A NOTE ON COMPOUND SIMPLE GAMES

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1. **Introduction.** A game is called simple if every coalition of players either wins or loses. Given any two simple games (players are assumed to be distinct in the two games) we define the product as one in which every winning coalition must include winning contingents from both the components. A typical example of a product game is any organization in which some member has veto power. L. S. Shapley has obtained the following theorem on product solutions [1].

THEOREM 1. *For $i = 1, 2$, let $\{ Y_i(\alpha) \,|\, 0 \le \alpha \le 1 \}$ be a monotonic family of solutions of $\Gamma(P_i, W_i)$ except that $Y_i(1)$ need not be externally stable, and let $X_i(\alpha) = A_{P_i} - \mathrm{dom}_i\, Y_i(\alpha)$. Then*

$$ X = \bigcup_{0 \le \alpha \le 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1 - \alpha) $$

is a solution of $\Gamma(P, W) = \Gamma(P_1, W_1) \otimes \Gamma(P_2, W_2)$.

In this connection L. S. Shapley has raised the following questions:

(1) Can the requirement of full monotonicity be relaxed outside a neighbourhood of $\alpha_i = 1$ for the validity of Theorem 1?

(2) If the answer to (1) is "Yes," then does there exist a solution to product simple games which need not have the property of full monotonicity?

In §3, we give an affirmative answer to the first question, while in the last section, we present an example of a product solution which is not fully monotonic in the sense of Shapley.

The author is greatly indebted to Dr. L. S. Shapley for several highly helpful comments. Thanks are also due to the referee and to Mr. A. R. Padmanabhan for some useful suggestions and criticism.

2. **Definitions and notations.**

Simple games. We shall denote a simple game by the symbol $\Gamma(P, W)$ where P is a finite set (players) and W is a collection of subsets of P (the winning coalitions). We demand that $P \in W$ and empty set not an element of W. Let $\Gamma(P_1, W_1)$ and $\Gamma(P_2, W_2)$ be two simple games with $P_1 \cap P_2 = \varnothing$ and let $P = P_1 \cup P_2$. Then the product $\Gamma(P_1, W_1) \otimes \Gamma(P_2, W_2)$ (for simplicity we will write $P_1 \otimes P_2$) is defined as the game $\Gamma(P, W)$ where W consists of all $S \subseteq P$ such that $S \cap P_1 \in W_1$ and $S \cap P_2 \in W_2$.

Imputations. Let A_P denote the simplex of real nonnegative vectors x, such that $\sum x_i = 1$. These vectors are traditionally called "imputations" where P is the set of players in a simple game. Let us write $x(S)$ for $\sum_{i \in S} x_i$. Let $R_S x$ be the restriction of x to S thus:

$$R_S x = x_i \qquad \text{if } i \in S,$$
$$= 0 \qquad \text{if } i \notin S.$$

The barycentric projection of x on A_S is given by

$$B_S x = \frac{1}{x(S)} R_S x.$$

This is well defined provided $x(S) > 0$. Let P_1 and P_2 be fixed disjoint sets. Let $P = P_1 \cup P_2$ and

$$A_{P_i} = \{ x \mid x \in A_P \text{ and } x(P_i) = 1 \}.$$

If X and Y are subsets of A_{P_1} and A_{P_2} respectively, we define an operation $\times_\alpha$ by ($0 \leq \alpha \leq 1$)

$$X \underset{\alpha}{\times} Y = \{ z \mid z = \alpha x + (1 - \alpha) y \text{ for some } x \in X, y \in Y \}.$$

We recall that a solution of the game $\Gamma(P, W)$ is a set X of imputations such that $X = A_P - \operatorname{dom} X$ where $\operatorname{dom} X$ denotes the set of all $y \in A_P$ such that for some $x \in X$, the set $[i \mid x_i > y_i]$ is an element of W. The notation dom_1 and dom_2 will be used for domination w.r. to special classes W_1, W_2.

