The Rayleigh–Taylor instability of a viscous fluid layer with viscosity stratification

N. Rudraiah*, R. D. Mathod and Hameeda Betigeri

Department of Physics, Gulbarga University, Gulbarga 585 106, India

*UGC–DSA Centre in Fluid Mechanics, Department of Mathematics, Bangalore University, Bangalore 560 082, India and National Research Institute for Applied Mathematics, 492/G, 7th Cross, 7th Block (West), Jayanagar, Bangalore 560 082, India

The effects of viscosity stratification and surface tension on Rayleigh–Taylor (RT) instability in a finite thickness layer of an incompressible viscous fluid bounded above by a denser fluid and below by a rigid impermeable surface have been studied using linear stability analysis. A relation for the RT-instability growth rate is found by calculating the eigenvalues of the stability equation. It is shown that the shape of the dispersion curve is controlled by the ratio of surface tension to the stress gradient, with the layer thickness and the viscosity stratification affecting the rate of growth of the instability. The growth rate, infact, is shown to increase with increase in viscosity stratification and the thickness of the fluid layer. The results are shown to revert to those of the case of a viscous fluid layer in the absence of viscosity stratification.

THE study of instability of the interfaces is of great importance in view of its fundamental role in understanding of the control and exploitation of many of the basic physical, chemical and biological process. The interface in some of these instances occurs due to the superposition of two fluids of different densities. The instability of the interface occurring under gravity when a heavier fluid is supported by a lighter one or equivalently when a lighter fluid is accelerated towards the heavier fluid is widely known as the Rayleigh–Taylor (RT) instability[1]. These instabilities have been the subject of considerable interest in recent years mainly because of their frequent encounter in nature and in many practical applications[2,3]. They also occur in ocean tides, the accelerated interstellar clouds driven by new born stars, supernova events, sinking of slabs of tectonic plates, etc. The analysis of motion induced by the instability of the highly viscous layers in the earth's interior indeed provide rational models for the prediction of continental drifts and the volcanic activities. These instabilities are further known to occur in thin aqueous layers of biological fluids which form an integral part of the various organs and their stability plays an important role in the organ functions. In technological contexts, the RT-instabilities occur in the inertial confinement fusion (ICF) which is based on the implosion of thin deuterium (D)–tritium (T) targets driven by laser or ion beams. The target during the implosion process could be hydrodynamically unstable subject to these instabilities, which destroy the symmetry of compression of the D–T fuel preventing the high density and temperature required for thermonuclear burning with high gain in energy. These instabilities are considered responsible, in material processing applications, for grain boundary failure in metals and cracking of cross-linked polymers. The polymer failure, infact, has been shown to occur often by the formation and growth of planar defects known as crazes; both craze tip advance and craze widening are attributed to these instabilities. These instabilities are responsible for development of non-uniformities that lead to streaked and mottled layers in the growth of thin films in a variety of device fabrication applications. Thus there is evidently a considerable current interest in studying the dynamics of fluid layers from both fundamental and technological perspectives. On the fundamental level the study of layer dynamics concerns a more general problem of evolution of active and passive interfaces; the issues of interest being the formation of ordered interfacial structures and the long time asymptotic behaviour of the deformed interfaces. The understanding and controlling of the deformation of viscous layers, from the technological point of view, provides useful information for suppressing or eliminating these hydrodynamic instabilities. It is well known that various physical factors such as surface tension, viscosity, density gradients and shear rates in flowing films greatly influence the development of RT-instability. Brown[4] has recently investigated the linear instability of a highly viscous fluid layer of finite thickness and shown that such a film if stagnant would rupture as a consequence of this instability. Further, Newhouse and Pozrikidis[5] have studied the nonlinear viscous instability of a liquid layer as a function of surface tension and the ratio of viscosities of the two fluids and have shown that for moderate surface tension, the instability of the layer leads to the formation of periodic array of viscous plumes which penetrate into the overlaying fluid layers. However, the combined effect of surface tension and viscosity variation on RT-instability has not been given much attention in spite of its importance in many practical situations.

