FLOW BETWEEN ECCENTRIC SPHERES

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THE flow of a viscous incompressible fluid between two eccentric cylinders has been investigated by Rudraiah and Geetha (1969) whose results are of interest in the liquid bearing systems. The flow of a viscous incompressible fluid between eccentric spheres is not only of interest in the lubrication theory but also is of interest in the geophysical problems, particularly in the discussion of fluid in the earth core. This problem has not been given much attention earlier and it is discussed in this paper. The main difference between Rudraiah and Geetha problem and our problem is that in their problem the bearing system has a line contact, whereas in our problem the bearing system has a point contact. As a consequence of this point contact we get side leakage of liquid about the point of contact which develops the adverse pressure gradient. The physical reason for this is as follows. When one of the spheres gets displaced to an eccentric position, new effects appear. First, a pressure gradient is created which tends to restore concentricity and it may cause the back flow. A second, an eddy adjacent to the outer sphere, is created in the region of largest clearance. To study these effects, the steady flow of a viscous incompressible fluid between two eccentric spheres, when the inner sphere of radius R_1 is rotating while the outer sphere of radius R_2 is stationary is investigated under the narrow gap approximation. The flow in the axial direction is assumed to be small compared with the flow in the radial and azimuthal directions.

Let (α, β, γ) be the spherical bipolar co-ordinates. The transformation from the cartesian co-ordinates (x, y, z) to the spherical bipolar system (α, β, γ) is given by

$$x = \frac{a \sinh \alpha}{\cosh \alpha - \cos \beta}, \quad y = \frac{a \sin \beta \cos \gamma}{\cosh \alpha - \cos \beta},$$
$$z = \frac{a \sin \beta \sin \gamma}{\cosh \alpha - \cos \beta} \tag{1}$$

and the corresponding Lames constants (h_1, h_2, h_3) are given by

$$h_1 = h_2 = \frac{a}{\cosh \alpha - \cos \beta},$$
$$h_3 = \frac{a \sin \beta}{\cosh \alpha - \cos \beta}. \tag{2}$$

If ψ and ϵ are respectively the clearance and eccentricity ratios, it can be shown, since

$$R_1 = -\frac{a}{\sinh \alpha_1}, \quad R_2 = -\frac{a}{\sinh \alpha_2}$$

that

$$\psi = \frac{R_2 - R_1}{R_2} = \frac{C}{R_2} = \frac{\sinh \alpha_1 - \sinh \alpha_2}{\sinh \alpha_1} \tag{3}$$
$$\epsilon = \frac{\sinh(\alpha_1 - \alpha_2)}{\sinh \alpha_1 - \sinh \alpha_2}. \tag{4}$$

The required equations, under the above approximations, following the analysis of Rudraiah and Geetha (1969) and using the basic equations derived by Rudraiah and Janakamma (1970), are

$$\frac{\partial p}{\partial \zeta} = 0 \tag{5}$$
$$\frac{\partial^2 v}{\partial \zeta^2} + H\psi(1-\epsilon^2)^{\frac{1}{2}} \frac{\partial v}{\partial \zeta} = RH\psi(1 - \epsilon^2) \frac{\partial p}{\partial \beta} \tag{6}$$
$$H^2 \sin \beta \frac{\partial u}{\zeta \partial} + \frac{\partial}{\partial \beta}(H^2 v) = 0 \tag{7}$$

where u and v are dimensionless velocity components in α and β directions respectively, p is the pressure,

$$\zeta = \frac{\alpha - \alpha_1}{t}, \quad t = \psi(1 - \epsilon^2)^{1}.$$
$$H = \frac{(1 - \epsilon^2)^{\frac{1}{2}}}{1 - \epsilon \cos \beta}, \quad R = \frac{V_0 C}{\nu}$$

and V_0 is the rotation of the inner sphere. The boundary conditions are

$$u = 0 \quad \text{at} \quad \zeta = 0, 1 \tag{8}$$
$$v = 1 \quad \text{at} \quad \zeta = 0, \tag{9}$$
$$v = 0 \quad \text{at} \quad \zeta = 1 \tag{10}$$

