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A CLASS OF GLOBALLY UNIVALENT DIFFERENTIABLE MAPPINGS

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Dedicated to Academician Otakar Borůvka on his ninetieth birthday

Abstract. In this paper we prove global univalence for C^1 maps in R^n when the Jacobian matrix has its determinant of sign opposite to that of all its principal minors.

Key words. Univalence, Jacobian matrix, almost P-matrix, KKM theorem.

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INTRODUCTION

It is well known that non-vanishing of the Jacobian does not necessarily imply univalence of the mapping. Thus placing additional conditions on the Jacobian matrix to obtain global one-to-oneness has been a problem of interest for a long time. D. Gale and H. Nikaido in the celebrated paper [2] proved global univalence for maps F from a rectangular region from R^n into R^n for which the Jacobian matrix $F'(x)$ as a P-matrix for each x in the region; that is when all principal minors of the Jacobian matrix $F'(x)$ are positive for each x in the region. In the same paper they stated the following

Problem. Let F: $\Omega \to R^n$, where Ω is a rectangular region of R^n, be continuously differentiable and assume the Jacobian matrix $F'(x)$ has all principal minors non-vanishing on Ω. Is F globally one-to-one?

We present in this paper a contribution to the above problem in the sense that we are establishing univalence for two classes of differentiable maps F for which the Jacobian matrix $F'(x)$ is almost P (almost N) everywhere; that is for every x in the domain of F all principal minors of $F'(x)$ are positive (negative) except the determinant which is negative (positive). Our theorem extends a result of G. Ravindran [13] in which he proved the same but for $n = 3$. In fact he solved the Problem for $n = 3$ there, since in this dimension the non-vanishing of principal minors reduces essentially to three different cases: for each x the Jacobian matrix $F'(x)$ is either a P-matrix or an N-matrix or almost P. A square matrix is called a P-matrix (an N-matrix) if all its principal minors are positive (negative).

For other univalence results when the non-vanishing of principal minors is assumed see K. Inada [5], A. Mas-Colell [7], G. B. Garcia and W. I. Zangwill [3] and also T. Parthasarathy and G. Ravindran [11]. For $n = 2$ see C. Olech [9] and also a recent paper by G. H. Meisters and C. Olech [8].

In the next section we give a slightly modified version of the Gale–Nikaido result (see [2], Theorem 3) which we will use to prove our result in almost P case. We prove there also a property of almost P-matrices which is essential for our theorem. Another tool we use is a new version of KKM theorem which may be of interest by itself. This is Lemma 1 and will be given in section 3. Section 4 containes the proof of the result for the almost P case, while in section 5 we discuss the almost N case.

2. A MODIFICATION OF GALE–NIKAIDO THEOREM

If A is a P-matrix then Ax is non decreasing with respect to the order induced by R^n_+; that is if $x \leqq y$ and $Ax \geqq Ay$ then $x = y$ or equivalently if $x \leqq y$ and x different from y then $Ay - Ax$ does not belong to R^n_-. If Ax is non-decreasing with respect to each order induced by an orthant of R^n then A is a P-matrix. The main observation of Gale and Nikaido was that this property holds true for non-linear maps also; that is if all principal minors of $F'(x)$ are positive for each x then for any order induced by an orthant of R^n the inequalities $a \leqq b$ and $F(a) \geqq F(b)$ can hold simultaneously only if $a = b$. Compare [2], Theorem 3. If we fix the order; that is if we fix an orthant in R^n then we can relax slightly the assumptions. We have the following modification of the above result:

Theorem 1 (Gale–Nikaido). *Assume F is differentiable, the determinant of $F'(x)$ is different from zero for each $x \in \Omega = \{x: a \leqq x \leqq b\}$ and that*

(1) *for each $x \in \Omega$ there is $v > 0$ such that $F'(x)\, v > 0$,*

where the order is fixed and induced by an orthant of R^n. Then $F(x)$ is not decreasing; that is the inequalities

(*) $F(x) \geqq F(y)$ *and* $a \leqq x \leqq y \leqq b$ *imply that* $x = y$,

provided the same implication holds if $x_i = y_i$ for at least one $i = 1, \dots, n$.

