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On Stability of the Flow of Stratified Rotating Superposed Fluids

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Abstract

The effect of Coriolis force on the stability of two superposed fluids is examined under the assumption of ideal (incompressible, non-viscous and zero thermal conductivity) flow. The analysis reveals only one type of instability in contrast to two types of instabilities observed by SONTOWSKI, SEIDEL, and AMES (1969), in the absence of Coriolis force. Thus the effect of Coriolis force on the superposed flow is to make the flow more stable. In particular, the special case of upper stratified fluid rotating and the lower non-rotating is examined in detail and we find two separate and different types of instabilities. As the velocity of the rotating fluid relative to the non-rotating one increases from zero, there first appears an instability of a selective and relatively weak nature referred to as the initial instability. This is followed, at higher velocities, by a stronger type of instability called the gross instability. We found that the effect of Coriolis force is to suppress the region of initial instability in the sense that the instability occurs for large velocity differences. We also found that the Coriolis force has no effect on the critical wavenumber. These general stability results are applied to a particular problem of the Kelvin-Helmholtz instability with rotation and we find that the results are in agreement with those of CHANDRASEKHAR (1961). We determine the rate of growth of initial instabilities, which depends on the density stratification and the angular velocity of the rotating gas. This growth rate, in the case of rotation, is greater than that in the absence of rotation, which is in agreement with observations of MUNK (1947).

Key words: Coriolis force; Instability; Superposed fluids.

1. *Introduction*

This paper deals with the stability of the motion at the interface between two superposed fluids having different densities and velocities with Coriolis force. This problem has an application to planetary atmospheres, to aid in understanding the limitations on the propagation of internal-inertial-gravitational waves and transfer of momentum from the troposphere to the ionosphere. By internal-inertial-gravitational wave, we mean a wave under the influence of Coriolis force in which the group velocity propagates almost vertically upwards while the phase velocity is downward and the particle motions are nearly horizontal. In the study of these waves, GERBIER and GERENGER (1961) have experimentally observed that there should be little disturbance above the critical level (which may be taken as the interface between the fluids, in the present paper) but large amplitude perturbations of the horizontal velocity should build up near there and turbulent breakdown seems probable. In that situation, the study of the stability of superposed fluids will help to understand turbulent breakdown. Another important application of the present problem is towards an understanding of the reversals of the wind in the planetary atmosphere. For a long time these reversals were a rather mysterious occurrence, since it did not seem conceivable that the thermal structure of the planetary atmosphere could produce such patterns in the gradient or geostropic wind (RUDRAIAH, NARASIMHAMURTHY, and MARIYAPPA 1972). The most important development during the last decade concerning the motion of planetary atmospheres has been the recognition that the wind reversals in the atmosphere are due to internal gravity waves (HINES 1960). These internal gravity waves with and without the Coriolis force have been extensively investigated by HINES (1960), BOOKER and BRETHERTON (1967), JONES (1967), and BRETHERTON (1969). Recently, these waves have been extended to MHD by RUDRAIAH and VENKATACHALAPPA (1972a, b, c), ACHESON (1972), ACHESON and HIDE (1973), and RUDRAIAH and VENKATACHALAPPA (1974a, b). The question of internal-inertial-gravitational waves in two superposed fluids with an interface is inseparable from the question of their stability. With this motivation in mind, we investigate here the stability of the flow of a stratified rotating gas over a rotating fluid.

In the study of such problems, one usually makes an approximation that the effect of density stratification is small compared with the potential energy of the system, i.e., the Boussinesq fluids (see CHANDRASEKHAR [1961, p. 16]). Conditions under which superposed fluids are important in the planetary atmosphere are usually far removed from the idealization of Boussinesq fluids. Since the dimensions involved are very large, variations of ρ and p in the planetary atmosphere can scarcely be regarded as small, and an additional complication is the source of energy provided by condensation of vapour giving rise to states called "conditionally unstable" (KAUO 1961). Also, in the planetary atmosphere pressure, density, and temperature vary by several orders of magnitude and transmission of heat by radiation is important (see UNNO 1957; KATO and UNNO 1960; UNNO, KATO, and MAKITA 1960; SPIEGEL and UNNO 1962). Therefore, in the study of the stability of superposed fluids, we have to consider the general density stratification without the Boussinesq approximation, and this is done in this paper.

The stability of superposed flow of two different fluids, each having uniform density and velocity distribution and flowing parallel to each other in a horizontal direction with an interface of arbitrary surface tensions was studied by KELVIN (1910), and he obtained a stability criterion in terms of the relative velocity between the two fluids. Application of this result to the special case of gas blowing over incompressible fluid yields a critical velocity of 24 km hr^{-1} at which instability first occurs. This result has met with general dissatisfaction since disturbances actually arise on large bodies of gas at much lower values of the wind velocity. However, MUNK (1947) expresses the belief that although not coincident with the onset of instability, the value obtained by KELVIN (1910) is indeed a critical value as it marks the occurrence of several other phenomena such as sudden increases in evaporation, convection, and the number of breaking waves. Whether true or not, this does not explain instabilities observed at velocities below KELVIN's (1910) critical value. Recently, SONTOWSKI et al. (1969) predicted such type of instabilities considering the density stratification in the upper fluid flowing over an incompressible fluid. They found two separate and different types of instabilities, namely, the initial and gross instabilities. The study of SONTOWSKI et al. (1969) revealed that as the velocity of the gas relative to the fluid increases from zero, there first appears an instability of a selective and weak nature referred to as the initial instability. This is followed, at higher velocities, by a stronger type of instability called the gross instability. The initial instability takes the form of two distinct waves of different lengths, one superposed upon the other. This superposition of two waves at low velocities is in accord with the experimental observation of MUNK (1947). SONTOWSKI et al. (1969) have considered only the effect of density stratification in the upper fluid but not the Coriolis force. But the effect of Coriolis force and density stratification is of fundamental importance in planetary atmospheres because of the rotation of planets. This is discussed in this paper.

The flow configuration and assumption of SONTOWSKI et al. (1969) are re-examined here with the additional consideration of Coriolis force. In particular, the case of the upper fluid rotating and the lower non-rotating is discussed in detail. We found that, in general, the effect of rotation is to remove the initial instabilities. However, when the upper fluid is rotating and the lower one is non-rotating, there exist two types of instabilities, namely, initial and gross instabilities, as observed by SONTOWSKI et al. (1969). Finally, we note that the assumption of incompressible fluid is reasonable, for the fluid velocities to be considered are low. Viscosity can be neglected by reasoning in terms of the types of instabilities expected in the bounded flow of a gas over a fluid (see SONTOWSKI et al. [1969]).

2. *Formulation of Problem*

2.1. *Mathematical Formulation*

We consider a single-fluid model of a heterogeneous, incompressible, non-viscous rotating fluid. The basic equations of motion of this model are (see CHANDRASEKHAR [1961, ch. XI]);

$$\nabla\cdot\boldsymbol{q}=0\ , \tag{2.1}$$

$$\frac{D\rho}{Dt}=0\ , \tag{2.2}$$

$$\rho\frac{D\boldsymbol{q}}{Dt}+2\boldsymbol{\Omega}\times\boldsymbol{q}=-\nabla p+\rho\boldsymbol{g}\ , \tag{2.3}$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+(\boldsymbol{q}\cdot\nabla)\ ,$$

$\boldsymbol{q}$, ρ, p are velocity, density, and pressure of the fluid, respectively, and $\boldsymbol{\Omega}$ is the angular velocity about the z-axis. Equation (2.2) physically represents the fact that the density discontinuities are allowable in the solutions.