The motivation of the present study is to extend the work of Brown[4] to include the effect of viscosity stratification to know its influence on the RT-instability in a thin layer of an incompressible viscous fluid confined above by an interface with a denser fluid and below by a rigid impermeable boundary by using linear perturbation analysis. The formalism developed in the present work is primarily to elucidate the influence of the combined effects of stress gradient, surface tension, viscosity stratification and the layer thickness on the RT-instability. We found that the viscosity stratification

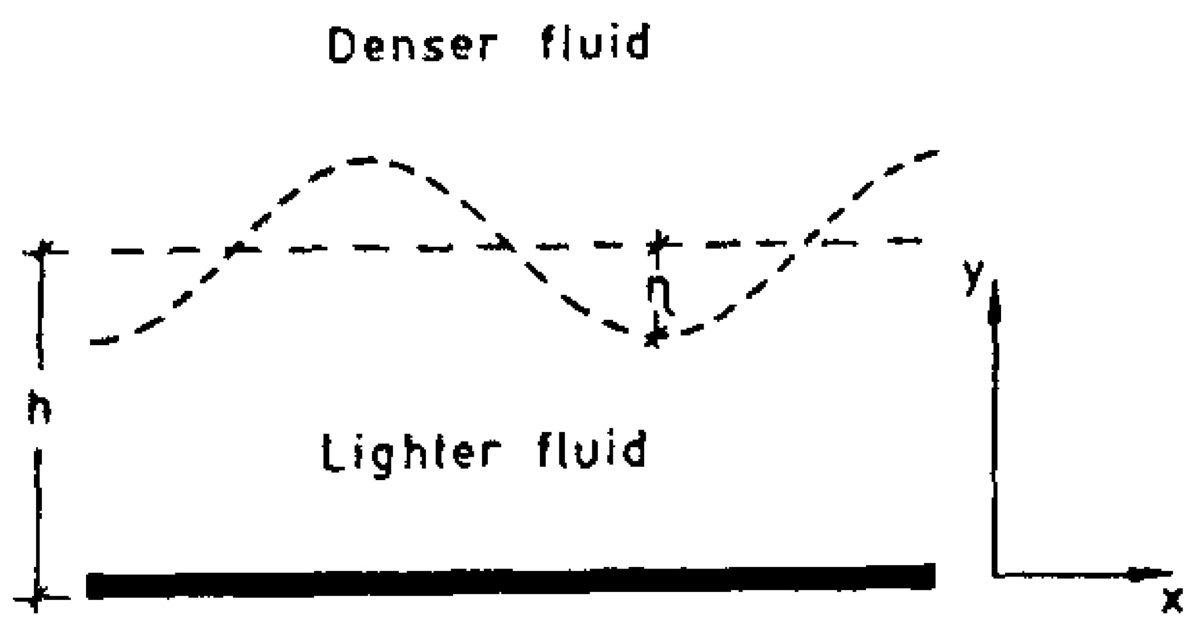

Figure 1. Physical configuration.

affects only the nature of the growth of the instability and does not change the shape of the dispersion curve.

We consider the case of two incompressible fluids with equal viscosities $\mu_F = \mu_L = \mu$ and densities ρ_L and ρ_F, where suffixes F and L refer to lower fluid (film) and upper denser fluid respectively as shown in Figure 1. The interface is characterized by surface tension γ and is governed by the layer thickness h and the interface elevation η. The flow within the viscous layer and the overlying fluid in the absence of any external force is governed by the following equations.

$$\rho \partial u_i/\partial t + \rho u_j(\partial u_i/\partial x_j) = -\partial p/\partial x_i + \partial/\partial x_j \{\mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i)\} \quad (1)$$

and

$$\partial u_j/\partial x_j = 0, \quad (2)$$

where ρ is fluid density, u_i, u_j are velocity components, p the pressure and μ the coefficient of viscosity. The following assumptions[6] enabled us to use the creeping flow approximation:

(i) The film thickness h, is much smaller than the thickness H, of the denser fluid above the film. That is

$$h << H. \quad (3)$$

(ii) The Strouhal number ε, which is a measure of the local acceleration to inertial acceleration in equation (1) is small. That is

$$\varepsilon = \frac{L}{TU} \ll 1 \quad (4)$$

Here $U = \nu/L$ is the characteristic velocity, $\nu = \mu/\rho_f$ is the kinematic viscosity, $L = \sqrt{\gamma/\delta}$ is the characteristic length, $\delta = g(\rho_L - \rho_F)$, γ is the surface tension, $T = \mu\gamma/h^3\delta^2$ is the characteristic time scale. This assumption enabled us to neglect local acceleration term $(\partial u_i/\partial t)$ in eq. (2).

(iii) We consider high viscous fluid so that inertial acceleration term in eq. (2) can be neglected in comparison with viscous term.

(iv) The interface elevation η is assumed to be small compared with the film thickness h. That is,

$$\eta/h << 1. \quad (5)$$

These assumptions enabled us to use the creeping flow approximations which allow us to neglect certain terms in the perturbation problem in order to arrive at closed form asymptotic equation for the interface evolution. In other words, we confine to the conditions of creeping flow in two dimensions and hence ignore the inertial terms. We superimpose on the basic static state of constant pressure a small symmetrical perturbation in both the velocity and pressure fields to obtain after linearizing, the following equations for the perturbed fields.