DEFINITION. A parametrized family of sets of imputations

$$[Y(\alpha) : 0 \leq \alpha \leq 1]$$

will be called semimonotonic if for every α, β, x such that $0 \leq \alpha \leq \beta \leq 1$ and $x \in Y(\beta)$, there exists $y \in Y(\alpha)$ such that

$$\alpha y \leq \beta x.$$

DEFINITION. A semimonotonic family $\{ Y(\alpha) \}$ is called fully monotonic if for every α, β, y such that $0 \leq \alpha \leq \beta \leq 1$ and $y \in Y(\alpha)$, there exists $x \in Y(\beta)$ such that

$$\alpha y \leq \beta x.$$

We will now generalize the concept of full monotonicity.

DEFINITION. A semimonotonic family $Y(\alpha)$ is called ∂-monotonic ($0 \leq \partial \leq 1$) if for every α, β, y such that $\partial \leq \alpha \leq \beta \leq 1$ and $y \in Y(\alpha)$, there exists $x \in Y(\beta)$ such that $\alpha y \leq \beta x$.

REMARK. Any ∂-monotonic family is necessarily ∂_1-monotonic if $\partial_1 \geq \partial$.

3. **On two theorems of Shapley.** Shapley has proved Theorem 1 and the following theorem in [1].

THEOREM 2. *Let* $\{X_i(\alpha)\}$ *be semimonotonic families of solutions of* $\Gamma(P_i, W_i)$. *Then*

$$X = \bigcup_{0 \leqq \alpha \leqq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(\alpha)$$

is a solution of $P_1 \otimes P_2$.

The purpose of this section is to prove a theorem which includes both of these theorems.

Let $Y_i(\alpha) \subseteq A_{P_i}$ for $i = 1, 2$. Further suppose the families to be semimonotonic. Let $X_i(\alpha) = A_{P_i} - \mathrm{dom}_i \, Y_i(\alpha)$. For every $S_j \in W_1$ consider the following sets

$$\Lambda_1(S_j) = \{\alpha \mid 1 > \alpha > 0, \quad \text{there exists } x_1 \in Y_1(\alpha)$$
$$\text{and } y_1 \in X_1(1) - Y_1(1) \text{ such that } \alpha x_1 > y_1 \text{ on } S_j\}.$$

Now choose one α_j from each $\Lambda_1(S_j)$ which is not empty. Let $\alpha_0 = \min \alpha_j$; we fix these α_j's and it is clear that $\alpha_0 > 0$. We wish to remark that there is certain amount of arbitrariness in choosing α_j's. Now we shall prove the following simple but useful lemma.

LEMMA 3. *If* $Y_1(\alpha)$ *is* α_0*-monotonic then there exist vectors* $z_j \in Y_1(1)$ *for those j for which* $\Lambda_1(S_j) \neq \varnothing$ *and a* ∂_1 *with* $0 < \partial_1 \leqq \alpha_0$ *such that*

$$z_j \geqq \partial_1 \text{ on } S_j.$$

Further if we take any $\alpha \in \Lambda_1(S_j)$, $\alpha \geqq \partial_1$.

PROOF. Since $\alpha_j \in \Lambda_1(S_j)$ there exists $x_j \in Y_1(\alpha_j)$ such that $\alpha_j x_j > 0$ on S_j. Since $\alpha_0 \leqq \alpha_j$ and the family is α_0-monotonic there exists $z_j \in Y_1(1)$ such that

$$z_j \geqq \alpha_j x_j.$$

Let e_j denote the least component among the S_j components of x_j. Define $\partial_1 = \min \alpha_j e_j$. Then we have

$$z_j \geqq \partial_1 \text{ on } S_j.$$

Also it is trivial to check that $0 < \partial_1 \leqq \alpha_0$. Hence the first part of the proof of the lemma is complete. We shall now prove that any $\alpha \in \Lambda_1(S_j)$ will satisfy the inequality $\alpha \geqq \partial_1$.

Take any $\alpha \in \Lambda_1(S_j)$. This means there exists an $x_1 \in Y_1(\alpha)$ and $y_1 \in X_1(1) - Y_1(1)$ such that $\alpha x_1 > y_1$ on S_j. We claim that at least one of the S_j components of y_1 must be greater than or equal to ∂_1;

otherwise $z_j > y_1$ on S_j and therefore $y_1 \in \text{dom}_1 z_j \subseteq \text{dom}_1 Y_1(1)$ which contradicts the assumption that $y_1 \in X_1(1) = A_{P_1} - \text{dom}_1 Y_1(1)$. Therefore we have $\alpha > y_1(S_j) \geq \partial_1$. Hence the proof of the lemma is complete.