2.2. *Basic State*

Figure 1 illustrates the problem. The fluids in the regions $z\gtrless 0$ are assumed to be of infinite extent below and above the common boundary. The basic state considered is of the form

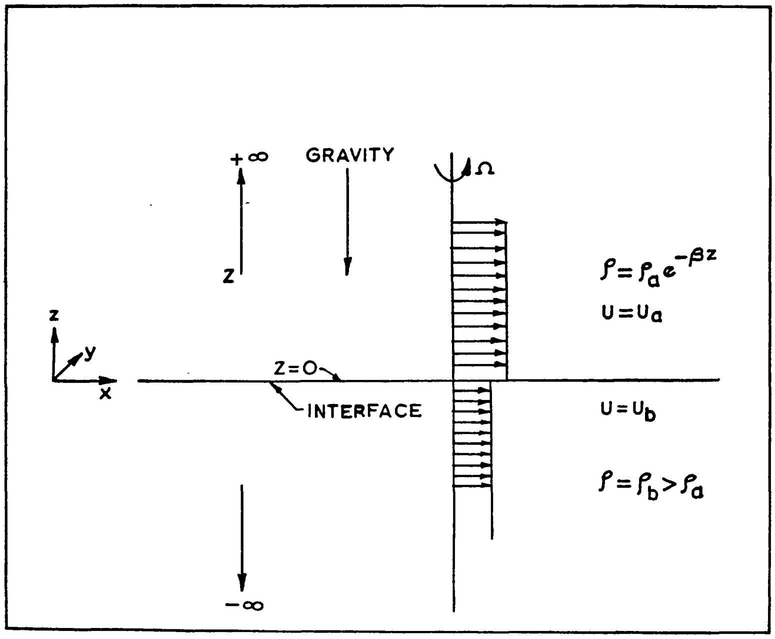

Fig. 1. Basic configuration.

$$\boldsymbol{q}=(U(z), 0, 0), \quad p=p_0, \quad \rho=\rho_0, \quad \boldsymbol{\Omega}=(0, 0, \Omega) . \tag{2.4}$$

For geostrophic balance

$$\frac{\partial p_0}{\partial y}=-2\Omega U\rho_0 , \tag{2.5}$$

and for hydrostatic balance

$$\frac{\partial p_0}{\partial z}=-g\rho_0 . \tag{2.6}$$

From equations (2.5) and (2.6), we get

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial y}=\frac{2\Omega}{g}\frac{dU}{dz}+\frac{2\Omega U}{g}\left(\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z}\right). \tag{2.7}$$

2.3. *The Perturbed State*

On the basic state discussed above we superimpose a small symmetrical disturbance of the form

$$(U+U', V', W'), \quad \rho_0+\rho', \quad p_0+p' , \tag{2.8}$$

where the primed quantities denote the perturbed quantities and are assumed to be small compared with the basic quantities. Linearizing equations (2.1) to (2.3) by using expressions (2.8) and seeking solutions of the form

$$(\text{Some function of } z) \exp\{i(kx+ly+\sigma t)\} , \tag{2.9}$$

it is possible to obtain, by a process of elimination, the equation governing vertical velocity W in the form

$$D\left\{\rho\Omega_{\rm d}\left(1-\frac{4\Omega^2}{\Omega_{\rm d}^2}\right)DW-\rho\left(k+\frac{2i\Omega l}{\Omega_{\rm d}}\right)(DU)W\right\}-\alpha^2\rho\Omega_{\rm d}W=g\alpha^2(D\rho)\frac{W}{\Omega_{\rm d}} , \tag{2.10}$$

where $\Omega_d=(\sigma+kU)$ is the Doppler-shifted frequency, $\alpha=(k^2+l^2)^{1/2}$ and for simplicity primes on ρ and W are neglected.

At the interface between the two fluids, we require that W/Ω_d is continuous and

$$\varDelta_0\left\{\rho\Omega_d\left(1-\frac{4\Omega^2}{\Omega_d^2}\right)DW-\rho\left(k+\frac{2i\Omega l}{\Omega_d}\right)(DU)W\right\}=g\alpha^2\left\{\varDelta_0(\rho)-\frac{\alpha^2 T_0}{g}\right\}\left(\frac{W}{\Omega_d}\right)_0, \quad (2.11)$$

where

$$\varDelta_0(f)=\lim_{\varepsilon\to 0}[f_{z=z_0+\varepsilon}-f_{z=z_0-\varepsilon}]$$

is the jump, a quantity f experiences at $z=z_0$, and $z_0(=0)$ represents the undisturbed interface position. (It is necessary to observe that $W/(\sigma+kU)$ must be continuous at an interface. If U is continuous at an interface, this condition simply reduces to a condition requiring the continuity of W; but if U is discontinuous at $z=z_0$, then we must require, instead, the uniqueness of the normal displacement at any point on the interface, i.e., the normal displacement at any point on the interface $\delta z_s\{=W_0/i(\sigma+kU_0)\}$ is to be continuous.)

Now, we consider equation (2.10) for regions $z>0$ and $z<0$. The region $z>0$ is occupied by a stratified gas and the region $z<0$ is occupied by an incompressible homogeneous fluid. In these regions, equation (2.10), assuming U=constant, takes the form:

$$\frac{d^2W}{dz^2}-\beta\frac{dW}{dz}-\alpha^2\left(1-\frac{g\beta-4\Omega^2}{\Omega_{da}^2-4\Omega^2}\right)W=0\,, \quad \text{for } z>0\,, \quad (2.12)$$

and

$$\frac{d^2W}{dz^2}-\alpha^2\left(\frac{\Omega_{db}^2}{\Omega_{db}^2-4\Omega^2}\right)W=0\,, \quad \text{for } z<0\,. \quad (2.13)$$

The solutions for the above equations are given by

$$W=C_a e^{m_{a+}z}+D_a e^{m_{a-}z}\,, \quad z>0\,, \quad (2.14)$$

$$W=C_b e^{m_{b+}z}+D_b e^{m_{b-}z}\,, \quad z<0\,, \quad (2.15)$$

where

$$m_{a\pm}=\frac{\beta}{2}\pm\left\{\left(\frac{\beta}{2}\right)^2+\alpha^2\left(1-\frac{g\beta-4\Omega^2}{\Omega_{da}^2-4\Omega^2}\right)\right\}^{1/2}, \quad (2.16)$$

$$m_{b\pm}=\pm\alpha\left(\frac{\Omega_{db}^2}{\Omega_{db}^2-4\Omega^2}\right)^{1/2}, \quad (2.17)$$

and C_a, D_a, C_b, D_b are arbitrary constants.

Boundary conditions disallow disturbances which increase exponentially as the outer bounds of the layers are approached. Thus

$$W=D_a e^{m_{a-}z}\,, \quad z>0\,, \quad (2.18)$$

$$W=C_b e^{m_{b+}z}\,, \quad z<0\,, \quad (2.19)$$

with the requirement that

$$\mathrm{Re}\left[\left(\frac{\beta}{2}\right)^2+\alpha^2\left(1-\frac{g\beta-4\Omega^2}{\Omega_{da}^2-4\Omega^2}\right)\right]^{1/2}\geq\frac{\beta}{2}\,. \quad (2.20)$$

Continuity of W/Ω_d and the interfacial boundary condition (2.11) yields the eigenvalue equation, which in dimensionless form may be written as

$$\rho^*\beta_k(\nu+k^*\bar{U}_a)^2+1-\rho^*+\sigma_k-4\rho^*\beta_k\Omega^{*2}$$
$$=\bar{\alpha}(\nu+k^*\bar{U}_b)^2\left\{1-\frac{4\Omega^{*2}}{(\nu+k^*\bar{U}_b)^2}\right\}\left\{1+\frac{4\Omega^{*2}}{(\nu+k^*\bar{U}_b)^2-4\Omega^{*2}}\right\}^{1/2}$$
$$+\rho^*(\nu+k^*\bar{U}_a)^2\left\{1-\frac{4\Omega^{*2}}{(\nu+k^*\bar{U}_a)^2}\right\}\left\{1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{(\nu+k^*\bar{U}_a)^2-4\Omega^{*2}}\right\}^{1/2}, \tag{2.21}$$

where

$$\nu=\frac{\{(\beta/2)^2+\alpha^2\}^{1/4}}{\alpha g^{1/2}}\sigma, \quad \Omega^*=\frac{\{(\beta/2)^2+\alpha^2\}^{1/4}}{\alpha g^{1/2}}\Omega, \quad k^*=k/\alpha, \quad \rho^*=\rho_a/\rho_b,$$

$$\beta_k=\frac{\beta/2}{\{(\beta/2)^2+\alpha^2\}^{1/2}}, \quad \bar{\alpha}=\frac{\alpha}{\{(\beta/2)^2+\alpha^2\}^{1/2}}, \quad \sigma_k=\frac{T_0\alpha^2}{g\rho_b},$$

$$\bar{U}_a=\frac{\{(\beta/2)^2+\alpha^2\}^{1/4}}{g^{1/2}}U_a, \quad \bar{U}_b=\frac{\{(\beta/2)^2+\alpha^2\}^{1/4}}{g^{1/2}}U_b,$$

and the requirement (2.20) now takes the form

$$\mathrm{Re}\left[1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{(\nu+k^*\bar{U}_a)^2-4\Omega^{*2}}\right]^{1/2}\geq\beta_k. \tag{2.22}$$

3. *Determination of the Instabilities*

Recalling the form of the disturbance in equation (2.9), it follows that the flow is unstable if and only if any one or more of the eigenvalues ν has a negative imaginary part. For a complete stability analysis the characteristic values of ν must be examined for all values of the wavenumber vector $\boldsymbol{k}=(k, l)$, which is done in this section.

