\omega t), \tag{30}$$

where we have taken note of the form of g_1 (see eq. (24)). It is interesting to observe that the transformed variable $\tilde{x}(t)$ is precisely the 'quasi-stationary' Kapitza coordinate $X(t)$ (see eqs (3) and (6)).

Gillary and Moiseyev exploit the above KH transformation and define an effective potential (in the spirit of Kapitza and Cook *et al*) by

$$V_0^{\mathrm{KH}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V^{\mathrm{KH}}(x,t)\mathrm{d}t, \tag{31}$$

where

$$V^{\mathrm{KH}}(x,t) = V(x - \hbar g_1(t)). \tag{32}$$

The strategy is to use *time-independent* perturbation theory in which the expansion is carried out in terms of the 'difference' potential energy

$$V_{\mathrm{pert}} = V^{\mathrm{KH}}(x,t) - V_0^{\mathrm{KH}}. \tag{33}$$

The term V_{pert} is further expanded into a Fourier series:

$$V_{\mathrm{pert}} = \sum_{m\neq 0} V_m(x) \mathrm{e}^{im\omega t}, \tag{34}$$

where

$$V_m(x) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \mathrm{d}t \mathrm{e}^{-im\omega t} V^{\mathrm{KH}}(x,t). \tag{35}$$

It is then straightforward to derive 'quasi-energy' eigenvalues in time-dependent perturbation theory, and as would be expected, the results are identical to those obtained earlier by the method of Cook *et al*.

2.4 *The method of Rahav et al*

To date the most systematic approach to derive effective Hamiltonians for periodically driven systems is due to Rahav *et al* [21]. The latter authors first go beyond the classical Kapitza treatment by extending the results to orders higher than $1/\omega^2$ and then apply a similar approach to the quantum domain. The resultant analysis is free from some of the restrictive conditions of Grozdanov and Raković, detailed in §2.2.

We begin with the classical analysis. Recall (see eq. (6)) that the rapidly varying coordinate $\xi(t)$ in the classical method, besides being proportional to $1/\omega^2$, is a function of X alone, the 'slow' coordinate. The first idea of Rahav *et al* is to bring $\dot{X}$ as well, on the same footing as X, wherein the dot denotes time derivative. That would be tantamount, in the Hamiltonian picture, to treating the momentum P, in addition to X, as a slow variable. Thus, as a generalization to eq. (3), one writes

$$x(t) = X(t) + \xi(X, \dot{X}, \omega t), \tag{36}$$

where the time t is replaced by the dimensionless variable $\tau = \omega t$. Correspondingly,

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \omega \frac{\partial \xi}{\partial \tau} + \frac{\partial \xi}{\partial X}\dot{X} + \frac{\partial \xi}{\partial \dot{X}}\ddot{X}. \tag{37}$$

The Newton's equation (1) then reads as

$$m\Big(\ddot{X}+\omega^2\frac{\partial^2\xi}{\partial\tau^2}+2\omega\Big[\frac{\partial^2\xi}{\partial X\partial\tau}\dot{X}+\frac{\partial^2\xi}{\partial\dot{X}\partial\tau}\ddot{X}\Big]$$
$$+\frac{\partial\xi}{\partial X}\ddot{X}+\frac{\partial\xi}{\partial\dot{X}}\ddot{X}+\frac{\partial^2\xi}{\partial X^2}\dot{X}^2+2\frac{\partial^2\xi}{\partial X\partial\dot{X}}\dot{X}\ddot{X}+\frac{\partial^2\xi}{\partial\dot{X}^2}\ddot{X}^2\Big)$$
$$=-V'(X+\xi)+f(X+\xi,\tau), \tag{38}$$

which generalizes eq. (4).