Similarly we can get hold of a ∂_2 by constructing $\Lambda_2(S_j')$ for $S_j' \in W_2$. If all the $\Lambda_1(S_j)$ and $\Lambda_2(S_j')$ are empty we define ∂_0 to be any positive number in the interval $(0, 1]$—in fact we can take $\partial_0 = 1$. If $\Lambda_1(S_j) \neq \varnothing$ for some j and $\Lambda_2(S_j') \neq \varnothing$ for some j we define $\partial_0 = \min[\partial_1, \partial_2]$. If $\Lambda_1(S_j) \neq \varnothing$ for some j and $\Lambda_2(S_j') = \varnothing$ for all j, we define $\partial_0 = \partial_1$ and in the other case we take $\partial_0 = \partial_2$. In all these cases we see that $\partial_0 > 0$. Now we are in a position to state and prove our

THEOREM 4. *Let $Y_1(\alpha)$ and $Y_2(\alpha)$ be a semimonotonic family of solutions to $\Gamma(P_1, W_1)$ and $\Gamma(P_2, W_2)$ respectively except that $Y_1(1)$ and $Y_2(1)$ need not be externally stable. Further suppose the family to be δ_0-monotonic. Then*

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1 - \alpha)$$

is a solution of $P_1 \otimes P_2$, where $X_2(\alpha) = A_{P_i} - \text{dom}_i Y_i(\alpha)$.

REMARK. Since $\{Y_i(\alpha)\}$ is semimonotonic it follows that $Y_i(1)$ is internally stable. If $Y_i(1)$ for $i = 1, 2$, is also externally stable, that is, if $Y_i(1)$ is a solution then it is not hard to check that $\Lambda_1(S_j)$ and $\Lambda_2(S_j')$ are all empty and we can take $\partial_0 = 1$. In other words Theorem 2 is included in our theorem. It is not hard to check that Λ-sets are empty if $Y_i(\alpha)$ are fully monotonic and hence Theorem 1 is also included in our theorem. We will show by giving an example that Theorem 4 is actually a generalization of Theorems 1 and 2.

PROOF OF THEOREM 4. Our proof follows along the same line as the one given for Theorem 2. Note that $X_i(\alpha) = Y_i(\alpha)$ for all α except $\alpha = 1$.

External stability. Take any $y \in A_P$ and define $\beta_i = y(P_i)$ and let y_i be the barycentric projection of y on A_{P_i}.

Case 1. $0 < \beta_1 < 1$.

Case 1a. $y_1 \in X_1(\beta_1)$, $y_2 \in X_2(\beta_2)$, then $y = \beta_1 y_1 + \beta_2 y_2 \in X$.

Case 1b. $y_1 \notin X_1(\beta_1)$, $y_2 \notin X_2(\beta_2)$, this means there exists $x_1 \in Y_1(\beta_1)$, $x_2 \in Y_2(\beta_2)$ such that $y_i \in \text{dom}_i x_i$ for $i = 1, 2$.

Then the imputation defined by

$$x = \beta_1 x_1 + \beta_2 x_2$$

which is in X, clearly dominates y or $y \in \text{dom} X$.

Case 1c. $y_1 \in X_1(\beta_1)$, $y_2 \notin X_2(\beta_2)$.

Find $x_2 \in Y_2(\beta_2)$ such that $y_2 \in \text{dom}_2 x_2$. Let $x_2 > y_2$ on $S_2' \in W_2$.

