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LINEAR AND NONLINEAR STABILITY OF HELICAL FLOW OF A HETEROGENEOUS CONDUCTING FLUID

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The problem of linear and nonlinear stability of a helical flow of a perfectly conducting heterogeneous fluid between two coaxial cylinders in the presence of an azimuthal magnetic field and a radial gravitational force is discussed. In the case of linear stability the problem has been formulated by the normal mode method and the analysis has been carried out by reducing the perturbation equations to a Sturm-Liouville system.

It is found that a necessary condition for instability is that the algebraic sum of hydrodynamic, magnetohydrodynamic and swirling Richardson numbers must be less than one quarter somewhere in the fluid. A semi-circle theorem similar to that of Howard is also obtained. In particular it is found that when gravitational force balances the centrifugal force of the swirling motion, the heterogeneous conducting fluid behaves as if it is homogeneous as far as the condition for stability is concerned.

In the case of nonlinear stability the problem has been formulated by the energy method and a universal stability estimate, namely a stability limit for motions subject to arbitrary nonlinear disturbances is obtained in terms of Alfvén number and Richardson number, J, for the flow. In the case of hydrodynamic flow by letting magnetic field tend to zero, it is found that the motion is stable if $J \geqslant 0$.

1. INTRODUCTION

The investigation of heterogeneous conducting flows and their stability is of importance in such varied fields as the study of sun spots, interstellar matter, terrestrial magnetism and so on. A detailed survey of these applications has been given by Elsasser (1955, 56). From a meteorological view point considerable interest is attached to the study of stability of heterogeneous conducting fluid, to understand the momentum transport by gravity waves (Rudraiah and Venkatachalappa 1972a, b, c, 1974).

The theory of momentum transport by gravity waves in a conducting fluid in the presence of a magnetic field is an area of considerable interest in meteorologial, oceanographic, geophysical and astrophysical problems. Additional interest in this field stems from the attempt to simulate solar-wind geomagnetic interactions and geo-hydromagnetic secular variations. The momentum transport in a Boussinesq perfectly conducting fluid has been investigated by Rudraiah and Venkatachalappa (1972a, b, c and 1974) with velocity shear and by Acheson (1972) with magnetic shear. Rudraiah and Venkatachalappa have shown that the waves are attenuated by a factor $\exp\{-2\pi(J-\frac{1}{4})\}$ with $J > \frac{1}{4}$, where J is the algebraic sum of hydrodynamic and hydromagnetic Richardson numbers. This condition of $J > \frac{1}{4}$ has to be obtained by the stability analysis. Therefore in this paper we have attempted to investigate the linear and non-linear stability of helical flow in a perfectly conducting fluid; with the object that the results of such a study will be useful in the study of momentum transport by gravity waves in geophysical and astrophysical problems.

The linear stability of heterogeneous conducting fluid without swirling has been investigated recently by Rudraiah (1964, 1967, 1970) and Rudraiah *et al.* (1972). It has been generalized, using Sturm-Liouville analysis, Synge's proof of Rayleigh's point of inflexion and it has been shown that the circular magnetic field makes the flow more stable. The linear stability of swirling flows in an incompressible homogeneous perfectly conducting fluid with an axial current has been investigated for axisymmetrical disturbances by Howard and Gupta (1972) and they found that the circular magnetic field has an effect analogous to density stratification in a radial gravitational field and the effective Richardson numbers of circular magnetic field and swirl are additive. Swirling flows in channels as they relate to energy and mass separation devices, heat exchangers, and more recently to nuclear rocket engines have been extensively studied. However, the same problem with heterogeneous fluids has not been given much attention. The stability of heterogeneous swirling flows is of interest in the design of gas turbines, blowers and other rotating machinery. Therefore, in this paper we study the stability of heterogeneous perfectly conducting inviscid swirling flows between concentric cylinders with a circular magnetic field and a radial gravitational force. The analysis is divided into two parts.

The first is a development of normal mode technique which is the usual method for the investigation of stability of many systems. It consists of solving the linearized equations of motion for small symmetrical disturbances about the basic flow. Using this linear stability analysis systematic proofs for the following theorems are given :

(i) Extension of Miles (1961) proof of Taylor's conjecture that Richardson number must somewhere be less than one quarter if the flow is unstable to magnetohydrodynamics.

(ii) A semi-circle theorem similar to that of Howard.

The second part deals with the discussion of non-linear stability of swirling heterogeneous conducting fluids using the energy method which is a generalization of the energy method used earlier by Serrin (1959) and Joseph (1966) for the case of viscous incompressible thermo-convective flows. A universal stability estimate, namely a stability limit for motions subject to arbitrary nonlinear disturbances, is obtained.

2. Mathematical Formulation of the Linear Stability Analysis

Let (r, θ, z) be the cylindrical coordinates with z-axis as the common axis of the cylinders. We assume axial symmetry, infinite conductivity and infinite length of the cylinders. Under these approximations the basic equations to be solved in the fluid are (in rationalized MKS units) :

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} (\nabla \times \vec{H}) \times \vec{H} - g\hat{r} \qquad \ldots(1)$$

$$\nabla \cdot \vec{q} = 0 \qquad \ldots(2)$$

$$\frac{\partial \rho}{\partial t} + (\vec{q} \cdot \nabla) \rho = 0 \qquad \ldots(3)$$

$$\nabla \cdot \vec{H} = 0 \qquad \ldots(4)$$

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{q} \times \vec{H}) \qquad \ldots(5)$$

where $\vec{q} = u_r\hat{r} + u_\theta\hat{\theta} + u_z\hat{z}$ is the velocity, $\vec{H} = H_r\hat{r} + H_\theta\hat{\theta} + H_z\hat{Z}$ is the magnetic field, p the pressure, ρ the density, μ the magnetic permeability and g the gravitational field. The applied magnetic field H_0 is in the azimuthal direction and is a function of r only. The gravitational force acts in the radial direction and the acceleration due to gravity g may be a function of r.