Equations (2.21) and (2.22), using the new variables

$$\xi=\nu+k^*\bar{U}_a, \quad \eta=\nu+k^*\bar{U}_b, \quad \eta_{a0}^2=\frac{1-\rho^*+\sigma_k}{\rho^*} \tag{3.1}$$

become

$$(\xi^2-4\Omega^{*2})\left\{\left(1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right)^{1/2}-\beta_k\right\}+\frac{\bar{\alpha}}{\rho^*}(\eta^2-4\Omega^{*2})\left(1+\frac{4\Omega^{*2}}{\eta^2-4\Omega^{*2}}\right)^{1/2}=\eta_{a0}^2, \tag{3.2}$$

and

$$\mathrm{Re}\left[1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right]^{1/2}\geq\beta_k, \tag{3.3}$$

which, in conjunction with the auxiliary relationship

$$\xi-\eta=k^*(\bar{U}_a-\bar{U}_b), \tag{3.4}$$

is equivalent to the eigenvalue problem.

Equations (3.2), (3.3), and (3.4) define the eigenvalue problem and will be an eighth degree equation in ν, which is nonalgebraic because of the condition (3.3). To restore the algebraic character to the problem, we construct a parent algebraic system possessing the eigenvalue as a subsystem. We distinguish two branches of the parent system, calling our eigenvalue problem the principal or P-branch and the remainder of the system, where $\mathrm{Re}\,[1-(2\beta_k-4\bar{\alpha}^2\Omega^{*2})/(\xi^2-4\Omega^{*2})]^{1/2}<\beta_k$ as the subsidiary or S-branch. In mathematical form these can be expressed as:

$$\text{P-branch} \begin{cases} PA^{1/2}+QB^{1/2}=R\,, & (3.5)\\ \mathrm{Re}\,[A]^{1/2}\geq\beta_k\,, & (3.6)\end{cases}$$

$$\text{S}_1\text{-branch} \begin{cases} PA^{1/2}+QB^{1/2}=R\,, & (3.7)\\ 0<\mathrm{Re}[A]^{1/2}<\beta_k\,, & (3.8)\end{cases}$$

$$\text{S}_2\text{-branch} \begin{cases} -PA^{1/2}+QB^{1/2}=R\,, & (3.9)\\ \mathrm{Re}\,[A]^{1/2}\geq 0\,, & (3.10)\end{cases}$$

$$\text{S}_3\text{-branch} \begin{cases} PA^{1/2}-QB^{1/2}=R\,, & (3.11)\\ \mathrm{Re}\,[A]^{1/2}\geq\beta_k\,, & (3.12)\end{cases}$$

$$\text{S}_4\text{-branch} \begin{cases} PA^{1/2}-QB^{1/2}=R\,, & (3.13)\\ 0<\mathrm{Re}\,[A]^{1/2}<\beta_k\,, & (3.14)\end{cases}$$

$$\text{S}_5\text{-branch} \begin{cases} -PA^{1/2}-QB^{1/2}=R\,, & (3.15)\\ \mathrm{Re}\,[A]^{1/2}\geq 0\,, & (3.16)\end{cases}$$

where

$$A=1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\,,\qquad B=1+\frac{4\Omega^{*2}}{\eta^2-4\Omega^{*2}}\,,$$

$$P=(\xi^2-4\Omega^{*2})\,,\qquad Q=\frac{\bar{\alpha}}{\rho^*}(\eta^2-4\Omega^{*2})\,,$$

$$R=\eta_{a0}^2+\beta_k(\xi^2-4\Omega^{*2})\,.$$

The auxiliary equation (3.4) is to be satisfied simultaneously with each of the above basic branch equations. In comparison with the four equations given by CHANDRASEKHAR (1961, p. 503), we see that we have six equations in our case. This is because of the conditions placed on the expression $\{1-(2\beta_k-4\bar{\alpha}^2\Omega^{*2})/(\xi^2-4\Omega^{*2})\}^{1/2}$. Due to the presence of two square roots in the equations, analytical treatment as in the case of SONTOWSKI et al. (1969) cannot be carried out completely in this case. However, the eigenvalue equation is solved numerically and the results are shown in the real (ξ, η) locus for particular values and is marked in figure 2. Tangents to the curves are vertical at the point $(\pm 2\Omega^*, 0)$ and horizontal at $(0, \pm\eta_{a0})$. Except for the two significant points at $(\pm\xi_0, 0)$ on the ξ-axis, there is no real locus for $2\Omega^*<|\xi|<(2\beta_k+4\beta_k^2\Omega^{*2})^{1/2}$ and $|\eta|<2\Omega^*$. The principal branch and the subsidiary branches (S_1, S_2, S_3, S_4, S_5) are shown in figure 2 for the real (ξ, η)-locus. It is of interest to note that, in the absence of Coriolis force SONTOWSKI et al.

(1969) have shown that there exists one P-branch and two subsidiary branches S_1 and S_2, whereas in the problem discussed here we observe there exist one P-branch and five subsidiary branches S_1, S_2, S_3, S_4, and S_5. Hence the effect of Coriolis force is to introduce three subsidiary branches S_3, S_4, and S_5. These additional subsidiary branches remove the initial instabilities as explained below.

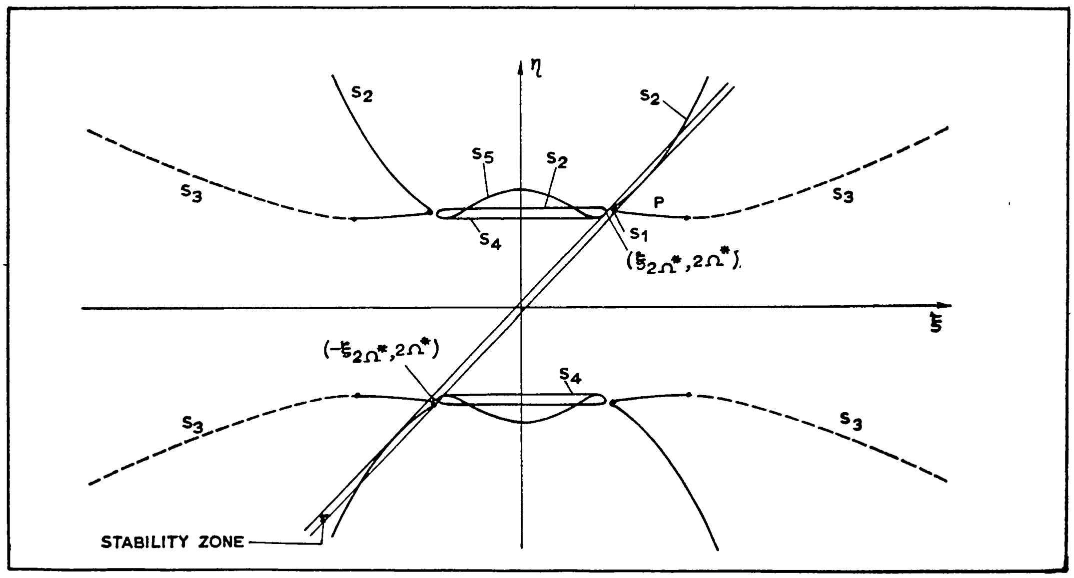

Fig. 2. Stability zones in the real (ξ, η)-plane (when both the fluids are rotating).

The intersections of the auxiliary line in the real (ξ, η)-plane with the locus of the basic branch equations represent the real roots of the parent algebraic system. From the figure, we observe that there is only one possible stability zone where the auxiliary line will have eight intersections. It is also of interest to observe that this stability zone is confined to a very small region symmetrical about $\xi=\eta$. So, there exists only gross instability, but not the initial instability.

We note that the "possible stability" zone lies between the auxiliary lines passing through $(-\xi_{2\Omega^*}, 2\Omega^*)$ and $(\xi_{2\Omega^*}, 2\Omega^*)$. Hence the principal result that can be established in the case when both the fluids are rotating is that, for a system partaking in rotation, the Kelvin-Helmholtz instability cannot occur so long as

$$|k^*(U_a-U_b)| \leq \xi_{2\Omega^*}-2\Omega^* , \tag{3.17}$$

where

$$\xi_{2\Omega^*}=\left[2\Omega^{*2}+\frac{\beta_k(1+\eta_{a0}^2)}{\bar{\alpha}^2}+\left[4\Omega^{*4}+\beta_k\left(\frac{1+\eta_{a0}^2}{\bar{\alpha}^2}\right)\left\{\beta_k\left(\frac{1+\eta_{a0}^2}{\bar{\alpha}^2}\right)-4\Omega^{*2}\right\}+\frac{\eta_{a0}^4}{\bar{\alpha}^2}\right]^{1/2}\right]^{1/2} .$$

As pointed out earlier, since equation (2.21) is an eighth degree equation in ν and because of the curious restrictions on the signs of the square roots in equation (2.21), the analytical treatment of the stability problem is very difficult and cumbersome. However, we note that when the upper fluid is rotating and the lower fluid is nonrotating, analytical conditions for stability can be obtained and this is done in the subsequent section.