and u, v, p are periodic in β. The solutions of (5) to (7) using the boundary conditions (8) to (10) are:

$$u = K_1\left(\frac{\zeta^2}{2} - \frac{\zeta^3}{3}\right) + K_2\left(K_3\zeta^2 - \frac{K_4}{3}\zeta^3\right) + K_5\left(\frac{\zeta^2}{2} - \zeta\right)\left\{(1 - \epsilon\cos\beta)\left(2 - \frac{\psi(1-\epsilon^2)}{2a_4}\right)\right\}$$
$$+ K_5\left(\zeta - \frac{\zeta^3}{3}\right)\left(\frac{3}{2}\psi(1-\epsilon^2) - \frac{\psi^2(1-\epsilon^2)^2}{4a_4}\right) \tag{11}$$

$$v = K_6(\zeta - \zeta^2) + \frac{1}{a_4}\left\{\zeta(1 - \epsilon\cos\beta) - \frac{\psi(1-\epsilon^2)}{2}\zeta^2 + a_4\right\} \tag{12}$$

$$p - p_0 = \frac{1}{R\psi(1-\epsilon^2)^{3/2}}\left[a_1\epsilon\sin\beta + a_2\epsilon^2\sin 2\beta + a_3\epsilon^3\sin^3\beta\right] \tag{13}$$

where p_0 is the static pressure.

$$a_1 = \left\{\frac{4(1 - 2K + \epsilon^2)(1 + 4K)}{[2K + \epsilon^2(K-2)]} - 6\right\}$$

$$a_2 = \left\{\frac{(K-2)(1+4K)}{[2K + \epsilon^2(K-2)]}\right\},$$

$$a_3 = \left\{-\frac{4}{3}\frac{(1+4K}{[2K + \epsilon^2(K-2)]}\right\},$$

$$a_4 = (K + \epsilon\cos\beta),$$

$$2K = [\psi(1-\epsilon^2) - 2],$$

$$K_1 = \left\{\frac{a_1\epsilon + 8a_2\epsilon^2\cos\beta - \epsilon^3(\cos^2\beta + \cos 2\beta)}{2a_1}\right\},$$

$$K_2 = \frac{\epsilon}{1 - \epsilon \cos\beta}\left\{a_1 \epsilon \cos\beta + 2a_2\epsilon^2 \cos 2\beta + \frac{3}{2}\sin\beta \sin 2\beta\right\},$$

$$K_3 = \left\{\frac{1}{\psi(1-\epsilon^2)} + \frac{2(1-\epsilon\cos\beta)}{a_1\psi(1-\epsilon^2)} - \frac{1-\epsilon\cos\beta}{2a_1^2}\right\},$$

$$K_4 = \left\{\frac{4a_1 - 2(1-\epsilon\cos\beta)}{4a_1^2}\right\},$$

$$K_5 = \frac{\epsilon}{a_1(1-\epsilon\cos\beta)},$$

$$K_6 = \left\{\frac{2a_1\epsilon\cos\beta + 4a_2\epsilon^2\cos 2\beta + 3\epsilon^3\sin\beta\sin 2\beta}{4a_1}\right\}.$$

The shearing stress τ is defined by the equation

$$\tau = \frac{\mu V_0}{a}\left[\frac{\partial}{\partial\alpha}(v\cosh\alpha - v\cos\beta) + t\frac{\partial}{\partial\beta}(u\cosh\alpha - u\cos\beta)\right]$$

The expression for skin-friction coefficient

$$c_f = \frac{\tau}{\frac{\mu V_0}{a}}$$

is

$$c_f = \frac{\partial}{\partial\alpha}(v\cosh\alpha - v\cos\beta) + t\frac{\partial}{\partial\beta}(u\cosh\alpha - u\cos\beta).$$

From equation (14), using equations (11) and (12), we can obtain skin friction coefficient at the inner and outer spheres.