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial \bar{\mu}}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad (6)$$

$$\frac{\partial p}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right) + 2\left(\frac{\partial \bar{\mu}}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right), \quad (7)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (8)$$

where u and v are the velocity components in x and y directions respectively and $\bar{\mu}$ is the mean viscosity which is a function of y only. The problem of concern is the growth rate of a small periodic perturbation of the interface. We therefore assume a variation in the velocity components, pressure and the interface elevation along x-axis to be of the form

$$[u,\ v,\ p,\ \eta] = [u(y),\ v(y),\ p(y),\ \eta(y)]\ \exp(i\alpha x + nt), \quad (9)$$

where α is the wave number and n, the growth rate. The eqs (3)–(5) in view of eq. (6) transform into the following equations

$$i\alpha p = \mu(-\alpha^2 u + D^2 u) + D\bar{\mu}\ (Du + i\alpha v), \quad (10)$$

$$Dp = \bar{\mu}(-\alpha^2 v + D^2 v) + 2D\bar{\mu}Dv, \quad (11)$$

$$i\alpha u + Dv = 0, \quad (12)$$

where $D = d/dy$. We further consider a realistic mean viscosity stratification because it pertains to stable stratification, to be given by

$$\bar{\mu} = \mu_0 \exp(-\beta y), \quad (13)$$

where μ_0 is a constant characterizing uniform viscosity and β the viscosity stratification factor. The elimination of pressure from eqs (10) and (11) and subsequent use of eqs (12) and (13) gives the following governing equation

$$D^4 v - 2\beta D^3 v - (2\alpha^2 - \beta^2)D^2 v + 2\alpha^2\beta Dv + (\alpha^2\beta^2 + \alpha^4)v = 0. \quad (14)$$

The foregoing equations have to be solved subject to the boundary and the interfacial conditions. The validity of the no-slip condition at the rigid boundary requires that not only normal component but also the horizontal component of velocity vanish on the bounding surface. This requirement in view of the mass conservation equation is ensured by the following conditions

$$v = \mathrm{D}v = 0 \text{ on the rigid surface at } y = 0. \quad (15)$$

The requirement of vanishing of the tangential stress at the free interface is

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0.$$

This, using eq. (9), becomes $\mathrm{D}u + i\alpha v = 0$.

The required surface condition, from this using $u = (i/\alpha)\mathrm{D}v$ obtained from eq. (12), is

$$\mathrm{D}^2 v + \alpha^2 v = 0 \text{ on the interface at } y = h. \quad (16)$$

However, the normal stress at the interface is discontinuous and it is compensated by the surface tension. Therefore the normal stress is determined by the combined effects of stress gradient, surface tension and shear on the interface and is given by

$$p = -\delta\eta - \frac{\gamma\partial^2\eta}{\partial x^2}+\bar{\mu}\ \mathrm{D}v \text{ at } y = h. \quad (17)$$

Further the kinematic condition governing the interface is given by

$$\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}+v\mathrm{D}\eta=0 \text{ at } y=h. \quad (18)$$

We consider the evolution of the interface which is described by the following interface equation

$$y = h - \eta. \quad (19)$$

The pressure balance eq. (17) in view of eq. (9) is given by

$$p = -\delta\eta + \alpha^2\gamma\eta + \bar{\mu}\mathrm{D}v. \quad (20)$$

Further, the kinematic condition expressed by eq. (18) in view of eq. (9) and the interface eq. (19) after linearizing yields

$$v = \frac{\partial\eta}{\partial t} \text{ at } y = h.$$

This using eq. (9) becomes

$$n = v/\eta. \quad (21)$$

The elimination of pressure from eqs (10) and (20) and subsequent use of eqs (10) and (18) lead to the following dispersion relation

$$n = (\alpha^2/\mu_0)v \exp(\beta h)(\delta - \alpha^2\gamma)/(2\alpha^2\mathrm{D}v - \mathrm{D}^3 v). \quad (22)$$

The growth rate n in the absence of the viscosity stratification ($\beta = 0$) may be seen to yield the result of that of the case of a fluid layer of constant viscosity considered by Brown[4]. We see that the shape of the entire dispersion curve is controlled by the ratio δ/γ, with the viscosity stratification factor β and the layer thickness h affecting only the rate of growth of the instability.