From eq. (6) we can anticipate that ξ can be expanded as a power series:

$$\xi=\sum_{i=1}^{\infty}\frac{\xi_i}{\omega^i}, \tag{39}$$

where the ξ_i's are so chosen that the equation for X emanating from eq. (38) is independent of τ. As in §1, we expand the right-hand side of eq. (38) in a Taylor series in ξ, substitute eq. (39) and solve for each ξ_i by integrating the corresponding equations of motion of ξ_i over a complete cycle of the perturbation, eq. (2). The resultant equation for the slow variable X can be derived from the 'effective' Hamiltonian

$$\mathcal{H}_{\rm eff}(P,X)=\frac{P^2}{2m}+\frac{3}{4m^3\omega^4}[a'(X)]^2P^2+V_{\rm eff}(X), \tag{40}$$

where

$$V_{\rm eff}(X)=V(X)+\frac{a^2(X)}{4m\omega^2}+\frac{a^2(X)a'(X)}{2m^2\omega^4}+\frac{a^2(X)}{4m^2\omega^4}V''(X)+O(\omega^{-6}). \tag{41}$$

Note that if the external force is uniform (see eq. (18)), eq. (40) would reduce to the result of Grozdanov and Raković (see eq. (20)).

We move on now to the quantum treatment which is identical in spirit to the time-dependent canonical transformation of Grozdanov and Raković [18], but is set up such that to every order in the $1/\omega$ expansion, the correct classical limit of eq. (40) is recovered. Recall that the unitary transformation effected by the operator $\hat{S}$ (see eq. (16)) is actually a *gauge transformation* in which the Hamiltonian operator $\hat{K}$ in the new gauge is time-independent (see eq. (17)) in the Schrödinger picture. The Schrödinger equation in the new gauge

$$i\hbar\frac{\partial}{\partial t}\phi=\hat{K}\phi, \tag{42}$$

has eigenfunction solutions that can be expressed as

$$\phi_\lambda(t,x)=\mathrm{e}^{-i\lambda t/\hbar}v_\lambda(x), \tag{43}$$

λ being the eigenvalue. Since the state function ϕ is related to the wave function ψ in the original gauge by $\phi=\mathrm{e}^{i\hat{S}}\psi$, the states in the original gauge correspond to

$$\psi_\lambda(t,x)=\mathrm{e}^{-i\hat{S}}\phi_\lambda=\mathrm{e}^{-i\lambda t/\hbar}\mathrm{e}^{-i\hat{S}}v_\lambda(x). \tag{44}$$

Because $\hat{S}$ has the time-periodicity described by eq. (15), the Floquet theory [22–26] tells us that

$$u_\lambda(x,t) = \mathrm{e}^{-i\hat{S}} v_\lambda(x) \tag{45}$$

has the same periodicity as that of the Hamiltonian operator $\mathcal{H}$ and the Hermitian operator $\hat{S}$ (see eq. (15)). Therefore, ψ_λ is a Floquet state with quasi-energy λ (mod $\hbar\omega$).

Introducing the reduced time $\tau(=\omega t)$ as before (see eqs (36) and (37)), the gauge transformation of eq. (17) can be re-expressed as

$$\hat{K} = \mathrm{e}^{i\hat{S}}\hat{\mathcal{H}}\mathrm{e}^{-i\hat{S}} - i\hbar\omega\mathrm{e}^{i\hat{S}}\Big(\frac{\partial \mathrm{e}^{-i\hat{S}}}{\partial\tau}\Big). \tag{46}$$

At high frequencies, $\hat{S}$ is assumed small, an assumption that is justified in hindsight, at the end of the calculation. The scheme is therefore to expand $\hat{K}$ and $\hat{S}$ in powers of $1/\omega$ and exercise the gauge freedom in choosing $\hat{S}$ so that $\hat{K}$ is time-independent, order by order. These expansions are given by

$$\hat{K} = \sum_{n=0}^{\infty} \frac{1}{\omega^n}\hat{K}_n, \tag{47}$$

and

$$\hat{S} = \sum_{n=1}^{\infty} \frac{1}{\omega^n}\hat{S}_n. \tag{48}$$

The computation is carried out by expressing $\hat{K}_m$ in terms of $\hat{S}_1$, $\hat{S}_2$,...,$\hat{S}_{m+1}$ and then choosing $\hat{S}_{m+1}$ so that $\hat{K}_m$ is time-independent.