The differential equation which determines the stability to axisymmetric disturbances of an heterogeneous inviscid conducting flow between the cylinders $r = a$ and $r = b$ $(b > a)$ with basic flow

$$\left.\begin{aligned} &\vec{q} = V(r)\,\hat{\theta} + W(r)\,\hat{z},\ \vec{H} = H_0(r)\,\hat{\theta},\ \rho = \rho_0(r),\ p = p_0(r), \\ &\frac{d\pi_0}{dr} = \rho_0\left(\frac{V^2}{r} - \frac{A^2}{r} - g\right),\ \pi_0 = p_0 + \frac{\mu H_0^2}{2}, \\ &A = H_0\sqrt{\frac{\mu}{\rho_0}},\ \text{the Alfvén velocity,} \end{aligned}\right\} \qquad \ldots(6)$$

is

$$D\,[\rho_0(W - C)^2 D_* F] - \left[\rho_0 k^2 (W - C)^2 + \mu r D\left(\frac{H_0}{r}\right)^2 - \rho_0 N - \Phi\right] F = 0 \qquad \ldots(7)$$

where

$$D = \frac{d}{dr}, \; D_* = \frac{d}{dr} + \frac{1}{r}$$

the perturbation velocities are given by

$$u = ik(W - C)\, F e^{ik(z-ct)} \quad \ldots(8)$$

$$v = -\,(D_* V)\, F e^{ik(z-ct)} \quad \ldots(9)$$

$$w = -\, D_* \,[(W - C)\, F]\, e^{ik(z-ct)} \quad \ldots(10)$$

$$h_\theta = -\, rD\left(\frac{H_0}{r}\right) F e^{ik(z-ct)} \quad \ldots(11)$$

$N = g\beta$ is the Brunt-Väissälä frequency, $\Phi = \frac{1}{r^3} D(\rho_0 v^2 r^2)$. The expression for u, given by eqn. (8), can be obtained in the following way. The equation of continuity (2) permits us to define the stream function ψ for axisymmetrical flow in the form

$$u = \frac{1}{r}\frac{\partial\psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial\psi}{\partial r}. \quad \ldots(12)$$

If η denotes the displacement of a line of constant density, then η is a function not only of z and t but also of r, and if higher order terms are neglected, we have

$$\frac{D\eta}{Dt} = u = \frac{1}{r}\frac{\partial\psi}{\partial z} \quad \ldots(13)$$

Since $\quad \eta = F(r)\, e^{ik(z-ct)} \quad \ldots(14)$

we have

$$\psi = r(W - C)\,\eta, \; u = ik(W - C)\,\eta$$

as stated in eqn. (8).

The general stability equation, from eqn. (7), using

$$F = ik\, U^{-n} H, \; U = W - C \quad \ldots(15)$$

is

$$D\,[\rho_0 U^{2(1-n)} D_* H] - \Big[\rho_0 k^2 U^{2(1-n)} + nU^{1-2n} rD\left(\frac{\rho_0}{r} DW\right) + \rho_0 U^{-2n}\left\{n(1-n)(DW)^2 + \frac{\mu_r}{\rho_0} D\left(\frac{H_0}{r}\right)^2 - N^2 - \frac{\Phi}{\rho_0}\right\}\Big]_H = 0. \quad \ldots(16)$$

The boundary conditions on H are

$$H(a) = H(b) = 0. \quad \ldots(17)$$

The stability equation in the form (16) is more useful than the form (7) in the sense that the equation required for the three models namely those of Synge (1933), Miles (1961) and Howard (1961) can easily be obtained taking $n = 1$, $n = \frac{1}{2}$ and $n = 0$ respectively in eqn. (15). The stability criteria will depend on the eigenvalue C of eqn. (16). If C is real, the steady motion is said to be neutrally stable, and if C is complex the motion is stable or unstable according as $C_i < 0$ or $C_i > 0$ respectively. Hence to discuss the stability of steady flow, we have to find the nature of C from the stability eqn. (16). We try to discuss the linear stability using Miles' and Howard's analysis.

3. Linear Stability Analysis Using the Energy Equation and Reynold Stress

We try to discuss the linear stability using energy and Reynolds stress consideration following closely the analysis of Miles (1961). The required stability equation may be obtained from the general stability eqn. (16) by setting $n = \frac{1}{2}$. Before this, we obtain the energy balance equation from the following linearized equations of motion.

$$\rho_0 \left[\frac{Du}{Dt} - \frac{2vV}{r}\right] + \rho\left(g - \frac{V^2}{r}\right) + \frac{2\mu H_0 h_\theta}{r} = \frac{-\partial\pi}{\partial r} \qquad \ldots(18)$$

$$\rho_0 \left[\frac{Dv}{Dt} + uD_*V\right] - \mu h_r D_* H_0 = 0 \qquad \ldots(19)$$

$$\rho_0 \left[\frac{Dw}{Dt} + uDW\right] = \frac{-\partial\pi}{\partial z} \qquad \ldots(20)$$

$$\frac{Dh_r}{Dt} = 0 \qquad \ldots(21)$$

$$\frac{Dh_\theta}{Dt} + u\left(DH_0 - \frac{H_0}{r}\right) - h_r\left(DV - \frac{V}{r}\right) = 0 \qquad \ldots(22)$$

$$\frac{Dh_z}{Dt} - h_r DW = 0 \qquad \ldots(23)$$

$$D_*u + \frac{\partial w}{\partial z} = 0 \qquad \ldots(24)$$

$$\frac{D\rho}{Dt} + uD\rho_0 = 0 \qquad \ldots(25)$$

where the lowered-key letters denote the perturbation quantities over the basic flow. The above eqns. (18) to (25) are obtained from eqns. (1) to (6) by the usual process of perturbation and linearization. Equations (21) and (23) indicate that the magnetic lines of force which were originally circular remain circular. That is, there is no distortion of the lines of force even after perturbing the field.

3.1. *The Energy Equations*

From eqns. (18) to (25), after making them dimensionless using the quantities :

$$t' = Nt,\ \vec{q}' = \vec{q}/U,\ \pi' = \pi/\rho_1 U^2,\ \vec{r}' = r/d,\ \rho_0' = \rho_0/\rho_1 \qquad \ldots(26)$$

we obtain

$$\int_a^b \int_{z_1}^{z_2} \frac{\rho_0}{2} \frac{D}{Dt}\left[u^2 + v^2 + w^2 + \beta\eta^2\left(g - \frac{V^2}{r}\right) + Sh_\theta^2\right] r\, dr\, dz$$

$$= -\frac{1}{\sqrt{J_H}} \int_a^b \int_{z_1}^{z_2} \left[D_*(\pi u) + \frac{\partial}{\partial z}(\pi w) + \rho_0 uwDW\right] r\, dr\, dz \qquad \ldots(27)$$

where

$J_H = \dfrac{N^2 d^2}{U^2}$ is the hydrodynamic Richardson number, $S = \dfrac{A^2}{U^2}$ is the Alfvén's number, $\eta = -\dfrac{\rho}{D\rho_0}$, $\beta = -\dfrac{D\rho_0}{\rho_0}$.