4. *On Stability of a Stratified Rotating Gas over a Non-Rotating Gas*

4.1. *The Dispersion Relation*

Of the two fluids that we are considering, we suppose that the upper stratified fluid is rotating in the z-direction with a constant angular velocity Ω and the lower fluid is non-rotating. For this case equation (2.21) reduces to

$$\rho^*\beta_k(\nu+k^*\bar{U}_a)^2+(1-\rho^*+\sigma_k-4\rho^*\beta_k\Omega^{*2})=\bar{\alpha}(\nu+k^*\bar{U}_b)^2 +\rho^*(\nu+k^*\bar{U}_a)^2\left\{1-\frac{4\Omega^{*2}}{(\nu+k^*\bar{U}_a)^2}\right\}\left\{1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{(\nu+k^*\bar{U}_a)^2-4\Omega^{*2}}\right\}^{1/2}. \quad (4.1)$$

To examine the nature of ν, we rewrite equation (4.1) using the new variables

$$\left.\begin{aligned}\xi&=\nu+k^*\bar{U}_a\,,\\ \eta&=\nu+k^*\bar{U}_b-\frac{\rho^*\beta_k}{(\bar{\alpha}-\rho^*\beta_k)}\{k^*(\bar{U}_a-\bar{U}_b)\}\,,\end{aligned}\right\} \quad (4.2)$$

in the form

$$\eta^2+\frac{\rho^*}{(\bar{\alpha}-\rho^*\beta_k)}(\xi^2-4\Omega^{*2})\left(1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right)^{1/2}=\eta_{a0}^2\,, \quad (4.3)$$

and

$$\mathrm{Re}\left[1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right]^{1/2}\geq\beta_k\,, \quad (4.4)$$

where

$$\eta_{a0}^2=\frac{\bar{\alpha}\rho^*\beta_k}{(\bar{\alpha}-\rho^*\beta_k)^2}\{k^*(\bar{U}_a-\bar{U}_b)\}^2+\frac{1-\rho^*+\sigma_k-4\rho^*\beta_k\Omega^{*2}}{(\bar{\alpha}-\rho^*\beta_k)}\,. \quad (4.5)$$

Also, the auxiliary equation, from equation (4.2) is

$$\xi-\eta=\frac{\bar{\alpha}}{\bar{\alpha}-\rho^*\beta_k}\{k^*(\bar{U}_a-\bar{U}_b)\}\,. \quad (4.6)$$

4.2. *The Eigenvalue Problem*

Equations (4.3) and (4.4) together with the auxiliary equation (4.6) defines the eigenvalue problem. This problem is non-algebraic because of the condition (4.4). However, we can restore the algebraic character by constructing a parent algebraic system possessing the eigenvalue problem as a subsystem. Equation (4.3), along with the auxiliary equation (4.6), is equivalent to a fourth-degree polynomial in ν and this is taken as a parent system. The parent system is distinguished into two branches. The eigenvalue problem is referred to as the principal or P-branch and the remainder of the system as the subsidiary or S-branch. In mathematical form, we have

$$\text{P-branch}\left\{\begin{aligned}&\eta^2+\frac{\rho^*}{(\bar{\alpha}-\rho^*\beta_k)}(\xi^2-4\Omega^{*2})\left(1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right)^{1/2}=\eta_{a0}^2\,, && (4.7)\\ &\mathrm{Re}\left[1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right]^{1/2}\geq\beta_k\,, && (4.8)\end{aligned}\right.$$

S$_1$-branch
$$\eta^2+\frac{\rho^*}{(\bar{\alpha}-\rho^*\beta_k)}(\xi^2-4\Omega^{*2})\left(1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right)^{1/2}=\eta_{a0}^2\,, \tag{4.9}$$
$$0<\mathrm{Re}\left[1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right]^{1/2}<\beta_k\,, \tag{4.10}$$

S$_2$-branch
$$\eta^2-\frac{\rho^*}{(\bar{\alpha}-\rho^*\beta_k)}(\xi^2-4\Omega^{*2})\left(1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right)^{1/2}=\eta_{a0}^2\,, \tag{4.11}$$
$$\mathrm{Re}\left[1-\frac{2\beta_k-4\bar{\alpha}^2\Omega^{*2}}{\xi^2-4\Omega^{*2}}\right]^{1/2}\geq 0\,. \tag{4.12}$$

The auxiliary equation (4.6) must be satisfied simultaneously, with each of the above basic branch equations and conditions.

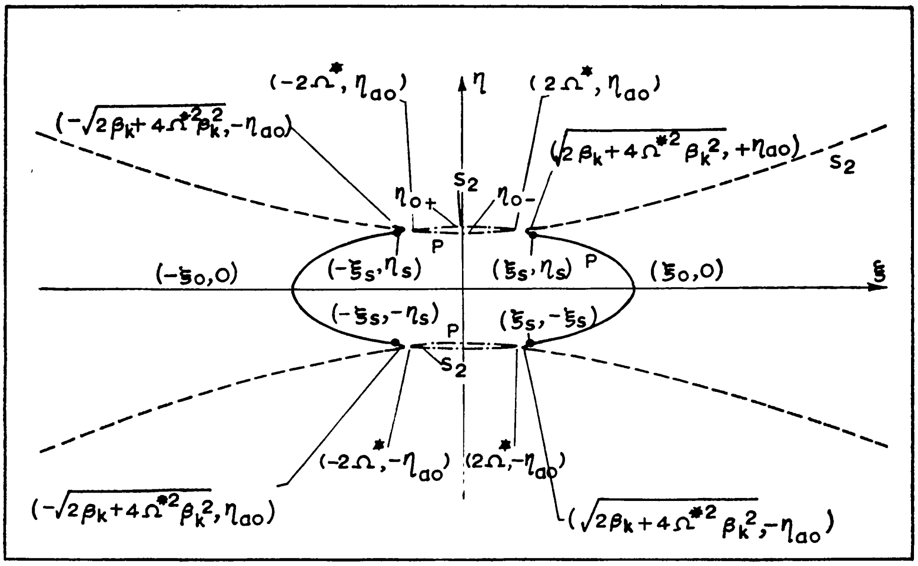

Fig. 3. The real locus of the basic branch equations.

The real loci of the basic branch equations are plotted in the (ξ,η)-plane for the case $\bar{\alpha}>\rho^*\beta_k$ (see figure 3), where

$$\xi_s=\frac{2\beta_k}{\bar{\alpha}^2}\,,\qquad \eta_s{}^2=\eta_{a0}^2-\frac{\rho^*\beta_k}{\bar{\alpha}-\rho^*\beta_k}\left(\frac{2\beta_k}{\bar{\alpha}^2}-4\Omega^{*2}\right),$$

$$\xi_0{}^2=\beta_k+2\Omega^{*2}(1+\beta_k{}^2)+\left\{(\beta_k-2\bar{\alpha}^2\Omega^{*2})^2+\eta_{a0}^4\left(\frac{\bar{\alpha}-\rho^*\beta_k}{\bar{\alpha}}\right)^2\right\}^{1/2},$$

$$\eta_0{}^2=\frac{\rho^*}{\bar{\alpha}-\rho^*\beta_k}4\Omega^{*2}(1+2\beta_k-4\bar{\alpha}^2\Omega^{*2})^{1/2}+\eta_{a0}^2\,.$$

Tangents to the curve are vertical at the points $(\xi_0, 0)$, $([2\beta_k+4\beta_k{}^2\Omega^{*2}]^{1/2}, \eta_{a0})$, and $(2\Omega^*, \eta_{a0})$. We see that there is no real locus for

$$2\Omega^*<|\xi|<(2\beta_k+4\beta_k{}^2\Omega^{*2})^{1/2}\,.$$

In the non-rotating case, discussed by SONTOWSKI et al. (1969), there exist two significant points on the η-axis, whereas, in the present problem, we observe that instead of these two significant points, there will be a loop form which is