The roots λ_i ($i = 1$ to 4) of the characteristic equation

$$\lambda^4 - 2\beta\lambda^3 - 2(\alpha^2 - \beta)\lambda^2 + 2\alpha^2\beta^2\lambda + \alpha^2(\beta^2 + \alpha^2) = 0 \quad (23)$$

of the differential equation (14) are

$$\lambda_1, \lambda_2 = \frac{1}{2}(\beta \mp \Delta_1), \lambda_3, \lambda_4 = (\beta \mp \Delta_2)\ldots, \quad (24)$$

where $\Delta_1 = [\beta^2 + 4\alpha(\alpha + i\beta)]^{1/2}$, $\Delta_2 = [\beta^2 + 4\alpha(\alpha - i\beta)]^{1/2}$.

We see that for the entire range of values of α and β relevant to the present problem, eq. (24) provides a pair of complex and distinct roots except at $\alpha = 0$. The solution of eq. (14) in this context satisfying the boundary conditions expressed in eqs (15) and (16) may be given by

$$v = C[\{\cos(b_1 y) + A\sin(b_1 y)\}\exp(a_1 y) - \{\cos(b_2 y) + B\sin(b_2 y)\}\exp(a_2 y)], \quad (25)$$

where A, B and C are constants and a_i's and b_i's are the real and imaginary parts of the roots of eq. (24) and we have further, $A = A_1/A_2$ and $B = A(b_2/b_1) - (a_1 - a_2)/b_1$, where A_1 and A_2 are given by

$$A_1 = \Big[\{(2a_2 - a_1)a_1 - b_1^2 + \alpha^2\}\cos(b_1 h) + \{((a_2 - a_1)/b_1)(a_1^2 - b_1^2 + \alpha^2) - 2a_1 b_1\}\sin(b_1 h)\Big]\exp(a_1 h) + \Big[\{(b_2^2 - a_2^2 - \alpha^2)\cos b_2 h + 2a_2 b_2 \sin b_2 h\Big]\Big\}\exp(a_2 h). \quad (26)$$

$$A_2 = \Big[\{b_2(b_1^2 - a_1^2 - \alpha^2)/b_1\}\sin(b_1 h) - 2a_1 b_2\cos(b_1 h)\Big]\exp(a_1 h) + \Big[(a_2^2 - b_2^2 + \alpha^2)\sin(b_2 h) + 2a_2 b_2\cos(b_2 h)\Big]\exp(a_2 h). \quad (27)$$

We employ the perturbation wavelength given by $\lambda^* = (\gamma/\delta)^{1/2}$ and the growth rate given by $n^* = n\mu_0/(\delta\gamma)^{1/2}$ as characteristic scales associated with the problem to render the equations dimensionless for stability analysis. In view of these parameters and the eq. (25), the dimensionless dispersion relation after omitting the asterisks for simplicity is given by

$$n = [(1 - \alpha^2)/\alpha][\alpha^3 Q/(2\alpha^2 R - S)]\exp(\beta h), \quad (28)$$