Denoting the z-integrals in eqn. (27) by bar and assuming either that the disturbance is periodic in z in which case the bar may be interpreted as implying an average over the wave length or that r and u vanish at $z = \pm\infty$ in which case the bar must be interpreted as implying integration from $z = -\infty$ to $z = +\infty$, we obtain

$$\frac{\partial E}{\partial t} = \frac{1}{\sqrt{J_H}}(R + Q) \qquad \ldots(28)$$

where

$$E = T + P + M, \text{ the total energy} \qquad \ldots(29)$$

$$T = \tfrac{1}{2}\int_a^b \rho_0(\bar{u}^2 + \bar{v}^2 + \bar{w}^2)\, r\, dr, \text{ the kinetic energy} \qquad \ldots(30)$$

$$P = \int_a^b \rho_0\left(g\beta - \frac{V^2\beta}{r}\right)\bar{\eta}^2\, r\, dr, \text{ the potential energy} \qquad \ldots(31)$$

$$M = \tfrac{1}{2}\int_a^b \mu \bar{h}_\theta^2\, r\, dr, \text{ the magnetic energy} \qquad \ldots(32)$$

$$Q = - \int_a^b \overline{D_*(\pi u)}\, r\, dr \qquad \text{...(33)}$$

the rate at which work is done on the perturbation flow by the external pressure (which is the algebraic sum of hydrodynamic pressure and magnetic pressure) at the boundaries a and b, and

$$R = \int_a^b \tau D W\, r\, dr \qquad \text{...(34)}$$

the rate at which energy is transferred from the mean flow to the perturbation flow by the Reynolds stresses

$$\tau = - \rho_0\, \overline{uw}. \qquad \text{...(35)}$$

The above energy equation (28) says that the rate of gain of energy of a cylindrical shell of fluid between $r = a$ and $r = b$ is equal to the flux of energy into the shell across the two cylindrical surfaces. The other way of interpreting eqn. (28) is to regard it as the gain of total energy under the action of pressure forces and Reynolds stresses on the cylindrical surfaces.

3.2. *Linear Stability Analysis Using Reynolds Stress Equation*

Even though the energy relation (28) is true for any arbitrary linear disturbances, in the subsequent part of this section we shall restrict ourselves to a specially periodic disturbance of the form (8). Thus the stability equation, obtained from eqn. (16), taking $n = 1$, is

$$\frac{d^2\varphi}{d\lambda^2} + h(\lambda)\, \varphi = 0 \qquad \text{...(36)}$$

where

$$\varphi = ru,\ \lambda = \int_a^r \frac{r\, dr}{\rho_0} \qquad \text{...(37)}$$

$$h(\lambda) = \frac{1}{(W - C)^2} \left[N^2 + \frac{\Phi}{\rho_0} - \frac{d}{d\lambda} \left(\frac{\mu H_0^2}{\rho_0^2\, (D\lambda)^2} \right) - (W - C) \frac{d^2 W}{d\lambda^2} + \frac{k^2 (W - C)^2}{(D\lambda)^2} \right]. \qquad \text{...(38)}$$

The boundary conditions are

$$\varphi(a) = \varphi(b) = 0. \qquad \text{...(39)}$$

Equation (36) has a singularity at the critical point $\lambda = \lambda_c$ at which $W = c$. If c is complex, then λ_c is a point in the complex r-plane. In a fluid of small viscosity, this point is replaced by a critical layer $| \lambda - \lambda_c | < \delta$ whose thickness δ vanishes with the viscosity.

The exponents of the singularity are

$$\tfrac{1}{2}(1 \pm \nu)$$

where

$$\nu = (1 - 4J_c)^{1/2} \qquad \ldots(40)$$

and J_c is the total Richardson number evaluated at $\lambda = \lambda_c$ and is given by

$$J_c = J_m(\lambda_c) + J_H(\lambda_c) + J_R(\lambda_c). \qquad \ldots(41)$$

In eqn. (41),

$$J_m = -\frac{1}{\rho_0}\frac{D(\mu H_0^2/r^2)}{(DW)^2}, \text{ the magnetic Richardson number}$$

$$J_H = \frac{N}{(DW)^2}, \text{ the hydrodynamic Richardson number}$$

$$J_r = -\frac{\Phi}{\rho_0(DW)^2}, \text{ the Swirling Richardson number.}$$

All are evaluated at $\lambda = \lambda_c$. Assuming that ν is not an integer and that $W(\lambda)$ and $\rho_0(\lambda)$ may be continued analytically into the complex neighbourhood of $\lambda = \lambda_c$, we may apply the method of Frobenius to obtain two linearly independent solutions to eqn. (36) in the form

$$\varphi_\pm(\lambda) = (\lambda - \lambda_c)^{(1\pm\nu)/2}\, \Theta_\pm(\lambda) \qquad \ldots(42)$$

where $\Theta_\pm$ are analytic functions in the neighbourhood of $\lambda = \lambda_c$ having the form

$$\Theta_\pm = 1 + \left[(1 + J_c)\frac{D(\rho_0 Dw)}{\rho_0 Dw} - \frac{J_c D^2\rho_0}{D\rho_0}\right]\frac{\lambda - \lambda_c}{1 \pm \nu} + \ldots . \qquad \ldots(43)$$

The Wronskian of these solutions is

$$W(\varphi_+, \varphi_-) = \varphi_+\varphi_-' - \varphi_+'\varphi_- = -\nu. \qquad \ldots(44)$$

If ν is an integer, the solution φ_- degenerates and must be modified, except, in some special conditions, to include a component proportional to $\varphi_+(\lambda)\log(\lambda - \lambda_c)$. The only statically stable flow in this category is that for which $J_c = J_m(\lambda_c) + J_N(\lambda_c) + J_R(\lambda_c) = 0$, $(\nu = 1)$ and the solutions are essentially similar to that of axisymmetric hydrodynamic flow. We note that the logarithmic component is proportional to the discontinuity in Reynolds stress at $\lambda = \lambda_c$ as calculated in eqn. (58) below.

Further, discussions of this section will be restricted to neutral wave motions for which $c_i = 0$. In this case we have the relations

$$\varphi^*_{\pm}(\lambda) = \varphi_{\pm}(\lambda) \exp(i\pi(1 \pm \nu)\, S(\lambda - \lambda_c)),\; J_c < \tfrac{1}{4} \qquad \text{...(45)}$$

$$= \varphi_{\mp}(\lambda) \exp(\pi(1 \pm \gamma)\, S(\lambda - \lambda_c)),\; J_c > \tfrac{1}{4} \qquad \text{...(46)}$$

where

$$\nu = i\gamma$$

$$S = 0 \text{ for } \lambda > \lambda_c$$

$$= \pm 1 \text{ for } k(DW)_c \gtrless 0,\; \lambda < \lambda_c$$

and the asterisks (*) denote the complex conjugate.

We remark that the singularity in φ_- at the $\lambda = \lambda_c$ for $0 < J_c < \frac{1}{4}$ (but not for $J_c = 0$) renders both the kinetic and potential energies of eqns. (30) and (31) infinite whereas the magnetic energy of eqn. (32) remains finite for singular neutral modes. Even though the magnetic energy is finite the energy E, which is the algebraic sum of kinetic, potential and magnetic energy, will be infinite for singular neutral modes. Here the energy becomes infinite mainly because of the non-dissipative nature of flow and infinite energy could be removed by considering dissipative conducting flow (Rudraiah and Venkatachalappa 1974).