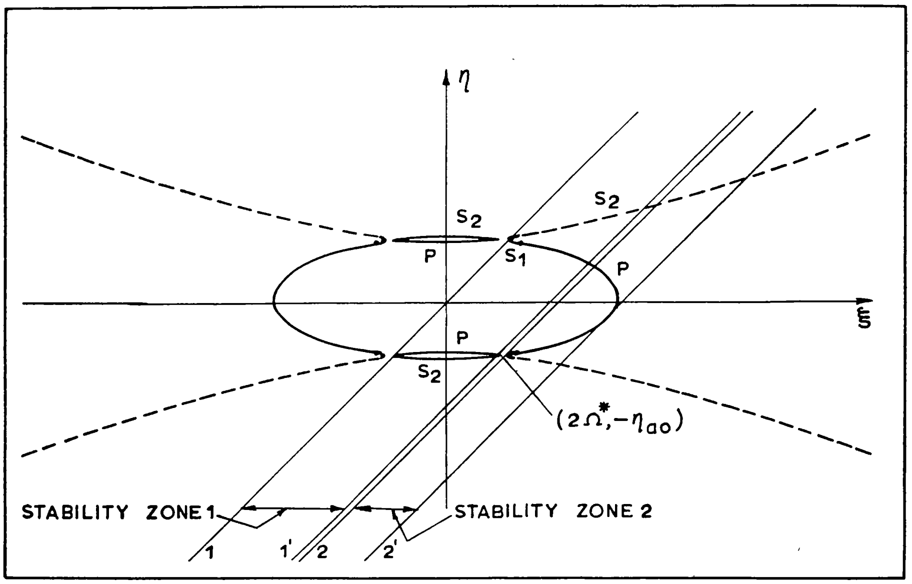

Fig. 4. Stability zones in the real (ξ, η)-plane $(\Omega^{*2}=0.25,\ \beta_k=0.5)$.

symmetrical about both axes.

The real locus of the auxiliary equation is a straight line having a slope of 1, as shown in figure 4. The locations of the intersections of the real (ξ, η)-plane of the auxiliary line with the locus of the basic branch equation indicate the particular branch to which each real part belongs. For situations where there are four real intersections, the eigenvalue problem represented by the P-branch can have no complex roots and the system is stable. Referring to figure 4, the auxiliary lines falling between 1 and 1′ and 2 and 2′ will always have four real intersections and hence the system is always stable.

Under the influence of a particular disturbance as the velocity $U=k^*(\bar{U}_a-\bar{U}_b)$ alone increases, the fluid system experiences consecutively conditions of definite stability (stability zone 1), possible instability (between stability zones 1 and 2), definite stability (stability zone 2), and finally possible instability (see figure 4).

However, for larger values of Ω^* (say $\Omega^{*2}=1.0$) we observe that the fluid system first experiences definite stability, possible instability, definite stability, possible instability, definite stability, and finally possible instability. So, for larger values of Ω^* the "possible instability zones" increase and hence the system may lead to an almost unstable situation.

4.3. *Discussion of Initial and Gross Instabilities*

The analytical examination of the loci, using Sturm's theorem, reveals that under the conditions

$$\left.\begin{aligned} &\frac{\rho^*}{\bar{\alpha}-\rho^*\beta_k}<1\,, \quad 3p_1^2p_2^2p^2+\eta_{a_0}^2(1-p^2)>p^2(1-p^2)[\beta_k+\Omega^{*2}(1+\beta_k^2)]\,, \\ &(3a_1^2-8a_0a_2)(a_2^2-2a_2a_3-4a_0a_4)>(6a_0a_3-a_1a_2)^2\,, \\ &b_2>\frac{b_1c_2^2+b_3c_1^2}{c_1c_2}\,, \end{aligned}\right\} \tag{4.13}$$

where

$$p_1=\frac{\bar{\alpha}}{\bar{\alpha}-\rho^*\beta_k}\,,\quad p_2=k^*(\bar{U}_a-\bar{U}_b)\,,\quad p=\frac{\rho^*}{\bar{\alpha}-\rho^*\beta_k}\,,$$

$$a_0=1-p^2\,,\quad a_1=4p_1p_2\,,$$

$$a_2=6p_1^2p_2^2+2\beta_kp^2-2\eta^2_{a_0}+4\Omega^{*2}p^2(1+\beta_k^2)\,,$$

$$a_3=4p_1p_2(p_1^2p_2^2-\eta^2_{a_0})\,,\quad a_4=\frac{a_3^2}{a_1^2}-8p^2\beta_k\Omega^{*2}(1+2\beta_k\Omega^{*2})\,,$$

$$b_1=(3a_1^2-8a_0a_2)/16a_0\,,\quad b_2=(6a_0a_3-a_1a_2)/8a_0\,,\quad b_3=(a_1a_3-a_4)/16a_0\,,$$

$$c_1=\frac{1}{b_1}\{4a_0b_3-2a_2b_1-b_2(3a_1+4a_0b_2/b_1)\}\,,$$

$$c_2=a_3-\frac{b_3}{b_1}(3a_1b_1+4a_0b_2)\,,$$

auxiliary lines falling between 1 and 1′ of figure 4 and lines falling between 2 and 2′ will have four real intersections. Thus there exist two possible stability zones as in the case of SONTOWSKI et al. (1969). Hence in the present problem we have both initial and gross instabilities as observed by SONTOWSKI et al. (1969).

For a given system, in a particular stationary state, the sufficient conditions (4.13) become a requirement on the range of $\bar{\alpha}$. In the case of normal gases, ρ^* and β_k are extremely small, and the above conditions are satisfied consecutively for all disturbances except those of extremely long wavelength.

The nature of the real part of $[1-(2\beta_k-4\bar{\alpha}^2\Omega^{*2})/(\xi^2-4\Omega^{*2})]^{1/2}$ gives information as to the branch to which a particular root belongs. Hence, according to the branch equations (4.7) to (4.12) it follows that a root belongs to the S_2-branch, if $\mathrm{Re}\,[(\eta^2_{a_0}-\eta^2)/(\xi^2-4\Omega^{*2})]<0$, to the S_1-branch if $0<\mathrm{Re}\,[(\eta^2_{a_0}-\eta^2)/(\xi^2-4\Omega^{*2})]<\rho^*\beta_k/(\bar{\alpha}-\rho^*\beta_k)$ and to the P-branch if

$$\mathrm{Re}\left[\frac{\eta^2_{a_0}-\eta^2}{\xi^2-4\Omega^{*2}}\right]\geq\frac{\rho^*\beta_k}{\bar{\alpha}-\rho^*\beta_k}\,.$$

Along the real locus of the combined S_1 and P-branches in the fourth quadrant, we note that $(\eta^2_{a_0}-\eta^2)/(\xi^2-4\Omega^{*2})$ is real and positive. Now, let $\bar{U}_s$ be that value of $\bar{U}$ for which the system enters stability zone 2 with the corresponding real root of multiplicity two being $(\xi_T, -\eta_T)$. Let $\xi_T=\xi_s+a$, where a is some positive real number, with the point $(\xi_T, -\eta_T)$ consecutively being on the P-branch. Therefore, it follows that

$$\frac{\eta^2_{a_0}-(-\eta_T)^2}{\xi_T^2-4\Omega^{*2}}=\frac{\rho_*\beta_k}{\bar{\alpha}-\rho^*\beta_k}+b\,, \tag{4.14}$$

where b is a positive number. For $\bar{U}<\bar{U}_T$ the auxiliary line is in the region between stability zones with a corresponding conjugate pair of complex roots that are destined to become a multiple root $(\xi_T, -\eta_T)$ when $\bar{U}=\bar{U}_T$. In conjunction with these complex roots

$$\frac{\eta^2_{a_0}-\eta^2}{\xi^2-4\Omega^{*2}}=\frac{\eta^2_{a_0}-(-\eta_T)^2}{\xi_T^2-4\Omega^{*2}}-\varepsilon=\frac{\rho^*\beta_k}{\bar{\alpha}-\rho^*\beta_k}+(b-\varepsilon)$$

and according to equation (3.14) the roots are zeros of the P-equation if

$$\mathrm{Re}\,[\varepsilon]\leq b\,.$$

Now, according to the theory on the geometry of the zeros (see MARDEN 1949), the pair of complex roots are always zeros of the parent algebraic system and hence they must be continuous functions of the coefficients of the parent equation. Hence, there must exist a region, immediately before stability zone 2, where $\varepsilon \leq b_T$, and the pair of complex roots belongs to the P-branch which must be the zone of unstable flow. Similarly, there exists another unstable region just after stability zone 2. The degree to which the imaginary part of the complex roots may become large is influenced by the width of the region between stability zones 1 and 2. Therefore, this width or the ξ-coordinate value $(2\beta_k+4\beta_k^2\Omega^{*2})^{1/2}$ is an indication of the rate of growth of the corresponding instabilities. Comparing this ξ-coordinate value to that of the non-rotating case, we see that the value of ξ is increased by an amount $4\Omega^{*2}\beta_k^2$, which means that the width of the unstable region increases as Ω^* increases and hence the effect of Coriolis force is to increase the growth rate of instabilities.