Choose $e > 0$ so that $\beta_2 x_2 - \beta_2 y_2 > e$ on S_2'. Using the semimonotonic property of $Y_2(\alpha)$ find $x_2' \in Y_2(\beta_2 - e)$ such that $\beta_2 x_2 - (\beta_2 - e) x_2'$ is nonnegative. This vector must be $\leq e$ in all components, since no component of a nonnegative vector can exceed the sum of all components. If follows that we have

$$(\beta_2 - e) x_2' > \beta_2 y_2 \text{ on } S_2'.$$

Let u_1 be any interior point of A_{P_1}. Then the imputation x defined by

$$x = \beta_1 y_1 + e u_1 + (\beta_2 - e) x_2'$$

dominates y on $P_1 \cup S_2'$. If $x \in X$ then $y \in \mathrm{dom}\, X$ and we are through. Suppose $x \notin X$. Then the barycentric projection of x on A_{P_1}, namely

$$x_0 = \frac{\beta_1}{\beta_1 + e} y_1 + \frac{e}{\beta_1 + e} u_1$$

must not be an element of $X_1(\beta_1 + e) = Y_1(\beta_1 + e)$. Hence we can find $x_1 \in Y_1(\beta_1 + e)$ such that $x_0 \in \mathrm{dom}_1 x_1$. Then it is clear that the imputation z defined by

$$z = (\beta_1 + e) x_1 + (\beta_2 - e) x_2'$$

dominates y. Since $z \in X$ it follows $y \in \mathrm{dom}\, X$.

Case 1d. $y_1 \notin X_1(\beta_1)$, $y_2 \in X_2(\beta_2)$ (like case 1c).

Case 2. $\beta_1 = 0$.

Case 2a. $y_2 \in X_2(1)$, then $y \in X$.

Case 2b. $y_2 \notin X_2(1)$. This means $y_2 \in \mathrm{dom}_2 Y_2(1)$ and hence argument of case 1c can be repeated with the understanding that $\beta_1 y_1 = 0$.

Case 3. $\beta_1 = 1$ (like case 2). This completes the proof of the external stability of X.

Internal stability of X. Suppose there exists x, $y \in X$ such that $x > y$ on $S \in W$. Let $S_1 = S \cap P_1$, $S_2 = S \cap P_2$; $x = \alpha x_1 + (1 - \alpha) x_2$ and $y = \beta y_1 + (1 - \beta) y_2$. Since $x > y$ on $S_1 \cup S_2 = S$, it follows that $0 < \alpha < 1$.

Case 1a. $1 > \beta \geq \alpha$. Since $\alpha > 0$, β is also strictly positive. Also $\alpha x_1 > \beta y_1$ on S_1. Since $Y_1(\alpha)$ is semimonotonic, there exists $x_1' \in Y_1(\alpha)$ such that $\beta y_1 \geq \alpha x_1'$. Hence, $\alpha x_1 > \alpha x_1'$ on S_1 or $x_1 > x_1'$ on S_1 contradicting the internal stability of $Y_1(\alpha)$.

Case 1b. $\alpha \geq \beta > 0$. This means $1 > 1 - \beta \geq 1 - \alpha$. Since $\alpha \neq 1$ this case is similar to the previous case except that we have to utilize the semimonotonic property of $Y_2(\alpha)$.

Case 2a. $\beta_1 = 1$ that is $\alpha x_1 > y = y_1$ on S_1. If $y_1 \in Y_1(1)$ then using the semimonotonic property of $Y_1(\alpha)$ we will arrive at a contradiction. If $y_1 \in X_1(1) - Y_1(1)$ then $\alpha \in \Lambda_1(S_1)$ and hence by Lemma 3, $\alpha \geq \partial_1 \geq \partial_0$. Since $Y_1(\alpha)$ is ∂_0-monotonic one can find $y_1' \in Y_1(1)$ such that $y_1' \geq \alpha x_1$

$> y_1$. This implies $y_1 \in \text{dom}_1 Y_1(1)$ or $y_1 \notin X_1(1)$ which contradicts our assumption regarding y_1.

Case 2b. $\beta = 0$ or $1 - \beta = 1$ and the argument can be carried over as in the case 2a.

Hence the proof of our theorem is complete.

4. An example of a solution of product simple games. The following example shows that solutions to product simple games can be found that satisfy the conditions of Theorem 4 but not that of Theorems 1 or 2. In other words every solution that satisfies the conditions of Theorem 4 need not have the property of full monotonicity.