Expressing u and w in terms of φ with the aid of eqns. (37), (8) and (10), and substituting the results in eqn. (35) we obtain

$$\tau = \frac{k}{2}\left(\psi^* \frac{d\psi}{d\lambda}\right)_i e^{2kC_i t} \qquad \text{...(47)}$$

where

$$\psi = rUF$$

and the subscript i implies the imaginary part. From eqn. (47), using eqn. (37), we have

$$\frac{\partial\tau}{\partial\lambda} = -\frac{kh_i}{2} \mid \psi \mid^2 e^{2kC_i t} = -\frac{h_i}{k} \rho_0 \bar{v}^2 \qquad \text{...(48)}$$

where

$$h_i = C_i \left[\frac{2(W - C_r)\left(N^2 - \frac{d}{d\lambda}\left(\frac{\mu H_0^2}{r^2}\right) + \Phi\right)}{\mid W - C \mid^4} - \frac{D(\rho_0 DW)}{\mid W - C \mid^2}\right]. \qquad \text{...(49)}$$

The boundary conditions on τ for spatially periodic wave motion is

$$\tau_1 = \tau_2 = 0. \qquad \text{...(50)}$$

If $C_i \neq 0$, then since $\partial\tau/\partial\lambda$ cannot vanish identically, the Reynolds stress must have an extremum and h_i must change sign in $(0, l)$ i.e., in (a, b). This result was proved earlier by Rudraiah and Narayana (1967) and they used it to prove that complex values of C must lie in one of the family of circles.

$$C_r^2 + C_i^2 - 2\left(W + \frac{\xi}{D(\rho_0 DW/r)}\right) C_r + W^2 + 2W \frac{\xi}{D(\rho_0 DW/r)} = 0 \quad \text{...(51)}$$

provided that $D\left(\frac{\rho_0 DW}{r}\right)$ and $\xi = D\left(\left(\frac{\mu H_0}{r}\right)^2 - N - \frac{\Phi}{\rho_0}\right)$ do not vanish simultaneously, and hence that

$$C_i \leqslant \mathrm{Max}\left|\frac{\xi}{D(\rho_0 DW/r)}\right|. \quad \text{...(52)}$$

Now the condition $C_i = 0$ implies, using eqns. (48) and (49), that the Reynolds stress must be constant except for the possible discontinuities at the critical point $\lambda = \lambda_c$. No such discontinuities can exist for non-singular motion and hence the condition (50) implies that $\tau \equiv 0$. Only one such discontinuity is possible if $W(\lambda)$ is monotonic in $(0, l)$ and a necessary condition for singular neutral mode subject to condition (50) is that this discontinuity vanish to yield $\tau \equiv 0$. Thus we have the following theorem:

Theorem 1 — The Reynolds stress for any neutral oscillation vanishes identically for monotonic $W(\lambda)$ if $J_c > 0$.

If $W(\lambda)$ is not monotonic, then more than one critical layer may exist, but the discontinuity in τ may cancel to satisfy the condition (50). Now the expressions for τ can be expressed in simple form using the general solution of eqn. (36) in the form

$$\psi(\lambda) = A\psi_+(\lambda) + B\psi_-(\lambda). \quad \text{...(53)}$$

Equation (47), using eqn. (53) and $C_i = 0$, becomes

$$\tau = \frac{k}{2}[\,|A|^2 \psi_+' \psi_+^* + |B|^2 \psi_-' \psi_-^* + AB^* \psi_+' \psi_-^* + A^* B \psi_+^* \psi_-']_i. \quad \text{...(54)}$$

The first two terms in parenthesis of eqn. (54), because of eqns. (42), (45) and (46), are real $J_c < \frac{1}{4}$(ν real), whereas the last two terms are complex conjugates for $J_c > \frac{1}{4}$ ($\nu = i\gamma$ imaginary). In the former case we have

$$\begin{aligned} \psi_+' \psi_-^* &= (\psi_+' \psi_-)^* e^{-i\pi(1+\nu)S} \\ &= (\nu + \psi_+ \psi_-')^* e^{-i\pi(1+\nu)S} \\ &= \nu e^{-i\pi(1+\nu)S} + \psi_+ \psi_-'^*. \end{aligned} \quad \text{...(55)}$$

Equation (54) using eqn. (55), becomes

$$\tau = \frac{k\nu}{2}\,[AB^* e^{-i\pi(1+\nu)S}]_i,\ J_c < \tfrac{1}{4}. \qquad \text{...(56)}$$

Similarly we obtain

$$\tau = \frac{k\gamma}{4}\,[\,|\,A\,|^{\,2} e^{\pi(1+\gamma)S} - |\,B\,|^{\,2} e^{\pi(1-\gamma)S}],\ J_c > \tfrac{1}{4}. \qquad \text{...(57)}$$

In hydrodynamic case $J_c = 0$ when $(\rho_0)'_c = 0$ and $\Phi = 0$ or $g = 0$ and $\Phi = 0$ or $N^2 + \Phi = 0$.

However, in magnetohydrodynamic case $J_c = 0$ when $N^2 + N_m^2 + \Phi = 0$. In this case integrating eqn. (48), between λ_c^- and λ_c^+ we obtain

$$\tau(\lambda_c^+) - \tau(\lambda_c^-) = \frac{\pi}{k}\left[\frac{D(\rho_0 DW)\,\bar{v}^2}{|\,DW\,|}\right]_c. \qquad \text{...(58)}$$

We know that when $W(\lambda)$ is monotonic ($DW \neq 0$) in $(0, \lambda)$ only a single critical layer can exist and eqns. (53) to (57) remain valid throughout $(0, l)$ for fixed A and B. Now, from eqns. (56), using the condition (50), we obtain $(AB^*)_i = 0$ and $AB^* \sin(\pi\nu) = 0$, which imply, since ν cannot be an integer, that either $A = 0$ or $B = 0$. Hence we have the following theorem :

Theorem 2 — A singular neutral mode ψ for which $0 < J_c < \frac{1}{4}$ must be simply proportional to either ψ_+ or ψ_-. Similarly, from eqn. (57), using the condition (50) we obtain $A = 0$, $B = 0$ and hence we have the following theorem :

Theorem 3 — Singular neutral modes cannot exist for monotonic $W(\lambda)$ if $J(\lambda) > \frac{1}{4}$ in $(0, l)$.

