# UNIVERSAL STABILITY OF THE LAMINAR DISPERSION OF SOLUTE IN A POROUS MEDIUM

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ABSTRACT

A universal stability limit for the laminar dispersion of a solute in a porous medium is obtained in terms of a Reynolds number and a Rayleigh number. This stability limit is improved using the techniques of calculus of variation. By increasing the Reynolds number $R_e$ towards unity, we find that the stability region is decreased by decreasing the Rayleigh number $R_a$ for the solvent.

## 1. INTRODUCTION

THE effects of buoyancy forces caused by a density difference between the solute and the solvent, when the solute is injected into the solvent through a porous medium, on the stability of concentrated flow is of particular interest in dye industry. In such circumstances to study the stability, one has to take into account the natural convection within the diffusion cell. In practice, such a convection would arise from either a thermal gradient, a solute gradient or a combination of these. In the present problem we are concerned only with the convection due to a solute gradient. The physical model we consider is that the solute is injected into the solvent, where the density of the solute is different from that of the solvent, so that, the buoyancy forces associated with the density gradients set up the secondary flows and the solute gives rise to definite dynamical effects which will be absent in the case of no buoyancy effect (*i.e.*, the solute and the solvent have the same density). Recently Emim Erdogan and Chatwin (1967) have studied the steady flow of a solvent in a tube and have correlated their theory with the experimental work of Reejhsinghani *et al.* (1966). These theoretical and experimental works were based on the nature of the steady flow and the stability of that flow has not been given much attention. The purpose of this paper is to study the universal stability, namely, a stability for motions subject to arbitrary non-linear disturbances of the flow of solvent in a porous medium into which a solute of different density is injected.

## 2. UNIVERSAL STABILITY ANALYSIS

We assume that the flow of solvent and the concentration factor C, in a bounded region V (t) with the boundary S, are governed by the following system of equations:

Darcy's law:

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla p - \rho g/k\,\vec{q} + \rho\vec{g}. \tag{1}$$

Fick's law:

$$\frac{\partial C}{\partial t} + (\vec{q}\cdot\nabla)\,C = D\nabla^2 C. \tag{2}$$

*Equation of continuity*:

$$\frac{\partial \rho}{\partial t} + (\vec{q}\cdot\nabla)\,\rho + \rho\,\nabla\cdot\vec{q} = 0 \tag{3}$$

where $\vec{q}$ is the fluid velocity, $\rho$ is the density, p is the pressure, $k$ is the permeability of the porous medium, $g$ is the acceleration due to gravity and D is the diffusion coefficient. The boundary conditions are:

$$\vec{q}\cdot\hat{n} = 0;\quad \frac{\partial C}{\partial n} + \sigma C = 0 \text{ on S} \tag{4}$$

where $\sigma$ is a continuous function of position and $\hat{n}$ is the outward unit normal vector.

The equations (1) to (3) under the Boussinesq approximation, *i.e.*, the density $\rho$ is replaced by a constant $\rho_0$ everywhere except in the last term of equation (1) and using $\rho = \rho_0[1 + \alpha C]$ where $\alpha$ is the coefficient of volume expansion, become

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} = -\frac{1}{\rho_0}\nabla p - \frac{g}{k}\vec{q} + [1 + \alpha C]\,\vec{g} \tag{1 a}$$

$$\frac{\partial C}{\partial t} + (\vec{q}\cdot\nabla)\,C = D\nabla^2 C \tag{2 a}$$

$$\nabla\cdot\vec{q} = 0. \tag{3 a}$$

The basic motion $(\vec{q}, C, p)$ at time $t = 0$, is altered to $\vec{q}^* = \vec{q} + \vec{u}$; $C^* = C + \theta$; $p^* = p + P$, where both the basic and altered flows satisfy the same basic equations (1) to

(3) and the same boundary conditions (4). Our aim is to determine whether the altered flow will approach the basic flow as $t \to \infty$. This we determine by using the energy principle as developed by Joseph (1966). Obtaining the difference equations by subtracting the equations for the basic flow from those of the altered flow and multiplying the resulting Darcy's law and Fick's law scalarly by $\vec{u}$ and $\theta$ respectively we obtain the energy equations.