The conditions that the continuity of the roots requires the coefficient of the highest degree term of the parent polynomial be different from zero, is satisfied for all disturbances except those nearly-infinite-length disturbances having the specific wavenumber $\alpha=\rho^*\beta/(1-\rho^{*2})$.

Now, we shall determine the conditions such that $d\eta/d\xi>1$ at the end point of the P-branch in the fourth quadrant, i.e., at $(\xi_s, -\eta_s)$, for the velocity $\bar{U}=\bar{U}_s$ locating the auxiliary line through $(\xi_s, -\eta_s)$. $\bar{U}_s$ is given by

$$\bar{U}_s=\frac{(2\beta_k)^{1/2}}{\bar{\alpha}}+\left(\frac{1-\rho^*+\sigma_k}{\bar{\alpha}}\right)^{1/2}, \tag{4.15}$$

which is the same as that of SONTOWSKI et al. (1969). For the velocity given in equation (4.15), the condition that $d\eta/d\xi>1$ at $(\xi_s, -\eta_s)$ is given by

$$\left[4-3\bar{\alpha}^2+\frac{4\Omega^{*2}\bar{\alpha}^2}{\beta_k^2}\left\{\Omega^{*2}(1+2\beta_k^2-3\beta_k^4)+\frac{8\beta_k^5}{\bar{\alpha}^2}+\beta_k(1+\beta_k^2)-\frac{2\beta_k}{\bar{\alpha}^2}(1+\beta_k^2)\right\}\right]$$
$$>\left[\frac{2\beta_k\bar{\alpha}}{\rho^{*2}}(1-\rho^*+\sigma_k)+\frac{4\beta_k^2}{\rho^*}\left\{\frac{(1-\rho^*+\sigma_k)2\beta_k}{\bar{\alpha}}\right\}^{1/2}\right]\left\{1+\frac{4\Omega^{*2}\bar{\alpha}^2}{\beta_k^2}(\bar{\alpha}^2\Omega^{*2}-\beta_k)\right\}. \tag{4.16}$$

This is a sufficient condition for instability prior to stability zone 2 and these instabilities are referred to as the initial instabilities. However, failure of the condition (4.16) in this range does not necessarily disallow initial instabilities. We note that in the absence of rotation, condition (4.16) reduces to that of SONTOWSKI et al. (1969).

Now, it is important to determine the value of $\bar{U}$, marking the occurrence of the initial instabilities. A readily accessible and accurate approximation for this velocity $\bar{U}$ is the value of $\bar{U}_i$ which locates the auxiliary line through the point $(2\Omega^*, -\eta_{a_0})$ as shown in figure 4. From the equations for η_{a_0} and the auxiliary line it follows that

$$\bar{U}_i=2\Omega^*+\left(\frac{1-\rho^*+\sigma_k}{\bar{\alpha}}\right)^{1/2},$$

which in dimension form is

$$[k_*(U_a-U_b)]_i=\frac{2\Omega}{\alpha}+\left\{\frac{g(1-\rho^*)}{\alpha}+\frac{T_0\alpha}{\rho_b}\right\}^{1/2}. \tag{4.17}$$

For the non-rotating system we have

$$[k_*(U_a-U_b)]_i=\left\{\frac{g(1-\rho^*)}{\alpha}+\frac{T_0\alpha}{\rho_b}\right\}^{1/2}. \tag{4.18}$$

Comparing equations (4.17) and (4.18), we see that the effect of Coriolis force is to delay the initial instabilities in the sense that the initial instabilities will occur for large velocity differences.

Further, we may expect that instabilities also occur after the stability zone 2. This instability is followed at higher U by an unlimited region of unknown stability conditions, which are assumed to be unstable. The above assumption may be justified on the basis of the nature of the results usually obtained in an analytic stability analysis and as well as in the results obtained in experimental investigations. The instabilities following the stability zone 2 will be referred to as the gross instability.

Now, we shall find the value $\bar{U}=\bar{U}_g$ which gives most of the gross intstability. $\bar{U}_g$ may be determined approximately by two methods. It may be approximated excellently by KELVIN's (1910) solution, or another good approximation is that value of U which makes the auxiliary line pass through the point $(\xi_0, 0)$. In view of the mathematical simplicity we follow the latter case. From the equations of ξ_0 and the auxiliary line, we have, for KELVIN's (1910) solution

$$[k^*(\bar{U}_a-\bar{U}_b)]_g=\left[2\Omega^{*2}+\left\{4\Omega^{*4}+\frac{\bar{\alpha}^2}{\rho^{*2}}\left(\frac{1-\rho^*+\sigma_k}{\bar{\alpha}}\right)^2\right\}^{1/2}\right]^{1/2},$$

or in dimensional form

$$[k^*(U_a-U_b)]_g=\frac{1}{(\rho^*)^{1/2}}\left[\frac{2\Omega^2}{\alpha^2}+\left\{\frac{4\Omega^4}{\alpha^4}+\left(\frac{(1-\rho^*)g}{\alpha}+\frac{T_0\alpha}{\rho_b}\right)^2\right\}^{1/2}\right]^{1/2}. \tag{4.19}$$

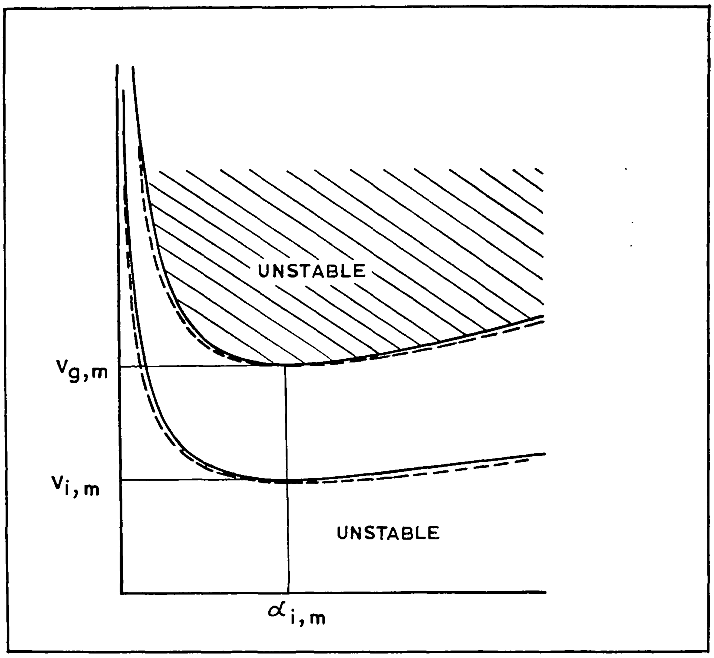

Fig. 5. Typical instabilities for superposition of rotating fluids.

We shall now consider the behaviour of a given fluid system under the influence of a general disturbance. In figure 5, the regions of stability and instability are plotted in a plane of velocity difference $V=[k^*(U_a-U_b)]$ versus wavenumber α.

The curve with dashes represents the non-rotating case. The minimum value of $[k^*(U_a-U_b)]$ for the initial instability is given by

$$[R^*(U_a-U_b)]_{i,m}=\frac{2\Omega}{\alpha_{i,m}}+\left[2\left\{\frac{g(\rho_b-\rho_a)T_0}{\rho_b^2}\right\}^{1/2}\right]^{1/2} \tag{4.20}$$

and the corresponding wavenumber $\alpha_{i,m}$ is

$$\alpha_{i,m}=\left\{\frac{g(\rho_b-\rho_a)}{T_0}\right\}^{1/2} . \tag{4.21}$$

We note that the minimum wavenumber at which the instabilities occur is the same as that of the non-rotating system. For velocities just above the value given by equation (4.21) the system is unstable. At such a point the instabilities are of a weak, selective nature. An additional instability is initiated at a velocity $[k^*(U_a-U_b)]_{g,m}$. This instability is of a much stronger nature. As the disturbance direction rotates toward that of the stationary state velocity, we observe that the tendency to become unstable increases.