Applying Liouville's method to eqn. (37), we obtain the asymptotic solutions

$$\varphi_\pm \sim K^{-1/2}\,(W - C)^{1/2} \exp\left[\pm i \int_0^\lambda \frac{K^{1/2}}{(W - C)}\,d\lambda\right],\ y \to \infty \qquad \text{...(59)}$$

where

$$K = N + \Phi - D\left(\frac{\mu H_0^2}{r^2}\right),$$

$$y = \frac{K_*^2\, l^2}{C_*^2},$$

l is the characteristic length, C_* the characteristic velocity, N_* the characteristic value of K. Equation (59), using the conditions (50) becomes

$$\sin\left[\int_0^\lambda \frac{K^{1/2}\,d\lambda}{(W-C)}\right] = 0, \ (y \to \infty). \qquad \ldots(60)$$

If $W(\lambda)$ is monotonic in $(0, l)$ we see that eqn. (60) can be satisfied only if C is real and not in (w_1, w_2), corresponding to a non-singular neutral mode. If we regard, C as a function of y and y_0 is the minimum value of y for which $J(y) \geqslant \frac{1}{4}$ in $(0, l)$, then it follows that $C_i(y) \equiv 0$ for $y > y_0$. Hene we have the following theorem :

Theorem 4 — Necessary condition for instability of a heterogeneous conducting flow between two rigid cylinders are $DW \neq 0$ and

$$J_m(r) + J_H(r) + J_R(r) < \tfrac{1}{4} \text{ in } (a, b).$$

3.3. *The Eigenvalue Problem*

In Section 3.1, we discussed the linear stability analysis using Reynolds stress concept. In this section we try to discuss the same problem by regarding C as an eigenvalue of equation :

$$D\,[\rho_0 U^2 D_* H] + [K - \rho_0 U^2 k^2]\,H = 0 \qquad \ldots(61)$$

where

$$U = W - C$$

$$K = N^2 + \frac{\Phi}{\rho_0} - \frac{\mu_r}{\rho_0} D(H_0/r)^2. \qquad \ldots(62)$$

Equation (61), is obtained from eqn. (16) by setting $n = 0$.

If l is the characteristic length, C_* is a characteristic velocity, and K_* is a characteristic measure of K, then from eqn. (61), using the boundary conditions (17), we have the secular equation for an eigenvalue problem of the form :

$$\Delta\left(\frac{C}{C_*}, \alpha, y, x\right) = 0 \qquad \ldots(63)$$

where

$$\alpha = kl, \ y = \frac{K_* l^2}{C_*^2} \equiv J_*, \ x = \frac{K}{K_*}$$

are dimensionless real parameters but $C = C_r + iC_i$ may be complex. The wave number is usually assumed to be positive. As in hydrodynamics, we define in magneto-hydrodynamics also a neutral surface as a locus of eigenvalues for which $C_i = 0$ in a (C_r, α, y, x)-space. Such a surface will be a stability boundary if and only if there exist continuous eigenvalues for which $C_i > 0$.

In the discussion of the stability boundary we assume following Miles (1961) that $W(r)$ and $\rho_0(r)$ are regular functions of r in (a, b) so that these functions may be continued analytically into a neighbourhood of (a, b) which includes the singular point $\lambda = \lambda_c$ and that $DW \neq 0$ and $K(r) \neq 0$ in this neighbourhood. This means that even the end points (a, b) are included as possible singularities of the differential eqn. (61). Even though these restrictions exclude some interesting problems but they guarantee that C_i is a continuous function of the remaining parameters of eqn. (63) and hence that the trajectory of a complex eigenvalue must terminate on a stability boundary.

From eqn. (42), it follows that

$$F_{\pm} \sim (W - C)^{-1} \varphi_{\pm} \qquad \ldots(64)$$

is consequence of which the boundary conditions (39) imply that :

Theorem 5 — The phase velocity C cannot be equal to W_1 or W_2.

Multiplying eqn. (61) by H^*, the complex conjugate of H, integrating between a and b and assuming $c_i > 0$, we obtain

$$\int_a^b K\rho_0 \mid H \mid^2 r\, dr = \int_a^b \rho_0 \mid U \mid^2 [\mid D_* H \mid^2 + k^2 \mid H \mid^2]\, r\, dr. \qquad \ldots(65)$$

The real part of (65) is

$$\int_a^b K\rho_0 \mid H \mid^2 r\, dr = \int_a^b \rho_0 [(W - C_r)^2 - C_i^2]\, [(D_* H^2 + k^2 \mid H \mid^2]\, r\, dr. \qquad \ldots(66)$$

From this we infer that "non-singular neutral modes cannot exist if $K(r) < 0$ in (a, b)".

In hydrodynamics (Miles 1961) this result is valid only when $\beta < 0$, i.e. the flow is basically unstable. However, in our analysis we note that even if $\beta > 0$, i.e. the flow is basically stable, the above result is still valid as long as $K < 0$. This is possible only when

$$N^2 + \frac{\Phi}{\rho_0} < \frac{\mu r}{\rho_0} D(H_0/r)^2. \qquad \ldots(67)$$

Thus we have the following theorem :

Theorem 6 — Non-singular neutral modes cannot exist if

$$N^2 + \frac{\Phi}{\rho_0} < \frac{\mu r}{\rho_0} D(H_0/r)^2.$$

The imaginary part of (65) is

$$C_i \int_a^b (W - C_r)\, [\mid D_* H \mid^2 + k^2 \mid H \mid^2]\, r\, dr = 0 \qquad \ldots(68)$$

from which we infer that :

Theorem 7 — The phase velocity C_r, for any unstable modes ($C_i > 0$), must lie between the maximum and minimum value of $W(r)$ in $[a, b]$. This implies

$$W_1 < C_r < W_2 \text{ if } DW \neq 0 \text{ in } (a, b).$$

A direct corollary of Theorems 5 and 7, together with the restrictions guaranteeing the continuity of C_i, is :

Theorem 8 — A stability boundary consists of singular neutral modes i.e., modes for which $C_i = 0$ and $W(r) = C_r$ in (a, b).

4. Generalization of Howard's Model to MHD

The linear stability analysis discussed in section 3 is true only when $DW \neq 0$ and that the velocity and the density profiles were analytic in a complex neighbourhood of the real flow domain. In this section, we try to give, following the analysis of Howard (1961), a simple proof for instability and it does not require the above assumptions. A semi-circle theorem similar to that of Howard and the growth rate are also obtained.

4.1. *Stability Analysis*

The required stability equation, taking $n = \frac{1}{2}$ in eqn. (16), is

$$D[\rho_0 U D_* H] - \rho_0\left[k^2 U + \frac{(DW)^2 - 4K}{4U}\right]H - \tfrac{1}{2} rD\left(\frac{\rho_0}{r} DW\right)H = 0 \qquad \text{...(69)}$$

where

$$K = N^2 + \Phi - \mu_r D\left(\frac{H_0}{r}\right)^2.$$

Multiplying eqn. (69) by the complex conjugate H^* and integrating between a and b, we obtain

$$\int_a^b \rho_0 U [\,|D_* H|^2 + k^2 |H|^2] \, r\, dr + \int_a^b \rho_0 U^* \left[\frac{1}{H}(DW)^2 - K\right] \times \frac{|H|^2}{|U|^2} r\, dr + \tfrac{1}{2}\int_a^b rD\left(\frac{\rho_0}{r} DW\right) |H|^2 r\, dr = 0. \qquad \text{...(70)}$$

The imaginary part of this equation $C_i > 0$ is

$$\int_a^b \rho_0 [\,|D_* H|^2 + k^2 |H|^2] \, r\, dr + \int_a^b \rho_0 \left[K - \left(\frac{DW}{2}\right)^2\right] \frac{|H|^2}{|U|^2} r\, dr = 0. \qquad \text{...(71)}$$

This is impossible if $K - \left(\frac{DW}{2}\right)^2$ is non-negative throughout so that a necessary condition for instability is that

$$K - \left(\frac{DW}{W}\right)^2 < 0 \qquad \text{...(72)}$$

or

$$J = J_H(r) + J_m(r) + J_e(r) < \tfrac{1}{4}. \qquad \text{...(73)}$$

Note that in obtaining an instability condition in the form (73), rather than (72), we have to be cautious about the fact that W is not strictly monotonic (i.e., $WD = 0$). Therefore, if one prefers the condition (73) rather than (72), we should allow J to approach ∞ when $DW = 0$.