Proof. This theorem can be proved exactly in the same way as Theorem 3 of 2. We will give here a proof which is direct. Assume that $x \leqq y$ are different. If $x_i = y_i$ for at least one i then the implication (*) holds by the assumption. Suppose that $x < y$. Then there is d such that $x \leqq d \leqq y$, $d_i = y_i$ for at least one i and $F(x) < F(d)$. If v in (1) is constant then putting $x(t) = x + tv$ we conclude from (1) that $F(x(t))$ is strictly increasing. Indeed, the derivative of the latter is $F'(x(t))\, v$

hence positive by the assumption (1). Putting $d = x(\tau)$ where $\tau = \max\{t \mid x(t) \leqq y\}$ we get d satisfying the desired properties. In the general case when v in (1) depends on x, $F(x(t)) > F(x)$ for $0 < t < \varepsilon$ for some $\varepsilon > 0$. In this case we can define, using the induction argument, a polygonal line $x(t)$ which is increasing and such that $F(x)(t)) > F(x(\alpha))$ if $t > \alpha$ where $x(\alpha)$ is any of the vertices of $x(t)$. Again $d = x(\tau)$ where τ is defined as above and we conclude that $F(x) < F(d)$. Either $d = y$ and then $F(x) < F(y)$ or by (*) the inequality $F_i(d) < F_i(y)$ holds for some of the coordinates. Therefore also $F_i(x) < F_i(y)$ for the same i. Thus the inequality $F(x) \geqq F(y)$ does not hold in both cases, which proves (*).

Remark. In the case of Gale–Nikaido $F'(x)$ is assumed to be a P-matrix. This implies (1) also for any principal submatrix of $F'(x)$. If $x_i = y_i$ for some i then (*) holds by the induction argument which is used there.

If $F'(x)$ is an almost P matrix then the assumption (1) may not be satisfied. However in this case we have the following

Proposition 1. *If $F'(x)$ is a continuous almost P-matrix for $x \in \Omega$ then we have the following alternative, either* (1) *holds or*

(2) $\quad F'(x)^{-1}v \leqq 0$ *for each* $x \in \Omega$ *and each* $v \geqq 0$.

Proof. We notice first that $F'(x)^{-1}$ is an N-matrix. Each entry of an N-matrix is different from zero. Indeed on the diagonal all entries are negative. Take an off-diagonal entry and take the two by two principal minor containing it. It is negative and the diagonal entries of it are negative. This is possible only if the off-diagonal entries are both non zero. Thus continuity of F' implies that each entry of $F'(x)^{-1}$ has constant sign. Therefore we hawe the alternative: either all entries of $F'(x)^{-1}$ are negative for each x then (2) holds or there is an entry of $F'(x)^{-1}$ which is positive for each x. In the latter case for each x there is a positive $w > 0$ such that $F'(x)^{-1}w > 0$, also (see [10], Theorem 2 on page 9). Putting $v = F'(x)^{-1}w$ we conclude that (1) holds true in this case. This completes the proof.

3. A VERSION OF KKM THEOREM

Let S be a closed simplex in R^n with vertices $p_1, \ldots, p_n$; that is

$$S = \{x: x = \mu_1 p_1 + \ldots + \mu_n p_n, \mu_1 + \ldots + \mu_n = 1, \mu_i \geqq 0\}$$

and denote by Q_i the face of S opposite to p_i; that is Q_i is the simplex with vertices $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$. The following result which we quote from Topology, vol. I by K. Kuratowski (see [6], Theorem 6 p. 311) is related closely to the KKM Theorem (compare [4]). KKM stands here for the abbreviation of three names: Knaster, Kuratowski and Mazurkiewicz.

Theorem 2. *If $A_1, \ldots, A_n$ are n closed sets such that*

$$S = A_1 \cup \ldots \cup A_n. \tag{3}$$

$$A_i \cap Q_i = \emptyset, \qquad i = 1, \ldots, n. \tag{4}$$

then $A_1 \cap \ldots \cap A_n$ is not empty.