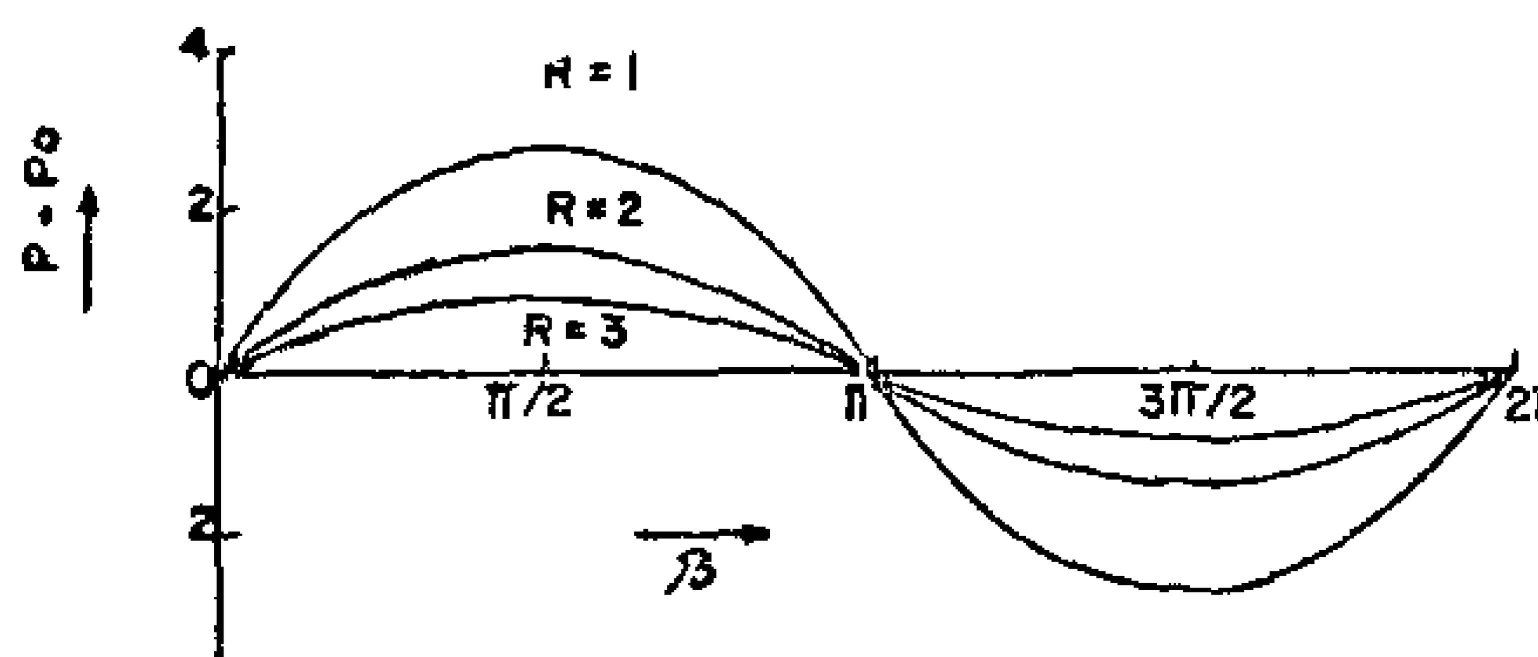

FIG: I PRESSURE DISTRIBUTION

The velocity and pressure distributions are plotted in Figs. 1 and 2. The velocity distribution shows that the velocity components in the radial and axial directions are much smaller than the azimuthal direction and the azimuthal velocity distribution exhibits the back flow for $\beta = 0$ and $\beta = \pi/4$. The reason for this back flow is mainly because of the adverse pressure gradient set up by the point leakage in the spherical bearing system. We find that the pressure decreases with increasing Reynolds number as shown in Fig. 2.