4.2. *Restrictions on the Complex Wave Velocity*

The real and imaginary part of eqn. (65), that lead to the Theorems 6 and 7, also place restrictions on the wave speed C_r and the amplification factor. Defining $Q = | D_* H |^2 + k^2 | H |^2$ and assuming $C_i > 0$, eqns. (66) and (68) can be written respectively in the form :

$$\int_a^b \rho_0 W^2 Q\, r\, dr = (C_r^2 + C_i^2) \int_a^b \rho_0 Q\, r\, dr + \int_a^b \rho_0 K\, | H |^2\, r\, dr \qquad \text{...(74)}$$

$$\int_a^b \rho_0 W Q\, r\, dr = C_r \int_a^b \rho_0 Q r\, dr. \qquad \text{...(75)}$$

Suppose now that $W_1 \leqslant W(r) \leqslant W_2$. Then from eqns. (74) and (75) we have

$$0 \geqslant \int_a^b \rho_0 (W - W_1)(W - W_2)\, Q\, r\, dr = [C_r^2 + C_i^2 - (W_1 + W_2) C_r$$

$$+ W_1 W_2] \int_a^b \rho_0 Q\, r\, dr + \int_a^b \rho_0 K\, | H |^2\, r\, dr$$

$$= \left[\left\{C_r - \frac{W_1 + W_2}{2}\right\}^2 + C_i^2 - \left\{\frac{W_1 - W_2}{2}\right\}^2\right] \int_a^b \rho_0 Q\, r\, dr$$

$$+ \int_a^b \rho_0 K\, | H |^2 r\, dr.$$

If $K > 0$, this implies

$$\left(C_r - \frac{W_1 + W_2}{2}\right)^2 + C_i^2 \leqslant \left(\frac{W_1 - W_2}{2}\right)^2. \qquad \text{...(76)}$$

Thus we have the following theorem :

Semi-circle theorem — The complex wave velocity c for $K > 0$ and $C_i > 0$ must lie inside the semi-circle in the upper-half plane which has the range of W for diameter.

In the case of rigid body rotation $V = \Omega r$ and if the applied circular magnetic field H_0 is due to the line current I in the axial direction such that $H_0 = I/r$, the expression for K is

$$K = \beta(g - \Omega^2 r) + 4\Omega^2 + \frac{4\mu I^2}{\rho_0 r^4}. \qquad ...(77)$$

The semi-circle theorem is valid in this case if $g \geqslant \Omega^2 r$, so that $K > 0$. However, for $g < \Omega^2 r$; $K > 0$ provided

$$g - \Omega^2 r < \frac{4\Omega^2}{\beta} + \frac{4\mu I^2}{\beta \rho_0 r^4}. \qquad ...(78)$$

In general in the case of rigid body rotation and with an axial current the semi-circle theorem is valid only when eqn. (78) is satisfied.

4.3. *The Growth Rate*

The stability analysis discussed in section 3 limits the value of the total Richardson number J. The semi-circle theorem discussed in section 4.2, restricts the complex wave velocity which is accessible to unstable modes. It is of interest to have a similar bound on the growth rate kc_i possible for an unstable wave. A bound of this type can be obtained from eqn. (71), by observing that

$$| U |^2 = | W - c |^2 \geqslant C_i^2 \cdot \qquad ...(79)$$

Thus, from eqn. (71), using eqn. (79), we have

$$k^2 \int_a^b \rho_0 \, | H |^2 r \, dr \leqslant \frac{1}{C_i^2} \, \mathrm{Max} \left[\left(\frac{DW}{2} \right)^2 - K \right] \int_a^b \rho_0 \, | H |^2 r \, dr$$

and so

$$k^2 C_i^2 \leqslant \mathrm{Max} \, [\tfrac{1}{4} - (J_H + J_m + J_R)] \, (DW)^2. \qquad ...(80)$$

5. Universal Stability of Heterogeneous Conducting Flows

The linear stability of a heterogeneous conducting flow has been investigated, using normal mode technique, in sections 3 and 4 of this paper. We found that the necessary condition for instability is that the total Richardson number J which is the algebraic sum of hydrodynamic, hydromagnetic and swirling Richardson numbers must be less than one-quarter somewhere in the fluid. We observe that linear theory can only predict instability and is unable to provide directly a stability criteria which can be tested experimentally. The non-linear theory provides directly stability criteria which can easily be tested experimentally for there is no linear approximation in this theory. Therefore, it is of great laboratory, meteorological and astrophysical interest to apply the non-linear theory to the present problem.

The method used is the generalization of Serrin (1959) and Joseph (1965) in the discussion of the stability of viscous incompressible homogeneous thermoconvective flows. We try to establish a universal stability estimate, which is a stability limit for motions subject to arbitrary non-linear disturbances, for a heterogeneous conducting swirling flows between two rigid concentric cylinders with a circular magnetic field and radial gravitational force.

5.1. *Mathematical Formulation*

The basic eqns. (1) to (5) are rewritten in the form :

$$\rho \frac{D\vec{q}}{Dt} = - \nabla \pi + \mu(\vec{H} \cdot \nabla) \vec{H} - \rho g \vec{r} \qquad \ldots(81)$$

$$\frac{D\vec{H}}{Dt} = (\vec{H} \cdot \nabla) \vec{q} \qquad \ldots(82)$$

$$\nabla \cdot \vec{q} = 0, \ \nabla \cdot \vec{H} = 0 \qquad \ldots(83)$$

$$\frac{D\rho}{Dt} = 0 \qquad \ldots(84)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{q} \cdot \nabla$$

and

$$\pi = p + \frac{\mu H^2}{2}.$$

We assume the Boussinesq approximation, which implies that ρ can be replaced by a constant ρ_0 except in the buoyancy term g in eqn. (81). We consider basic flow occupying a bounded domain $V = V(t)$ with preassigned normal components of velocity distribution and magnetic field on the rigid surface S of V. To analyse the universal stability we consider the basic motion with the magnetic field at time $t = 0$ is altered and we wish to determine whether the subsequent flow, consistent with the original basic boundary conditions, approaches the basic flow asymptotically as $t \to \infty$.