Moreover, the hypothesis that the sets A_i are closed can be replaced by the hypothesis that they are open in S. For our purpose we found usefull the following modification and an easy consequence of the above result.

Lemma 1. *If $A_1, \ldots, A_n$ are n closed sets such that* (3) *holds and*

$$\Big(\bigcap_{j \in I'} A_j\Big) \cap \Big(\bigcap_{i \in I} Q_j\Big) = \emptyset \qquad \textit{for each } I \subset \{1, \ldots, n\}, \tag{5}$$

where $I' = \{1, \ldots, n\} \setminus I$, then $A_1 \cap \ldots \cap A_n$ is not empty.

Proof. Put $B_i = A_i \cup Q_i$. The intersection of B_i and that of A_i are equal. Indeed, if $x \in \cap B_i$ then either $x \in \cap A_i$ or there is $I \subset \{1, \ldots, n\}$ such that x is in the intersection on the left-hand side of (5). Since the latter is assumed to be empty, we conclude that $\cap B_i \subset \cap A_i$. The opposite inclusion being obvious we have equality. Suppose now that $\cap B_i$ is empty. Then the union of $C_i = S \setminus B_i$ is equal S, $C_i \cap Q_i$ is empty fo each i. C_i are open in S therefore we can apply Theorem 2 and there is $x \in C_i$ for $i = 1, \ldots, n$. Hence x does not belong to any B_i and therefore the union of B_i does not cover S opposite to what was assumed. Hence a contradiction, the intersection of B_i as well as that of A_i is not empty and the proof of the lemma is completed.

4. THE MAIN RESULT

We are ready to prove our main result which reads:

Theorem 3. *Assume that $F\colon R^n \to R^n$ is of class C^1 and that for each x the Jacobian matrix $F'(x)$ is almost P. Then F is univalent on R^n.*

Remark. Theorem 3 remains true if the domain of F is any rectangular region instead of R^n.

Proof of Theorem 3. Let a and b be such that $F(a) = F(b)$. We have to prove that $a = b$. If one of the coordinate of a and b is the same, say $a_n = b_n = \alpha$ then Jacobian matrix of $G(y) = (F_1(y_1, \ldots, y_{n-1}, \alpha), \ldots, F_{n-1}(y_1, \ldots, y_{n-1}, \alpha))$ is a P-matrix in R^{n-1}, since principal minors of $G'(y)$ are equal to principal minors of F' evaluated at $x = (y, \alpha)$ and they all are positive by the assumption that $F'(x)$ is an almost P-matrix. Hence $G'(y)$ is a P-matrix. Thus G is univalent by Gale—

Nikaido Theorem 1 and therefore the remaining coordinates of a and b are equal, too. Suppose now that each coordinate of a and b are different. Then there is an orthant in R^n which contains $b - a$ in its interior. Without any loss of generality we may assume that it is R^n_+, thus $a < b$. Now we apply Proposition 1. If (1) holds true then $a = b$ by Theorem 1. Therefore only the case when (2) holds true remains. In what follows we assume without any loss of generality that $F(a) = = 0 = F(b)$. In this case we consider Ważewski's equation [14]

$$(6) \qquad x' = F'(x)^{-1}v, \qquad x(0) = b, \qquad v \in S,$$

where S is the simplex $\{x\colon x_1 + \ldots + x_n = 1,\ x_i \geqq 0\}$ of dimension $n - 1$. We denote by $x(t, v)$ the solution of (6). It exists and it has the property

$$(7) \qquad F(x(t, v)) = F(b) + tv = tv.$$

Indeed, the derivative of the left-hand side of (7) is constant equal v. Since F is locally a diffeomorphism, (7) defines $x(t, v)$ uniquely. Hence also from (7) it follows that $x(t, v)$ is continuous in v.