Now, we shall find the velocity of propagation γ for the initial waves. For the initial instabilities, from the (ξ,η)-graph it follows that $\eta=-\eta_{a_0}$. By the definition of η and η_{a_0} and letting $\gamma=-\sigma/\alpha$, the velocity of propagation γ is given by

$$\gamma=k^*U_b-\frac{\rho^*\beta_k}{\bar{\alpha}-\rho^*\beta_k}[k^*(U_a-U_b)] + \left\{\frac{\bar{\alpha}\rho^*\beta_k}{(\bar{\alpha}-\rho^*\beta_k)^2}[k^*(U_a-U_b)]^2+\frac{g(1-\rho^*+\sigma_k)}{\alpha-\rho^*\beta/2}-\frac{4\rho^*\beta_k}{\bar{\alpha}-\rho^*\beta_k}\frac{\Omega^2}{\alpha^2}\right\}^{1/2} , \tag{4.22}$$

which reduces to that of SONTOWSKI et al. (1969) in the non-rotating case. From equation (4.22) it follows that the effect of rotation is to reduce the velocity of propagation. For ρ^*, β, and Ω small, it follows that

$$\gamma\cong k^*U_b+\left(\frac{g\lambda}{2\pi}+\frac{2\pi T_0}{\lambda\rho_b}\right)^{1/2} , \tag{4.23}$$

which is the classical result for the propagation speed of surface waves on a fluid of infinite depth. It is also of interest to note the relationship of this propagation speed with that of the accompanying wind velocity. From the definition of ξ and for initial instabilities $\xi=2\Omega/\alpha$ it follows that

$$\gamma=k^*U_b+[k^*(U_a-U_b)]_i-\frac{2\Omega}{\alpha} . \tag{4.24}$$

Thus, the effect of rotation is such that the wind velocity is larger than the wave velocity, whereas for the non-rotating system wind and wave travel together.

Since the degree to which the imaginary part of the complex roots may become large is influenced by the width of the region between the possible stability zones, it is of interest to study the (ξ,η) locus for different values of β_k and Ω^*. The stability zones are marked in the (ξ,η)-plane for $\beta_k=0.5$, $\Omega^{*2}=0.25$, and $\beta_k=0.5$, $\Omega^{*2}=0.00$ in figures 4 and 6 respectively. Comparing the two figures, we observe that in the non-rotating system, the possible stability zone-1 is confined to a very small region, thereby the width of the initial instability is more. Hence,

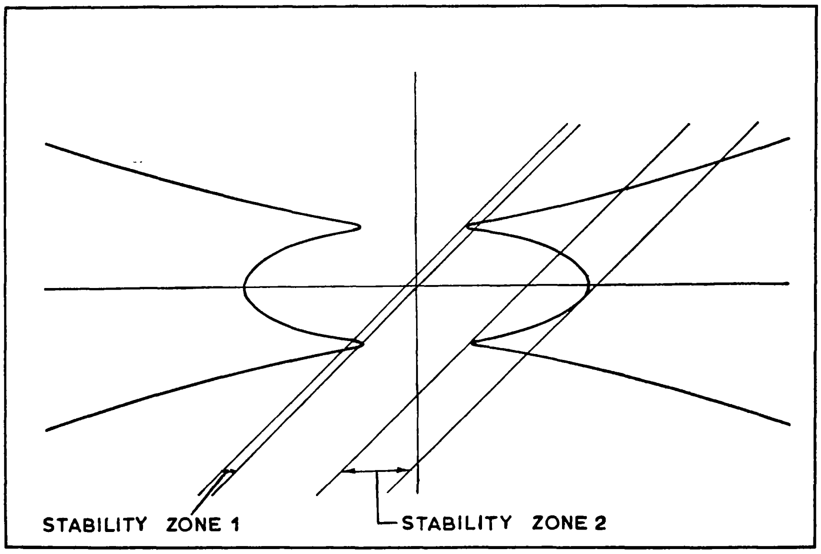

Fig. 6. Stability zones in the real (ξ, η)-plane for $\Omega^{*2}=0.0$ $(\beta_k=0.5)$.

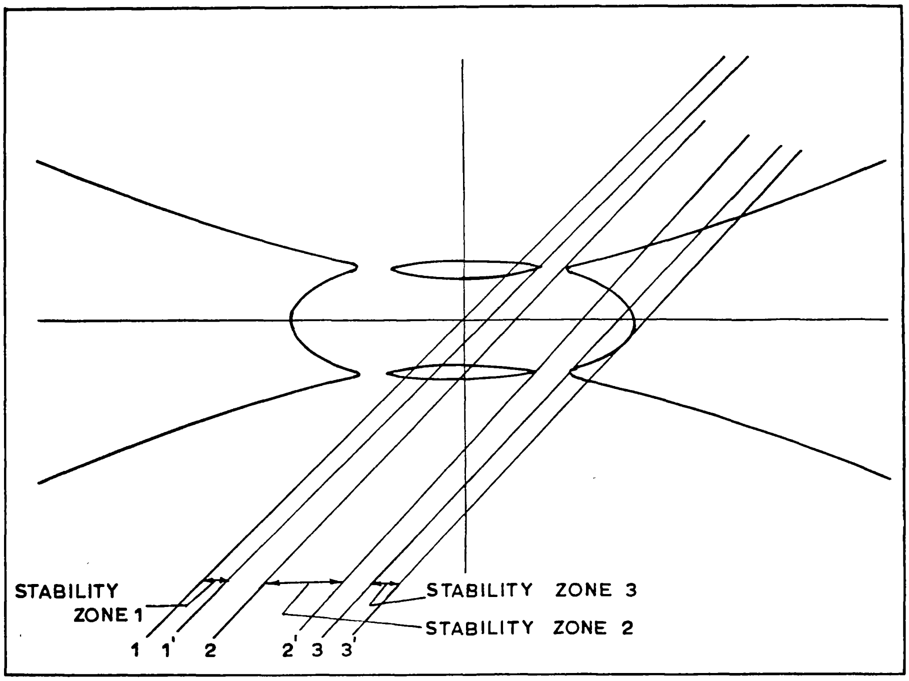

Fig. 7. Stability zones in the real (ξ, η)-plane for $\beta_k=0.5$, $\Omega^{*2}=1.0$.

the effect of rotation is to reduce the initial instability zone. In figure 7, the stability zones are marked for $\beta_k=0.5$ and $\Omega^{*2}=1.0$. As $\bar{U}$ alone increases, in this case, the fluid system experiences consecutively conditions of definite stability (stability zone 1), possible instability-1, definite stability (stability zone 2), possible instability-2, definite stability (stability zone 3), and finally possible instability (to the right of stability zone 3).

5. *KELVIN's (1910) Solution in the Rotating System*

The above general theory is applied to the particular problem of $\beta=0$. From the real (ξ, η)-locus of the basic branch equations, when $\beta=0$, we have

$$\xi_0{}^2=0\ ,\quad \eta_s{}^2=\eta^2_{a_0}\ ,\quad \eta^2_{a_0}=1-\rho^*+\sigma_k\ .$$

The most significant of these results is the location of the point $([2\beta_k+4\beta_k{}^2\Omega^{*2}]^{1/2},\ \eta_{a0})$ on the η-axis. The stability zones for this case are shown in figure 8. Comparing figures 4 and 8, we observe that the stability zone-1 is reduced when $\beta_k=0.0$, and thereby the initial instability is increased. On the basis of the (ξ,η)-locus plotted in figures 4, 6, and 8, we can also conclude that the combined effect of density stratification and rotation is to reduce the initial instabilities and hence to increase the stability zone. When both β_k and Ω^* are zero, the stability zones are plotted in figure 9 and we see that there are no initial instabilities.

Further, when $\beta=0$ the branch equations are independent of basic steady state velocities. The stability bound is obtained by determining the point along the P-branch where $d\eta/d\xi=1$, passing the auxiliary line through the point, and then solving the resulting equation for the velocity differences. This velocity is given by

$$(U_a-U_b)^2=\frac{\rho_b}{\rho_a}\left[\frac{2\Omega^2}{\alpha^2}+\left\{\frac{4\Omega^4}{\alpha^4}+\left(\frac{(1-\rho^*)g}{\alpha}+\frac{T_0\alpha}{\rho_b}\right)^2\right\}^{1/2}\right]^{1/2}, \tag{5.1}$$

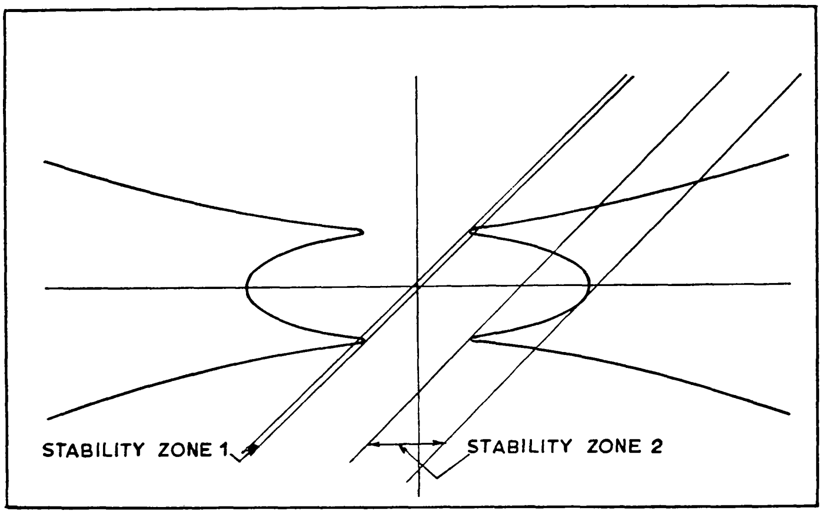

Fig. 8. Stability zones in the real (ξ,η)-plane for $\beta_k=0.0$ $(\Omega^{*2}=0.25)$.