We exemplify the method by taking up the case of the leading order, $O(\omega^0)$, in which

$$\hat{K}_0 = \frac{\hat{p}^2}{2m} + \hat{V}(x) + v(x)\cos\tau - \hbar\frac{\partial\hat{S}_1}{\partial\tau}. \tag{49}$$

Hence, in order to cancel the time-dependence of $\hat{K}_0$, we choose

$$\hat{S}_1 = \frac{1}{\hbar}\int_0^\tau \mathrm{d}\tau' \cos(\tau')v(x), \tag{50}$$

which yields

$$\hat{K}_0 = \frac{\hat{p}^2}{2m} + \hat{V}(x), \tag{51}$$

as desired. Similarly, $\hat{S}_2$ is chosen such that $\hat{K}_1$ equals zero. Indeed all odd (in $1/\omega$) terms disappear because of the even (in ω) nature of the perturbation. The calculation of $\hat{K}_2$ proceeds in a straightforward manner. On the other hand, in

the next higher order, $\hat{S}_4$ is chosen in such a manner that the Hamiltonian $\hat{K}$ so calculated reduces in the classical limit to eq. (40).

Collecting all the terms it is found that up to fourth order in $1/\omega$, the Hamiltonian reads

$$\hat{K} = \frac{\hat{p}^2}{2m} + \hat{V}_{\text{eff}} + \frac{1}{4\omega^4}(\hat{p}^2 g(\hat{x}) + 2\hat{p}g(\hat{x})\hat{p} + g(\hat{x})\hat{p}^2) + \frac{\hbar^2}{\omega^4}\hat{V}_x + O(\omega^{-6}). \tag{52}$$

In eq. (52), $\hat{V}_{\text{eff}}$ is the quantum operator form of eq. (41),

$$\hat{g}(\hat{x}) = \frac{3}{m^3}[a'(\hat{x})]^2, \tag{53}$$

which appears as the coefficient of the $\hat{P}^2$ term in the classical Hamiltonian of (40), and $\hat{V}_x$ is a new term that is a quantum correction to eq. (40), the form of which is dictated by the chosen ordering of operators (as done in eq. (52)), and is given by

$$\hat{V}_x = \frac{3}{16m^3}[a''(\hat{x})]^2. \tag{54}$$

Therefore, eq. (52) is an improvement over the result (20) derived by Grozdanov and Raković, the latter result being equivalent to the limiting form of eq. (52) in the special case of a homogeneous external force (see eq. (18)).

3. A comparative analysis

3.1 *Exactly solvable model of a driven oscillator*

In order to make a comparison between the different approaches to periodically driven quantum systems, detailed in §2, it is useful to have as a benchmark a model system that can be handled exactly. As would be expected, examples of such model systems are rare, with the anticipated exception of a quantum harmonic oscillator driven by a space-independent force. We shall first discuss the exact treatment below, and then use it as a basis to cross-check the different methods.

The Schrödinger equation for the one-dimensional driven oscillator can be written from eq. (9) as

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2 - ax\cos\omega t\right]\psi(x,t). \tag{55}$$

A three-dimensional generalization with $\omega_0 = 0$, in which the constant a is taken positive for two axes and negative for the third, constitutes what is called the quadrupole trap. Following Husimi [27], we note that the classical equation of motion for the coordinate, now denoted by ζ, is given by

$$m(\ddot{\zeta} + \omega_0^2\zeta) = a\cos\omega t. \tag{56}$$

The idea therefore is to shift the coordinate about the classical trajectory and introduce

$$y = x - \zeta(t). \tag{57}$$

In terms of y and $\zeta(t)$, eq. (55) reads

$$i\hbar\frac{\partial\psi(y,t)}{\partial t} = \Big[- i\hbar\dot{\zeta}(t)\frac{\partial}{\partial y} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_0^2 y^2 + my\ddot{\zeta}(t) + \frac{1}{2}m\omega_0^2\zeta^2(t) - a\zeta(t)\cos\omega t\Big]\psi(x,t). \tag{58}$$

We now make use of the following unitary transformation [28]:

$$\psi(y,t) = \mathrm{e}^{-im\dot{\zeta}y/\hbar}\phi(y,t), \tag{59}$$

to reduce eq. (58) to

$$i\hbar\frac{\partial}{\partial t}\phi(y,t) = \Big[- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_0^2 y^2 - L(\zeta,\dot{\zeta},t)\Big]\phi(y,t), \tag{60}$$

where $L(\zeta,\dot{\zeta},t)$ is the classical Lagrangian of a driven oscillator

$$L = \frac{1}{2}m\dot{\zeta}^2 - \frac{1}{2}m\omega_0^2\zeta^2 + a\zeta\cos\omega t. \tag{61}$$