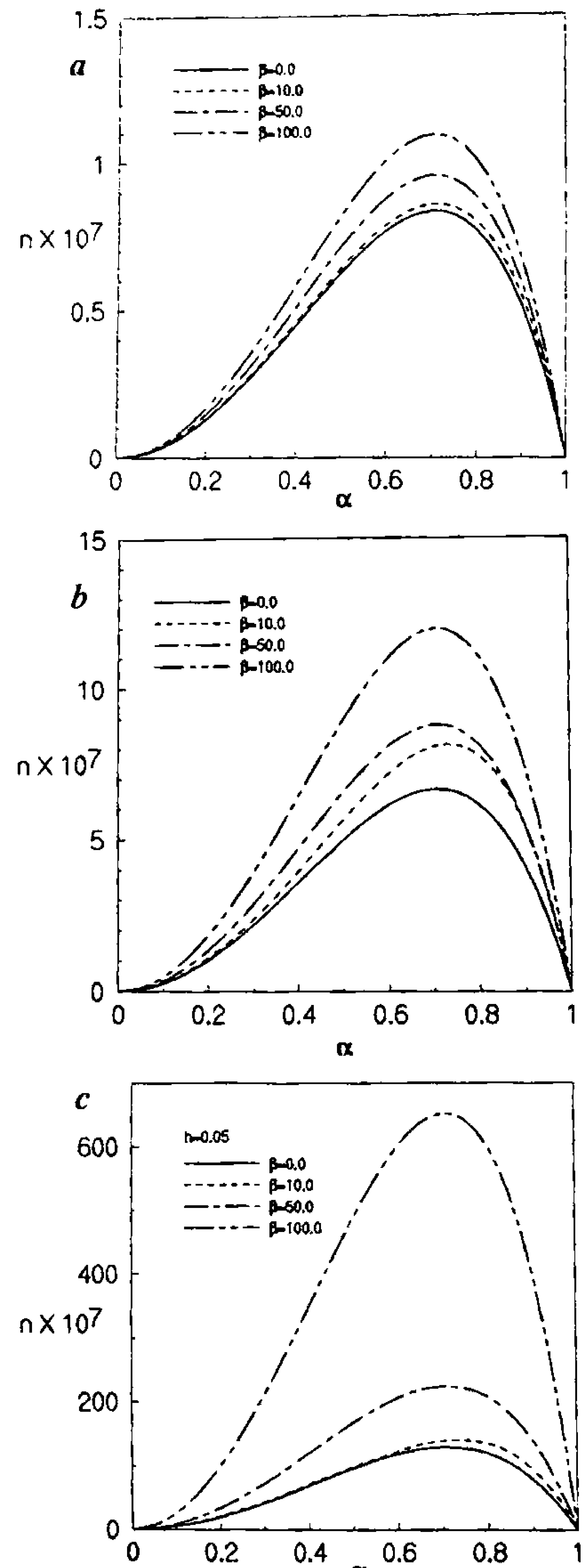

Figure 2 *a–c*. A few typical plots of the growth rate n against the wave number α.

where

$$Q = C[\{\cos(b_1 h) + B\sin(b_1 h)\} \exp(a_1 h) - \{\cos(b_2 h) + A\sin(b_2 h)\}\exp(a_2 h)]. \quad (29)$$

$$R = C[\{(a_1 + Bb_1)\cos(b_1 h) + (Ba_1 - b_1)\sin(b_1 h)\} \exp(a_1 h) - \{(a_2 + Ab_2)\cos b_2 h + (Aa_2 - b_2) \sin(b_2 h)\}\exp(a_2 h)]. \quad (30)$$

$$S = C\Big[\{(a_1^3 - Bb_1^3 + 3a_1 b_1(Ba_1 - b_1))\cos(b_1 h) + (Ba_1^3 + b_1^3 - 3a_1 b_1(a_1 + Bb_1))\sin(b_1 h)\}\exp(a_1 h) - \{(a_2^3 - Ab_2^3 + 3a_2 b_2(Aa_2 - b_2))\cos(b_2 h) + (Aa_2^3 + b_2^3 - 3a_2 b^2(a_2 + Ab_2)) \sin(b_2 h)\}\exp(a_2 h)\Big]. \quad (31)$$

A few typical plots of the growth rate n against the wave number α for a range of viscosity stratification factor β and the layer thickness h of the film are given in Figure 2.

It may be seen from Figure 2 that the whole dispersion curve is controlled by the ratio of stress gradient to surface tension, δ/γ, with the viscosity stratification factor β and the layer thickness h just affecting the nature of the growth of the instability. Eq. (28) reveals that the characteristic length L is of the order of critical wavelength and the maximum growth rate occurs at a wavelength of the same order as L. We note that our results on the growth rate in the absence of viscosity stratification ($\beta = 0$) revert to the case of Brown[4]. When $\beta \neq 0$, the frequency of oscillations of RT instability tend to amplify with increasing β in proportion to exp (βh), where the viscosity stratification $\beta = (-1/\bar{\mu})$ ($\mathrm{d}\bar{\mu}/\mathrm{d}y$) is the reciprocal of the film thickness h. This variation can readily be interpreted physically in terms of energy flux of oscillations, since it just compensates for the upward decrease of viscosity of thin film in proportion to exp ($-\beta y$) in maintaining the flux constant. It may be observed from the plots given in Figure 2 that for given viscosity stratification factor β the RT-instability increases with increase in the layer thickness h. This behaviour of the instability growth rate may be because at higher values of the stratification factor β, the fluid viscosity exp($-\beta y$) is very small at the interface and hence due to the reduced damping of the disturbance the surface becomes increasingly unstable. Initiallv for $\alpha < 0.7$, buoyancy dominates and every point of the interface moves faster with a velocity proportional to the elevation η of that point and elevation grows. For $\alpha > 0.7$ the surface tension action which was negligible initially becomes enhanced and elevations are reduced, the steepening process is slowed down. We note that though the values of β and h help in affecting the nature of the RT-instability growth rate, no complete change in nature of instability is possible. It is evident that although the present work is limited to the linear analysis, it throws much light on the size scale and behaviour of RT-instabilities in thin viscous layers relevant in numerous problems of natural and practical applications discussed earlier.

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