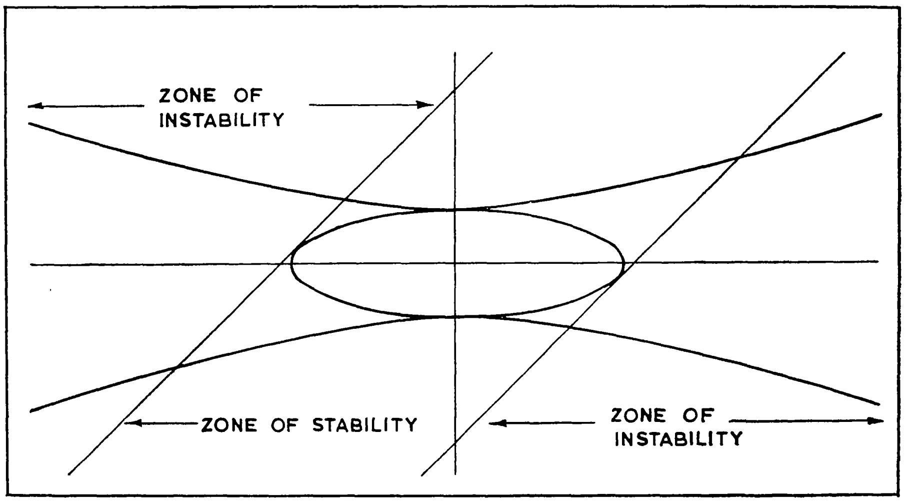

Fig. 9. Stability zones in the real (ξ,η)-plane for $\beta_k=0.0$, $\Omega^{*2}=0.0$.

which is the critical velocity for the case when the upper fluid is rotating and the lower one is non-rotating. This is similar to the critical velocity obtained by CHANDRASEKHAR (1961, p. 507).

We note that some indication of the rate of growth of initial instabilities is given by the value of $(2\beta_k)^{1/2}(1+2\Omega^{*2}\beta_k^{2})^{1/2}$. Therefore, for the initial instability it is expected that

$$\mathrm{Im}\,\sigma \approx (\alpha g)^{1/2}\,\mathrm{Im}\,\xi \sim (\alpha g)^{1/2}(2\beta_k)^{1/2}(1+2\Omega^{*2}\beta_k)^{1/2}\ ,$$

and as the imaginary part of σ characterizes the growth rate, it is indicated that for rotating gas blowing over a fluid the growth rate increases by an amount of $(2\beta_k)^{1/2}\Omega^*$. This is in agreement with the experimental observations of MUNK (1947).

6. *Conclusions*

The effect of Coriolis force on the stability of two superposed fluids is investigated. It is found that the effect of rotation is to stabilize the flow in the sense that there is no initial instability, but there exists only gross instability. In particular, in the case of stratified rotating gas flowing over non-rotating fluid there exist two types of instabilities, one relatively weak instability called the initial instability and other, the stronger classical instability of KELVIN (1910) called gross instability as observed by SONTOWSKI et al. (1969) in the absence of rotation. However, the analysis indicates that the effect of Coriolis force is to delay the onset of initial instabilities in the sense that the initial instabilities will occur for large velocity differences. Comparing our results with those of SONTOWSKI et al. (1969), we conclude that the effect of rotation is to reduce the velocity of propagation. We also found that the combined effect of density stratification and rotation is to reduce the initial instabilities and hence to increase the stability zone.

It is also of interest to compare our analysis with that of CHANDRASEKHAR (1961, p. 498) who investigated the Kelvin-Helmholtz instability in a rotating fluid in the absence of stratification. CHANDRASEKHAR's (1961, p. 503) solution to the eigenvalue equation is based on the exchange of roots between the branches which necessitates determining the singular points of the eigenvalue equation. Later on, HUPPERT (1968) has shown that CHANDRASEKHAR's (1961) solution is incorrect and HUPPERT (1968) has enumerated the unstable modes correctly with the aid of Cauchy's principle of the argument. However, in the present analysis, although we use CHANDRASEKHAR's (1961, p. 501) graphical analysis, the method of finding the roots of the eigenvalue equation is quite different from that of CHANDRASEKHAR (1961, p. 501). Instead of using the concept of exchange of roots, we determine the number of real roots of the eigenvalue equation based on the number of intersections of the auxiliary line with the basic branch equations. This method does not involve the process of determining the singular points and hence gives the correct instability modes.

Regarding the nature of the initial instability, the following physical mechanism is suggested.

In the absence of Coriolis force, the density stratification gives rise to internal waves (see BOOKER and BRETHERTON 1967) whereas, in the absence of density stratification, the Coriolis force gives rise to the inertial waves (see JONES 1967; BRETHERTON 1969). However, the combined effect of density stratification and Coriolis force is to give rise to inertial-internal waves (see RUDRAIAH and VENKATACHALAPPA 1972a). Hence an interaction occurs between the internal-inertial waves in the rotating, stratified upper fluid and capillary-gravity waves in the lower one. The wave length and phase speed of the two waves are such that they are less than those in the absence of Coriolis force, and the waves are slowly amplified. This behaviour is indicated by the eigenvalue calculation which shows the propagation speed and wave length to be related as in the classical form of capillary-gravity waves. Since the Coriolis force and density stratification are weak and the inertial-internal wave travels very slowly relative to the upper fluid, and because the rotating gas and waves travel together, the inertial-internal wave can match up and resonate with the capillary-gravity wave. This resonance leads to an enhanced instability.

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References

ACHESON, D. J. 1972, *J. Fluid Mech.*, **52**, 529.
ACHESON, D. J., and HIDE, R. 1973, *Rept. Prog. Phys.*, **36**, 159.
BOOKER, J. R., and BRETHERTON, F. P. 1967, *J. Fluid Mech.*, **27**, 513.
BRETHERTON, F. P. 1969, *Quart. J. Roy. Met. Soc.*, **95**, 313.
BRETHERTON, F. P., CARRIER, G. F., and LONGUIET HIGGINS, M. S. 1966, *J. Fluid Mech.*, **26**, 393.
CHANDRASEKHAR, S. 1961, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, London).
GERBIER, N., and GERENGER, M. 1961, *Quart. J. Roy. Met. Soc.*, **87**, 13.
HINES, C. O. 1960, *Can. J. Phys.*, **38**, 1441.
HUPPERT, H. E. 1968, *J. Fluid Mech.*, **33**, 353.
JONES, W. L. 1967, *J. Fluid Mech.*, **30**, 430.
KATO, S., and UNNO, W. 1960, *Publ. Astron. Soc. Japan*, **12**, 127.
KAUO, H. L. 1961, *Tellus*, **13**, 441.
KELVIN, L. 1910, *Mathematical and Physical Papers, IV, Hydrodynamics and General Dynamics* (Cambridge), p. 76.
MARDEN, M. 1949, *Math. Surveys, No. 3* (American Mathematical Society, Providence).
MUNK, W. H. 1947, *J. Marine Res.*, **6**, 203.
RUDRAIAH, N., NARASIMHAMURTHY, S., and MARIYAPPA, B. V. 1972, *Appl. Sci. Res.*, **26**, 285.
RUDRAIAH, N., and VENKATACHALAPPA, M. 1972a, *J. Fluid Mech.*, **52**, 193.
RUDRAIAH, N., and VENKATACHALAPPA, M. 1972b, *J. Fluid Mech.*, **54**, 209.
RUDRAIAH, N., and VENKATACHALAPPA, M. 1972c, *J. Fluid Mech.*, **54**, 217.
RUDRAIAH, N., and VENKATACHALAPPA, M. 1974a, *J. Fluid Mech.*, **62**, 705.
RUDRAIAH, N., and VENKATACHALAPPA, M. 1974b, submitted to *Quart. J. Mech. Appl. Math.*
SONTOWSKI, J. F., SEIDEL, B. S., and AMES, F. W. 1969, *Quart. Appl. Math.*, **27**, 325.
SPIEGEL, E. A., and UNNO, W. 1962, *Publ. Astron. Soc. Japan*, **14**, 28.
UNNO, W. 1957, *Astrophys. J.*, **126**, 259.
UNNO, W., KATO, S., and MAKITA, M. 1960, *Publ. Astron. Soc. Japan*, **12**, 192.