If $\vec{q}$, $\vec{H}$, ρ_0 and π_0 are respectively the velocity, magnetic field, density and the pressure of the basic flow, then the corresponding quantities in the altered flow are defined by

$$\vec{q}^* = \vec{q} + \vec{u},\ \vec{H}^* = \vec{H} + \vec{h},\ \rho^* = \rho_0 + \rho,\ \pi^* = \pi_0 + \pi. \qquad ...(85)$$

We define

$$K_1 = \int_V \frac{u^2}{2}\, dV, \quad K_2 = \int \frac{\mu h^2}{2}\, dV \qquad ...(86)$$

kinetic and magnetic energies respectively and we say that the basic flow is asymptotically stable in the mean if both K_1 and K_2 tend to zero as $t \to \infty$. We see that the difference motion represented by $\vec{u}$ and $\vec{h}$ must satisfy the boundary conditions

$$\vec{u} . \hat{n} = 0 \quad \vec{h} . \hat{n} = 0 \qquad ...(87)$$

on S, where $\hat{n}$ is the unit outward normal vector to S. Since both the basic and altered equations must satisfy eqns. (1) to (4), we find on subtraction and using the Boussinesq approximation, that

$$\frac{\partial \vec{u}}{\partial t} + (\vec{q}^* . \nabla)\, \vec{u} + (\vec{u} . \nabla)\, \vec{q} = -\frac{1}{\rho_1} \nabla p - \frac{\rho}{\rho_1} g\hat{r}$$
$$+ \frac{\mu}{\rho_1} [(\vec{H}^* . \nabla)\, \vec{h} + (\vec{h} . \nabla)\, \vec{H}] \qquad ...(88)$$

$$\frac{\partial \vec{h}}{\partial t} + (\vec{q}^* . \nabla)\, \vec{h} + (\vec{u} . \nabla)\, \vec{H} = (\vec{H}^* . \nabla)\, \vec{u} + (\vec{h} . \nabla)\, \vec{q} \qquad ...(89)$$

$$\nabla . \vec{u} = 0,\ \nabla . \vec{h} = 0 \qquad ...(90)$$

$$\frac{D\rho}{Dt} + (\vec{u} . \nabla)\, \rho = 0. \qquad ...(91)$$

Multiplying eqns. (88) and (89) scalarly by $\vec{u}$ and $\vec{h}$ respectively and using eqn. (90), we obtain

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + (\vec{q}^* . \nabla)\, \frac{u^2}{2} = -\ [(\vec{u} . \nabla)\, \vec{q}] . \vec{u} - \frac{\rho}{\rho_1} g\hat{r} . \vec{u}$$
$$+ \frac{\mu}{\rho_1} (\vec{H}^* . \nabla\, \vec{h} . \vec{u} + \vec{h} . \nabla\, \vec{H} . \vec{u}) - \nabla\left(\frac{\pi \vec{u}}{\rho_1}\right) \qquad ...(92)$$

$$\frac{\partial}{\partial t}\left(\frac{h^2}{2}\right) + (\vec{q}^* . \nabla)\, \frac{h^2}{2} = [(\vec{h} . \nabla)\vec{q}] . \vec{h} + \vec{H} . \nabla\, \vec{u} . \vec{h}$$
$$- \vec{u} . \nabla \vec{H} . \vec{h} - \nabla . \left(\vec{u}\, \frac{h^2}{2}\right). \qquad ...(93)$$

We integrate eqns. (92) and (93) over V to obtain the rate of change of K_1 and K_2 in the form

$$\frac{DK_1}{Dt} = -\int_V [\vec{u}\,.\,\vec{D}\,.\,\vec{u} + \frac{\rho}{\rho_1}\,g\hat{r}\,.\,\vec{u} - \frac{\mu}{\rho_1}(\vec{h}\,.\,\nabla\,\vec{H}\,.\,\vec{u} + \vec{h}\,.\,\nabla\,\vec{H}^*\,.\vec{u})]\,dV \quad \text{...(94)}$$

$$\frac{DK_2}{Dt} = \int_V [\vec{h}\,.\,\vec{D}\,.\,\vec{h} + \vec{h}\,.\,\nabla\,\vec{H}\,.\,\vec{u} + \vec{u}\,.\,\nabla\,\vec{H}^*\,.\,\vec{h}]\,dV \quad \text{...(95)}$$

where $\vec{D}$ is the strain rate tensor of the basic flow.

Equations (94) and (95) are made dimensionless using the quantities

$$\left.\begin{array}{l}
\bar{K}_1 = \dfrac{k_1 d}{U_0^2},\ U_0 \text{ is the characteristic velocity} \\[2ex]
\bar{K}_2 = \dfrac{k_2 d}{H_0^2},\ H_0 \text{ is the characteristic magnetic field} \\[2ex]
\bar{\vec{r}} = \dfrac{\vec{r}}{d},\ d \text{ is the characteristic length} \\[2ex]
\bar{t} = \dfrac{tU_0}{d},\ \bar{\vec{u}} = \dfrac{\vec{u}}{u_0},\ \bar{\vec{D}} = \dfrac{d}{U_0}\,\vec{D},\ \bar{\vec{h}} = \dfrac{\vec{h}}{H_0} \\[2ex]
\bar{\vec{H}} = \dfrac{\vec{H}}{H_0},\ \bar{\vec{H}} = \dfrac{\vec{H}^*}{H_0},\ \bar{\rho} = \rho/\rho_1.
\end{array}\right\} \quad \text{...(96)}$$

Substituting eqn. (96), into eqns. (94) and (95) and for simplicity neglecting the bars (—), we get

$$\frac{DK_1}{Dt} = -\int_V \left[\vec{u}\,.\,\vec{D}\,.\,\vec{u} + \frac{\rho}{F}\,\vec{r}\,.\,\vec{u} - S(\vec{h}\,.\,\nabla\,\vec{H}\,.\,\vec{u} + \vec{h}\,.\,\nabla\,\vec{H}^*\,.\,\vec{u})\right] dV \quad \text{...(97)}$$

$$\frac{DK_2}{Dt} = \int_V [\vec{h}\,.\,\vec{D}\,.\,\vec{h} + \vec{h}\,.\,\nabla\,\vec{H}\,.\,\vec{u} + \vec{u}\,.\,\nabla\,\vec{H}^*\,.\,\vec{h}]\,dV \quad \text{...(98)}$$

where

$$S = \frac{\mu H_0^2}{\rho_1 U_0^2}, \text{ the Alfvén's number}$$

$$F = \frac{U_0^2}{dg}, \text{ the Froude number.}$$

5.2. *Universal Stability Estimate*

Let $\alpha = \max|\nabla\,\vec{H}^*|$, $\beta = \max|\nabla\,\vec{q}\,|$, $\gamma = \max|\nabla\,\vec{H}|$

$-m$ be the least eigenvalue of D and n be the largest number such that $\int \rho \vec{r} . \vec{u}\, d\tau \geqslant 2nK$ where the relation between ρ and $\vec{u}$ can be obtained from eqn. (91). The integrals in eqn. (97) and (98) have no definite signs and therefore can potentially destabilize the flow for critical values of the parameters F and N. Therefore, in the following theorem we establish a condition for which the motion is certainly stable.