Because of (2) the right-hand side of (6) is negative, therefore $x(t, v)$ is decreasing in t for each fixed v. Thus it follows that there is $t(v)$ such that $a < x(t, v) < b$ if $0 < t < t(v)$ and there is i such that $x_i(t(v), v) = a_i$. Such $t(v)$ is uniquely defined and continuous. Put

$$A_i = \{v\colon a_i = x_i(t(v), v)\}.$$

A_i is closed for each $i = 1, \ldots, n$ and (3) of Theorem 2 holds. We shall prove now that also (5) is satisfied. For this purpose let us fix $I \subset \{1, \ldots, n\}$. Suppose the opposite is true that is the set in (5) is not empty and let v belongs to it. Then $x(t(v), v) = \sum_{i \in I'} x_i(t(v), v)\, e_i + \sum_{i \in I} a_i e_i$ and $v = \sum_{i \in I} v_i e_i$, where e_i is i-th vector of the standard basis in R^n and $I' = \{1, \ldots, n\} \setminus I$. The above and (7) implies that

$$(8) \qquad \sum_{i \in I'} F_i(x(t(v), v))\, e_i = 0.$$

Consider the map $G(y) = \sum_{i \in I'} F_i(\sum_{i \in I'} y_i e_i + \sum_{i \in I} a_i e_i)\, e_i$ from a proper subspace of R^n into itself. The Jacobian matrix $G'(y)$ of G is a P-matrix since it is a proper principal submatrix of $F'(x)$ and the latter is almost P-matrix. Thus by Gale–Nikaido theorem the map G is globally univalent and is equal zero only if $y_i = a_i$ for each $i \in I'$. Thus (8) implies that $x(t(v), v) = a$. Because of (7) this is possible only if $t(v) = 0$ and thus $a = b$. Therefore if $a < b$ then (5) holds and we can apply Lemma 1. So there is $v \in A_i$ for each i which again means that $x(t(v), v) = a$ and by (7) it implies that $t(v) = 0$ and $a = b$. This completes the proof.

5. THE CASE OF ALMOST N-MATRIX

We will give now an analogue of Theorem 3 for the case when the Jacobian matrix is almost N. Before we will prove the following

Proposition 2. *Let $n \geqq 4$. Assume that $F'(x)$ is continuous almost N-matrix with some entries positive for each x from a rectangular region Ω, then*

(i) *the entries of F' are non-zero, the sign pattern of F' is symmetric and each row (column) contains at least one positive entry,*

(ii) *either* (1) *holds or* (2) *holds.*

Proof. We notice first that the first two claims of (i) are true also if $n \geqq 3$ since they follow from the assumption that each 2 by 2 principal minor is negative. Indeed the diagonal entries are negative thus off-diagonal entries have to be different from zero and of the same sign. The third part of (i) follows if we noticed that a 3 by 3 determinant with principal minors negative cannot be negative if only two of its entries are positive. This can be checked directly and on the other hand if there were one column thus also one row of $F'(x)$ with all entries negative then there would be a 3 by 3 principal minor with only two positive entries. This would contradict the assumptions that $F'(x)$ is almost N and $n \geqq 4$. Hence (i) is established.

Because of (i) it is enough to prove the (ii) part of Proposition 2 for a fixed x in Ω since if (2) holds for one $x \in \Omega$ it holds for each x by continuity and (i).

Assume therefore that (1) does not hold for a fixed x; that is there in no $w \geqq 0$ such that $F'(x)\, w > 0$. Then by one version of the theorem of alternatives we know (see for example Gale [1]) that there is a non-zero $v \geqq 0$ such that $v^T F'(x) \leqq 0$ Thus the set

$$W = \{w \colon w = -F'(x)^T v, \qquad v \geqq 0, \qquad w \geqq 0, \qquad w \neq 0\}$$

is not empty. We will show that $W = R^n \backslash \{0\}$. The latter set we denote by U. Clearly W is a subset of U and is closed in U. If $w \in W$ then the corresponding v is strictly positive. Indeed, suppose the opposite; that is one of the coordinates of v is zero, say $v_n = 0$. Then we have the following relation