EXAMPLE. The four person game J is defined by

$$\Gamma(\overline{1234}, \{\overline{124}, \overline{134}, \overline{234}, \overline{1234}\})$$

where members in the curly brackets denote the winning coalitions of the game J.

$$J = M_3 \otimes B_4.$$

M_3 denotes the 3-person simple majority game and B_4 the 1-person pure bargaining game. Define, for $0 \leq \alpha \leq 3/4$

$$X_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta)$$

where

$$Y(\beta) = \{(\beta^2/2, \beta t, \beta(1 - t) - \beta^2/2, 1 - \beta)/0 \leq t \leq 1 - \beta/2\}$$

for $0 \leq \beta < 1$ and

$$Y(1) = \{(1/2, t, 1/2 - t, 0)/0 \leq t \leq 1/2\} \cup (0, 1/2, 1/2, 0).$$

Define for $3/4 < \alpha \leq \alpha_0$, where α_0 is so chosen that

$$\alpha_0(1 - 7/8(1 + \alpha_0)) = \tfrac{1}{2},$$

the family:

$$X_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta),$$

where

$$Y(\beta) = \left\{ \left(\frac{7}{8} \cdot \frac{\beta^2}{1 + \alpha}, \beta t, \beta(1 - t) - \frac{7}{8} \frac{\beta^2}{1 + \alpha}, 1 - \beta \right) \right.$$

$$\left. \text{where } 0 \leq t \leq 1 - \frac{7\beta}{8(1 + \alpha)} \right\} \text{ for } 0 \leq \beta \leq 1.$$

For $\alpha_0 < \alpha < 1$ define

$$X_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta)$$

where

$$Y(\beta) = \left\{ \left(\left(1 - \frac{1}{2\alpha}\right)\beta^2, \beta t, \beta(1 - t) - \left(1 - \frac{1}{2\alpha}\right)\beta^2, 1 - \beta \right) \right.$$

$$\left. \text{where } 0 \leq t \leq 1 - \left(1 - \frac{1}{2\alpha}\beta\right) \right\} \text{ for } 0 \leq \beta \leq 1.$$

For $\alpha = 1$ define $X_1(1) = \bigcup_{0 \leq \beta \leq 1} Y(\beta)$ where

$$Y(\beta) = \left\{ (\beta^2/2, \beta t, \beta(1 - t) - \beta^2/2, 1 - \beta)/0 \leq t \leq 1 - \beta/2 \right\}.$$

Now it is not hard to check that the family $X_1(\alpha)$ is semimonotonic and that each $X_1(\alpha)$ is a solution to the game J except $X_1(1)$ because the element $(0, 1/2, 1/2, 0) \notin X_1(1) \cup \mathrm{dom}_1 X_1(1)$. If $X_1'(\alpha) = X_1(1) \cup (0, 1/2, 1/2, 0)$ then $X_1'(1)$ is a solution to J. But $\{X_1(\alpha): 0 \leq \alpha < 1\}$ together with $X_1'(1)$ is not semimonotonic for corresponding to $(0, 1/2, 1/2, 0)$ there exists no element $x \in X_1(\alpha)$ for any $\alpha > 3/4$ with $\alpha x \leq (0, 1/2, 1/2, 0)$. Further the family $\{X_1(\alpha): 0 \leq \alpha \leq 1\}$ is not fully monotonic because corresponding to the element $(0, 1/2, 1/2, 0) \in X_1(3/4)$ there exists no element $y \in X_1(1)$ with the property that

$$y \geq 3/4 \ (0, 1/2, 1/2, 0).$$

Hence it is clear that the conditions of Theorem 1 as well as Theorem 2 are violated. Moreover it is not difficult to check that $\Lambda_1(S_j) = \varnothing$ for all j and as such we can take $\partial_1 = 1$. Hence this family $\{X_1(\alpha)\}$ can be used to produce product solution to arbitrary games of the form $J \otimes K$—solutions which Theorem 1 or Theorem 2 cannot predict.

REFERENCE

1. L. S. Shapley, *Solutions of compound simple games*, Annals of Mathematical Studies No. 52, pp. 267–305, Princeton Univ. Press, Princeton, N. J., 1964.

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