It is natural then to make another unitary transformation via the classical action as

$$\phi(y,t) = \mathrm{e}^{\frac{i}{\hbar}\int_0^t \mathrm{d}t' L(\zeta,\dot{\zeta},t')}\chi(y,t), \tag{62}$$

to finally yield the well-known Schrödinger equation of a stationary harmonic oscillator

$$i\hbar\frac{\partial}{\partial t}\chi(y,t) = \Big[- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_0^2 y^2\Big]\chi(y,t). \tag{63}$$

The eigenfunctions associated with eq. (63) are proportional to the well-known Hermite polynomials and the corresponding eigenvalues are

$$E_n = \Big(n + \frac{1}{2}\Big)\hbar\omega, \quad n = 0,1,2,3,.... \tag{64}$$

Collecting eqs (57), (59) and (62), the solution of the Schrödinger equation (55) can be written as

$$\psi_n(x,t) = \mathrm{e}^{-im\dot{\zeta}(x-\zeta(t))/\hbar}\mathrm{e}^{i/\hbar\int_0^t \mathrm{d}t' L(\zeta,\dot{\zeta},t')}\mathrm{e}^{-(i/\hbar)E_n t}\chi_n(x-\zeta(t),t), \tag{65}$$

where $\chi_n(x)$ is the eigenfunction solution of eq. (63).

Our aim, of course, is to elicit the energy eigenvalue, appropriately called 'quasi-energy' eigenvalue in the present context which, apart from E_n, is another term to be extracted from the exponent of eq. (65), that is proportional to $(-i/\hbar)t$. To this end we note that the solution of the classical equation (56) reads as

$$\zeta(t) = \frac{a\cos\omega t}{m(\omega_0^2 - \omega^2)}. \tag{66}$$

The choice of the initial conditions implied in eq. (66) means that (see eqs (59) and (62))

$$\psi(y, t = 0) = \chi(y, t = 0). \tag{67}$$

Because $\chi(y)$ satisfies the stationary state Schrödinger equation (63), the above initial condition is equivalent to our premise that the time-dependent perturbation is switched on at time $t = 0$ and the subsequent time evolution from the harmonic oscillator wave function is investigated. The Lagrangian in eq. (61) can then be explicitly integrated over time to yield

$$\int_0^t L\,\mathrm{d}t' = -\frac{a^2 t}{4m(\omega_0^2 - \omega^2)} - \frac{a^2(\omega^2 + \omega_0^2)}{8m\omega(\omega_0^2 - \omega^2)^2}\sin 2\omega t + \frac{a^2}{m\omega(\omega_0^2 - \omega^2)}\sin\omega t. \tag{68}$$

Hence, apart from the first term in eq. (68) that grows linearly in time, the rest of the terms in the exponent of eq. (65) are oscillatory and average out to zero over a complete cycle of the perturbation.

Thus the quasi-energy eigenvalue reads as

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{a^2}{4m(\omega^2 - \omega_0^2)}, \quad n = 0, 1, 2\ldots\,. \tag{69}$$

Evidently the eigenfunction

$$\psi_n(x, t) = \mathrm{e}^{-i/\hbar\left\{m\dot{\zeta}(t)(x-\zeta(t)) - \left(\int_0^t \mathrm{d}t' L - \frac{t}{T}\int_0^t \mathrm{d}t' L\right)\right\}}\chi_n(x - \zeta(t), t), \tag{70}$$

in which the first term in the right-hand side of eq. (68) has been split off from the exponent in eq. (70), has the correct Floquet property for quasi-energy eigenfunction, discussed in §2.4. Expanding eq. (69) in a power series in ω_0/ω we may write

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{a^2}{4m\omega^2} + \frac{a^2}{4m\omega^4}\omega_0^2 + \frac{a^2}{4m\omega^6}\omega_0^4 + \cdots. \tag{71}$$

Therefore, within the limitations mentioned against each method in §2, the exact answer in eq. (69) matches with the approximate answers, up to $O(1/\omega^4)$. A further graphical comparison is facilitated in figure 1 for the choice of $\frac{\omega_0}{\omega} = \frac{1}{3}$. It is seen that the simple ansatz of Cook *et al* does not fare so badly although the method due to Rahav *et al* [21] provides more accurate answers.