$$\frac{dK_1}{dt} + \frac{2g}{k} K_1 = - \langle \vec{u}.\vec{\vec{E}}.\vec{u} - \alpha \vec{g}.u\theta \rangle \quad (5)$$

$$\frac{dK_2}{dt} = - \langle \vec{u}.\nabla C\theta + D\nabla\theta.\nabla\theta \rangle - D\langle \sigma\theta^2 \rangle' \quad (6)$$

where $\vec{E}$ is the basic strain rate tensor,

$K_1 = \langle \frac{1}{2} u^2 \rangle$ is the Kinetic energy,

$K_2 = \langle \frac{1}{2} \theta^2 \rangle$ is the concentration energy.

$\langle$ denotes the volume integral over V and $\langle \rangle'$ denotes the surface integral over S.

Defining $-m$ as the least eigenvalue of E, $\beta =$ max. $|\nabla C|$, applying the Schwartz's inequality and using the conditions

$$\vec{u} \cdot \hat{n} = 0; \frac{\partial\theta}{\partial n} + \sigma\theta = 0; \nabla \cdot \vec{u} = 0 \quad (7)$$

we obtain

$$\frac{1}{g\alpha} \frac{d\hat{K}_1}{dt} + \frac{(1 - Re)}{k\alpha} \hat{K}_1 - \hat{K}_2 \leqslant 0 \quad (8)$$

$$\frac{1}{\beta} \frac{d\hat{K}_2}{dt} + \frac{\Sigma D}{\beta d^2} \hat{K}_2 - \hat{K}_1 \leqslant 0 \quad (9)$$

where

$$K_1 = \hat{K}_1^2; K_2 = \hat{K}_2^2; Re = \frac{mk}{g} \text{ and } \Sigma$$

is such that

$$\langle \nabla\theta \cdot \nabla\theta \rangle + \langle \sigma\theta^2 \rangle' \geqslant \frac{2\Sigma}{d^2} K_2.$$

We note that

$$X = \left(\mu + \frac{\Sigma D}{d^2}\right) \exp.(-\mu t)$$

and

$$Y = \beta \exp.(-\mu t)$$

where

$$\mu = -\frac{1}{2}\left[\frac{\Sigma D}{d^2} + \frac{g}{k}(1 - Re)\right] + \frac{1}{2}\left[\left\{\frac{\Sigma D}{d^2} + \frac{g}{k}(1 - Re)\right\}^2 - \frac{4Dg}{d^2 k}\{\Sigma(1 - Re) - Ra\}\right]^{\frac{1}{2}} \quad (10)$$

$R_a = \alpha\beta d^2 k/D$ is the Rayleigh number for the solvent, satisfy the differential equations

$$-\frac{1}{g\alpha}\frac{dX}{dt} + \frac{(1 - Re)}{k\alpha} X - Y = 0 \quad (11)$$

$$-\frac{1}{\beta}\frac{dy}{dt} + \frac{\Sigma D}{\beta d^2} Y - X = 0 \quad (12)$$

and X and Y are both positive.

Multiplying equation (8) by X and (11) by $\hat{K}_1$ and then subtracting, we obtain

$$\frac{1}{g\alpha}\frac{d}{dt}(\hat{K}_1 X) - \hat{K}_2 X + \hat{K}_1 Y \leqslant 0. \quad (13)$$

By a similar process from equations (9) and (12) we obtain

$$\frac{1}{\beta}\frac{d}{dt}(\hat{K}_2 Y) - \hat{K}_1 Y + \hat{K}_2 X \leqslant 0. \quad (14)$$