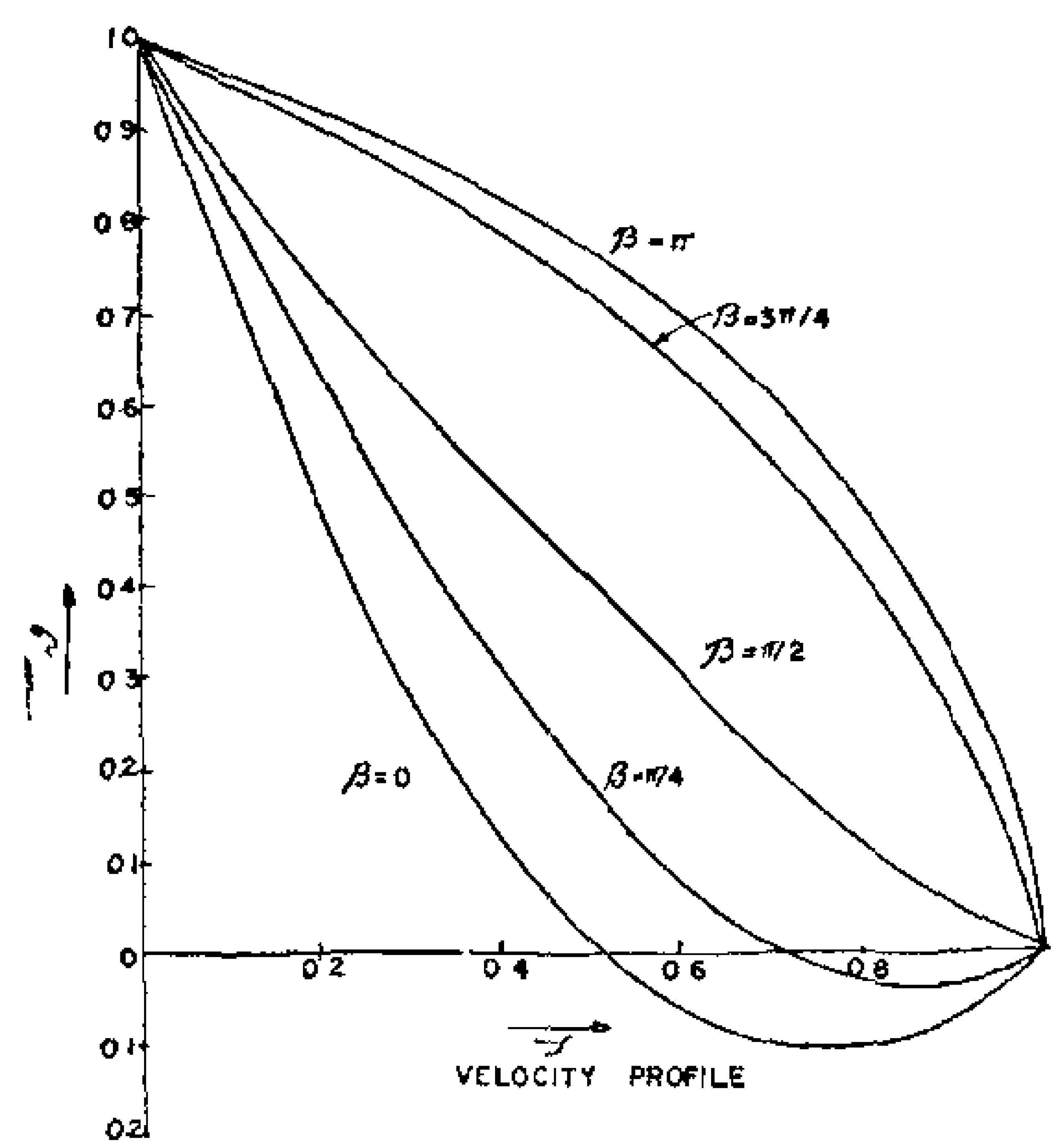

FIG 2

ACKNOWLEDGEMENT

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