Theorem 9 — Let $V = V(t)$ be a bounded region of space of q, H the velocity and magnetic field vectors respectively, satisfying the prescribed conditions on S. Then K_1 and K_2 satisfy

$$(\delta + m)\, K_1^{1/2} + (\gamma + d)\, K_2^{1/2} \leqslant [(\delta + m)\, K_{10}^{1/2} + (\gamma + d)\, K_{20}^{1/2}] \exp \delta t$$

where K_{10} and K_{20} are the initial values of K_1 and K_2 due to the initial disturbance and

$$\delta = -\frac{J}{2} + \tfrac{1}{2}\, [J^2 - 4\, \{(J - m)m - S(\gamma + \alpha)^2\}]^{1/2} \qquad ...(99)$$

$$J = \frac{N}{F} = \text{Richardson number.}$$

PROOF : We note that

$$\int_V \vec{u} . \vec{D} . \vec{u}\, dV \geqslant - m \int_V u^2\, dV = - 2mK_1$$

$$\int_V \vec{h} . \vec{D} . \vec{h}\, dV \geqslant - m \int_V h^2\, dV = - 2mK_2.$$

By the Schwarz inequality we have

$$\int_V \vec{h} . \nabla \vec{H} . \vec{u}\, dV \leqslant \gamma\, |\, \vec{h}\, |\ \ |\, \vec{u}\, |\, dV \leqslant 2\gamma(K_1K_2)^{1/2}$$

$$\int_V \vec{u} . \nabla \vec{H} . \vec{h}\, dV \leqslant \alpha\, |\, \vec{u}\, |\ \ |\, \vec{h}\, |\, dV \leqslant 2\alpha(K_1K_2)^{1/2}.$$

Thus we may write eqns. (97) and (98) in the form

$$\frac{DK_1}{Dt} + 2(J - m)\, K_1 \leqslant 2(\gamma + \alpha)\, S(K_1K_2)^{1/2} \qquad ...(100)$$

$$\frac{DK_2}{Dt} + 2mK_2 \leqslant 2(\gamma + \alpha)\, (K_1K_2)^{1/2}. \qquad ...(101)$$

Equations (99) and (100), using $K^2 = K_1$, and $M^2 = K_2$ become

$$\frac{1}{S(\gamma + \alpha)} \frac{DK}{Dt} + \frac{J - m}{S(\gamma + \alpha)} K - M \leqslant 0 \qquad ...(102)$$

$$\frac{1}{(\gamma + \alpha)} \frac{DM}{Dt} + \frac{m}{\gamma + \alpha} M - K \leqslant 0. \qquad ...(103)$$

We can easily see that

$$x = (\delta + m) \exp(-\delta t) > 0$$

$$y = (\gamma + \alpha) \exp(-\delta t) > 0$$

where δ is given by eqn. (99), are the solutions of the differential equations :

$$-\frac{1}{S(\gamma + \alpha)} \frac{Dx}{Dt} + \frac{J - m}{S(\gamma + \alpha)} x - y = 0 \qquad ...(104)$$

$$-\frac{1}{\gamma + \alpha} \frac{Dy}{Dt} + \frac{m}{\gamma + \alpha} y - x = 0. \qquad ...(105)$$

Combining these equations, we get,

$$D^2 y - Jy + [(J - m) m - S(\gamma - \alpha)^2] y = 0 \qquad ...(106)$$

One of the possible characteristic roots of eqn. (106) is given by eqn. (99).

Multiplying eqn. (102) by x and eqn. (104) by K, and then subtracting, we obtain

$$\frac{1}{S(\gamma + \alpha)} \frac{D(Kx)}{Dt} + yK - xM \leqslant 0. \qquad ...(107)$$

Similarly from eqns. (104) and (105) we get

$$\frac{1}{(\gamma + \alpha)} \frac{D(My)}{Dt} + xM - yK \leqslant 0. \qquad ...(108)$$

Adding eqns. (107) and (108), we have

$$\frac{D}{Dt} [(\delta + m) K + (\gamma + \alpha) M] \exp(-\delta t) \leqslant 0. \qquad ...(109)$$

The Theorem 9 follows by integrating eqn. (109) from $t = 0$ to the current value of t. From eqn. (99), it is clear that

$$(\delta + m) = \left(m - \frac{J}{2}\right) + \left[\left(m - \frac{J}{2}\right)^2 + S(\gamma + \alpha)^2\right]^{1/2} \geqslant 0.$$

From Theorem 9 we conclude that if δ is negative K_1 and K_2 tend asymptotically to zero as $t \to \infty$. δ is negative if

$$0 \leqslant S(\gamma + \alpha)^2 \leqslant (J - m) m \qquad ...(110)$$

and hence the condition for universal stability is eqn. (110).

For hydrodynamic heterogeneous flow ($S = 0$), eqn. (99) takes the form

$$\delta = -\frac{J}{2} + \left[\left(m - \frac{J}{2}\right)^2\right]^{1/2} = m - J$$

and the condition for universal stability is

$$J \geqslant m \geqslant 0. \qquad \ldots(111)$$

From eqn. (111) it follows that the motion is universally stable if $J \geqslant 0$. In the case of linear stability theory Miles (1961) proved that the motion is stable if $J \geqslant \frac{1}{4}$ which follows from eqn. (111) when $m = \frac{1}{4}$.

6. Conclusion

The linear and nonlinear stability of a heterogeneous conducting swirling flow between two rigid concentric cylinders is investigated. In the case of linear theory the stability is discussed using the normal mode technique and systematic proof of the following theorems are given :

(i) Extension of Miles' proof of Taylor's conjecture, that the Richardson number must somewhere be less than one-quarter if the flow is unstable to magnetohydrodynamics.

(ii) A semi-circle theorem similar to that of Howard (1961). This theorem restricts the value of the phase velocity C_r for an unstable motion. A growth rate kC_i is also obtained.

In the case of nonlinear arbitrary disturbances, the stability criterion is derived using the energy method and we found that the bound on the stability is

$$0 \leqslant S(\gamma + \alpha)^2 \leqslant (J - m)\, m.$$

In particular, we see that the bound for stability for hydrodynamic flow ($S = 0$) is $J \geqslant m \geqslant 0$, whereas the bound for stability obtained (Miles 1961) by a small perturbation analysis is $J \geqslant \frac{1}{4}$.

Finally, we conclude that the linear and nonlinear theories are complimentary to each other in a sense that the linear theory can predict only instability whereas the nonlinear theory predicts the criterion for stability which can be tested experimentally.

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