$$v = (z, 0), \qquad z \in R^{n-1}, \qquad -w = (G^T z, g^T z),$$

where G is obtained from $F'(x)$ by deleting the last column and the last row and g is the last column of $F'(x)$ without the last coordinate. Since $F'(x)$ is an almost N-matrix, therefore G is an N-matrix. Suppose now that G is of the first kind (has a positive entry), then by Theorem 1, page 7 of [10] we get that $z = 0$ which means $w = 0$, a contradiction. Suppose G is of the second kind; that is $G < 0$. Then by (i) g has all coordinates strictly positive. But $g^T z \leqq 0$ and $z \geqq 0$, which is possible only if $z = 0$. A contradiction again, thus we proved that $v = -\{F'(x)^{-1}\}^T w$ is strictly positive if $w \in W$. But this implies that there is a neighbourhood in U of any $w \in W$ contained in W. Hence W is also open in U, thus $W = U$. Therefore $v = -\{F'(x)^{-1}\}^T w \geqq 0$ for each $w \geqq 0$ which is possible only if $F'(x)^{-1} < 0$. Hence (2) holds and the proof is completed.

Remark. One can prove a stronger statement than (i) in Proposition 2. Namely, that there exists a signature matrix S such that $SF'(x)\,S < 0$ for each x (see [12]).

Theorem 4. *Assume that F: $R^n \to R^n$ is of class C^1 and that for each x the Jacobian matrix $F'(x)$ is almost N. Then F is univalent on R^n.*

Proof. Similarly as in the proof of Theorem 3 we can reduce the argument to the case that the following is not possible

$$a < b, \qquad F(a) = F(b) = 0. \tag{9}$$

The only difference is that instead of Gale – Nikaido, we need to refer to the result of K. Inada [5] where he proved univalence for the case the Jacobian matrix is an N-matrix.

The result is true for $n = 2$, since the non-vanishing of a principal minors itself is enough for the univalence (see [9]). For $n = 3$ it is contained in [13]. In which follows we assume $n \geqq 4$. We notice that none of the entries of $F'(x)$ can be equal zero if the latter is an almost N-matrix. Consider first the case $F'(x) < 0$. In this case $F(x)$ is strictly decreasing and hence (9) is not possible.

Suppose now that $F'(x)$ has positive entries. Then by Proposition 2 we have two cases: either (1) holds or (2) is satisfied. Suppose (1) holds. We shall apply Theorem 1. Thus we have to prove that the implication

$$F(a) \geqq F(b), \qquad a \leqq b \text{ implies } a = b \tag{10}$$

holds true if $a_i = b_i$ for an i. Without any loss of generality we assume that $i = n$ and we put

$$G(y) = (F_1(y_1, \ldots, y_{n-1}, a_n), \ldots, F_{n-1}(y_1, \ldots, y_{n-1}, a_n)).$$

Condition (1) implies that each row of $F'(x)$ contains a positive entry. Hence by (i) of Proposition 2 each column of $F'(x)$ contains a positive entry also. $G'(y)$ is obtained from $F'(x)$ by deleting the last column and the last row and putting the last coordinate of x constant equal a_n. Thus since F' is almost N, $G'(x)$ is an N-matrix. If $G'(y)$ contains a positive entry (an N-matrix of the first kind) then Inada's result gives the implication (10). If $G(y) < 0$ (an N-matrix of the second kind) then the first $n - 1$ entries of the last row of $F'(x)$ are positive. This means that the inequalities

$$a \leqq b, \qquad F_n(a_1, \ldots, a_{n-1}, a_n) \geqq F_n(b_1, \ldots, b_{n-1}, a_n)$$

are possible only if $a = b$. Hence again the implication (10) holds true in this case and Theorem 1 implies that (9) is not possible also if $F'(x)$ contains positive entries and (1) holds.

Suppose now that $F'(x)$ contains some positive entries and (2) holds true. In this case we proceed the same way as in the proof of Theorem 3. Because of (2) solutions of Ważewski equation (6) are decreasing, $t(v)$ and the cover $\{A_i\}$ can be defined similarly and condition (5) of Lemma 1 holds since F restricted to $\cap\{Q_i\colon i \in I\}$ is one-to-one due to the fact that the corresponding Jacobian matrix is an N-matrix. Thus Inada's result is applicable. In this way Lemma 1 completes the proof of Theorem 4.

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