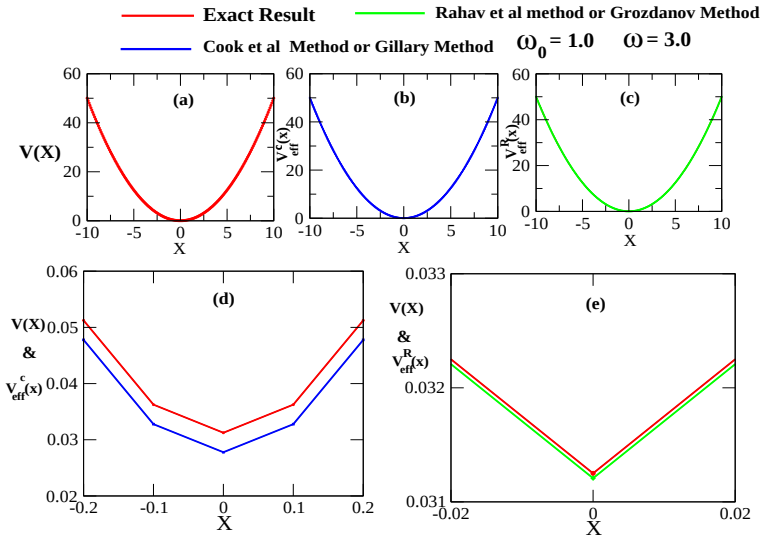

Figure 1. The effective potential. (**a**) Exact result, (**b**) Cook *et al* method and Gillary *et al* method ($V^c_{\rm eff}(x)$), (**c**) Rahav *et al* method and Grozdanov and Raković method ($V^R_{\rm eff}(x)$), (**d**) differences between exact result and Cook *et al* approach, (**e**) differences between exact result and Rahav *et al* method. Figures (**d**) and (**e**) are on a fifty-fold expanded scale in order to accentuate even minor differences.

3.2 *The quantum pendulum*

The next case that is amenable to an exact analysis is the quantum analogue of the Kapitza pendulum [1]. Kapitza showed that if the point of suspension of a pendulum is rapidly vibrated, the pendulum can perform stable vibrations around the point where the pendulum points upwards, whereas otherwise, the pendulum is unstable. We are thus led to consider what is appropriately called the quantum Kapitza pendulum [29]. The essential ideas are contained in a simpler Hamiltonian (14), in which

$$\mathcal{H}_0 = \frac{\hat{p}^2}{2m}, \quad v(\hat{x}) = V_0 \exp(-\beta \hat{x}^2). \tag{72}$$

It is evident that a simple averaging over the period of the perturbation yields free particle motion. Yet, the effective time-independent Hamiltonian has an extremely interesting dynamics, as discussed below.

Applying the method of Rahav *et al* (see eq. (52)) the effective Hamiltonian reads as

$$\hat{K} = \frac{\hat{p}^2}{2m} + \frac{\beta^2 V_0^2 \hat{x}^2}{m\omega^2} \exp(-2\beta \hat{x}^2) + O(\omega^{-4}). \tag{73}$$

Up to orders of ω^{-2} the methods discussed in §§2.1 and 2.4 would lead to eq. (73). Curiously then, the effective potential acquires a double hump character as revealed

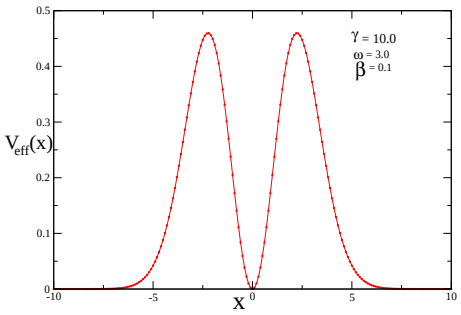

Figure 2. Effective time-independent potential for eq. (73).