Adding (13) and (14) and integrating it from 0 to $t$, we get

$$\left(\mu + \frac{\Sigma D}{d^2}\right) K_1^{\frac{1}{2}} + g\alpha (K_2)^{\frac{1}{2}} \leqslant \left[\left(\mu + \frac{\Sigma D}{d^2}\right)(K_{10})^{\frac{1}{2}} + g\alpha (K_{20})^{\frac{1}{2}}\right] \exp.(\mu t)$$

where $(\mu + \Sigma D/d^2) > 0$, $K_{10}$ and $K_{20}$ are the initial values of $K_1$ and $K_2$. $K_1 \to 0$ and $K_2 \to 0$ as $t \to \infty$ if $\mu < 0$. From equation (10) it follows that $\mu < 0$ if R $< \Sigma$ (1 − R ). Hence the condition for universal stability is

$$0 \leqslant Ra \leqslant \Sigma(1 - Re). \quad (15)$$

The number $\Sigma$ will depend on the geometrical configuration, *i.e.*, $\Sigma = 3$ for a spherical geometry, $\Sigma = 1$ for a channel flow.

The limit (15) can be further improved using the techniques of the calculus of variation. For this, equations (5) and (6) are re-written in the form

$$\frac{d}{d\tau}(K_1 + \lambda Pr K_2) = -[Ra^{\frac{1}{2}}(I_1 + \lambda I_2) + (I_3 + \lambda I_4)] \quad (16)$$

where

$$I_1 = \langle \mu_1 \vec{v}.\vec{F}.\vec{v} + \hat{k}.\vec{v}\theta \rangle; \quad I_2 = \langle \vec{v}.\nabla\psi\theta \rangle;$$

$$I_3 = \langle v^2 \rangle; \quad I_4 = \langle \nabla\theta.\nabla\theta \rangle + \langle \sigma\theta^2 \rangle'.$$

$$\left(\frac{\alpha Dk}{\beta d^2}\right)^{\frac{1}{2}} \vec{v} = \vec{u}; \quad m\vec{F} = \vec{E}; \quad \beta d\psi = C; \quad \tau = \frac{gt}{k};$$

$$\vec{g} = -g\hat{k}; \mu_1 = \frac{Re}{\sqrt{Ra}}; Pr = \frac{gd^2}{kD}.$$

The motion is stable if $d/d\tau\ (K_1 + \lambda p\ K_2) < 0$, which is possible only when

$$Ra^{\frac{1}{2}}(I_1 + \lambda I_2) + (I_3 + \lambda I_4) \geqslant 0. \quad (17)$$

The problem now is the following: First we consider $\lambda > 0$ as given and look for the smallest value of $R_a^{\frac{1}{2}}$ for which (17) holds. We will call this $\bar{R}_\lambda(\mu_1)$. Next we seek the value of $\lambda$ for which $\bar{R}_\lambda(\mu_1)$ is greatest and we call this $R(\mu_1)$. We see that for a fixed $\mu_1$, if $R_a^{\frac{1}{2}} < R(\mu_1)$ we will have stability. By changing $\mu_1$ $[0 \leqslant \mu_1 (\infty)]$ one can establish an open region of certain stability near the origin of the $(R_a^{\frac{1}{2}}, R_\theta)$ cartesian plane. This can be achieved by discussing the maximum problem for a fixed $\mu_1$ and $\lambda$ in the form

$$-(I_1 + \lambda I_2) = \text{Max.}$$

with the normalizing condition $I_3 + \lambda I_4 = 1$.

By introducing the Lagrangian multipliers $1/R_\lambda$ and $P(r, t)/R_\lambda$ we obtain the Euler-Lagrange equations

$$\vec{v} + \mu_1 R_\lambda \vec{F} . \vec{v} + \frac{R_\lambda}{2} (\hat{k} + \lambda \nabla \psi) \theta = -\nabla P \quad (18)$$

$$\frac{R_\lambda}{2\lambda} (\hat{k} + \lambda \nabla \psi) . \vec{v} = \nabla^2 \theta. \quad (19)$$

Solving these equations we get the required eigenvalue. It may be seen (comparing with Westbrook, 1969) that the variational problem has a solution and that

$$\text{max.} (-I_1 - \lambda I_2) = \frac{1}{\bar{R}_\lambda}$$

where $\bar{R}_\lambda$ is the greatest lower bound of $R_\lambda^{i}$ and $R_\lambda^{i}$ are the eigenvalues of (18) and (19).

The motion is stable if $R_a^{\frac{1}{2}} < \bar{R}_\lambda$.


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