in figure 2. There are two potential barriers separated by a potential well, and consequently, the quantum particle can be trapped in definite resonance states. Essentially similar ideas have been employed in the Paul trap [30]. The observed resonances describe long-lived unstable states each being characterized by a complex energy $\varepsilon - i\gamma$, where ε is the location of the resonance while γ is the width which is inversely proportional to the lifetime of the resonance. A plot of ε vs. ω is shown in figure 3 for the exact Hamiltonian of eqs (14) and (72) (solid line) and for the effective Hamiltonian of eq. (73) (dashed line). The corresponding plots of γ are indicated in figure 4. The properties of these resonance states are discussed in detail by Moiseyev [31]. The effective Hamiltonian described by eq. (73) and its variants are relevant for manipulation of cold atoms by electromagnetic fields. Resonant coupling between an electromagnetic field and an atom yields a potential proportional to the intensity on the center of mass of the atom. This potential is expected to oscillate at a frequency that is much larger than the natural frequencies characterizing the centre of mass motion, although the laser frequency can be still larger. The ensuing mechanism describes how atoms are trapped in an effective light billiard [9,10]. The model may also be relevant for the treatment of the electronic motion of atoms and molecules in the presence of intense laser fields.

4. Conclusions

In this paper we have provided an overview of various treatments of quantum dynamics under the influence of rapidly varying periodic forces. Most of these treatments are motivated by an incisive idea of Kapitza who showed that the corresponding classical dynamics can be discussed on the basis of an effective time-independent Hamiltonian, over time-scales longer than the period of the external force. The results of all the different treatments match when the external force is space-independent, up to orders ω^{-2}. The analysis becomes progressively more complicated beyond orders ω^{-2}. The method of Cook *et al* [17] stands out for its elegance and simplicity, and is also valid for arbitrary space-dependence of the external force, though restricted to orders ω^{-2}. We have presented theories for one-dimensional cases only, though generalization to three dimensions are straightforward.

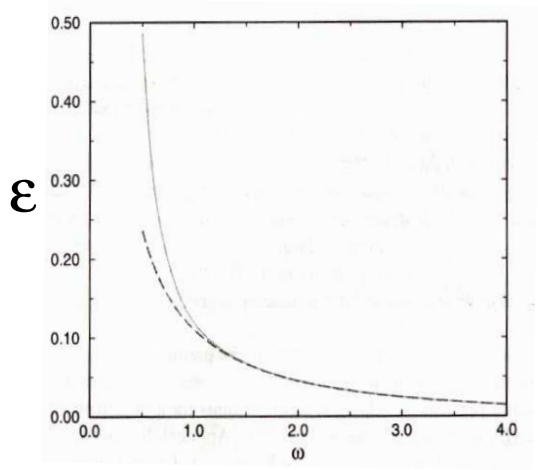

Figure 3. The energy position ε of the lowest resonance as a function of the driving frequency ω. The solid line is a result of the exact calculation for (72) and the dashed line for the effective potential of eq. (73).

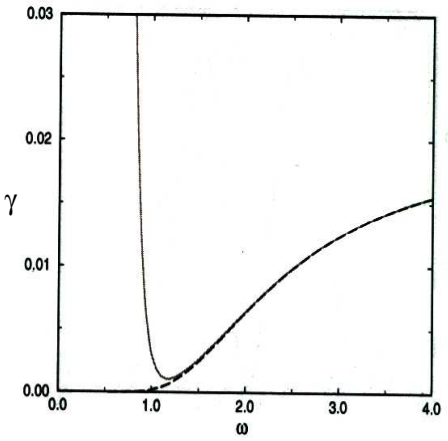

Figure 4. The resonance width γ of the lowest resonance as a function of the driving frequency ω. The solid line is a result of the exact calculation for (72) and the dashed line for the effective potential of eq. (73).

We have applied the results for the effective Hamiltonian to two exactly treatable cases: (i) a linearly driven quantum harmonic oscillator and (ii) a nonlinearly (and Gaussian space-dependent) driven system. These applications are relevant for atomic traps and also quantum optic properties of atoms in intense radiation fields.

In future work we intend to present applications of the reviewed concepts to distinct phenomena: (i) nucleation of superconductivity in bulk and in surface, for Type II superconductors, under external magnetic fields [32] and (ii) Schrödinger cat states [33] in a quantum double-oscillator [34]. The latter has a potential energy

which is not differentiable in space, but the method by Cook *et al* can be easily incorporated, especially in very high-frequency fields for which a treatment up to $O(\omega^{-2})$ suffices.

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