ON THE MIQUEL-CLIFFORD CONFIGURATION.

By A. Narasinga Rao.

(From the Department of Mathematics, Annamalai University, Annamalainagar.)

Received December 7, 1937.

1. *Introduction.*

The classical chain of theorems connected with the names of Miquel and Clifford associate with an even number of lines in a plane a point, called their "Miquel Point", and with an odd number of lines a circle, called their "Clifford Circle", with the well-known incidence relations between them. The resulting configuration could be viewed from the standpoint of the projective geometry of the plane, or as one belonging to inversive geometry. Clifford's own proof was based on considerations connected with the foci of n-fold parabolas and belongs to the former category.[1] It was, however, pointed out by J. H. Grace[2] and later by E. H. Neville[3] that if the n straight lines which generate the figure be placed by concurrent circles, the configuration becomes symmetric in the sense that it consists of 2^{n-1} circles and 2^{n-1} points such that n of the circles pass through each point and n of the points lie on each circle. It would thus appear, that it would be more appropriate to view it as one of circle (inversive) geometry, and I have recently[4] discussed from this standpoint certain transformations in circle-space connected with the configuration.

In this paper it is seen that by combining both these points of view, fresh light is thrown on the structure of the configuration. The results obtained are :

(i) a proof of the following theorem announced without proof[5] by the late V. Ramaswami Aiyar, Founder of the Indian Mathematical Society,—" The curves that I desire to bring to the notice of the reader are of class $n+1$. Each touches the line at infinity n times—the circular points at infinity being always two of the points

[1] W. K. Clifford, "Synthetic Proof of Miquel's Theorem", *Mathematical Papers*, p. 38.

[2] Grace, "Circles, Spheres and Linear Complexes", *Trans. Camb. Phil. Soc.*, 1898, **16**, 31.

[3] Neville, "The Inverse of the Miquel-Clifford Configuration," *Jour. Ind. Math. Soc.*, 1926, p. 241.

[4] A. Narasinga Rao, " On certain Cremona-Transformations in Circle-Space connected with the Miquel-Clifford Configuration," *Proc. Camb. Phil. Soc.*, 1937, **33**, § 31.

[5] V. Ramaswami Aiyar, "Note on a Class of Curves," *The Mathematics Student*, 1936, **4**, p. 106.

at contact. It can be determined when $2n$ tangents are assigned. If $2n+1$ tangents of such a curve be taken, the Clifford-Miquel Circle of the $2n+1$ tangents becomes A STRAIGHT LINE" ;

(ii) a generalisation of Ramaswami Aiyar's result to cover the case when the Clifford Circle of the $2n+1$ lines is a circle cutting a fixed circle orthogonally, or is of constant radius ;

(iii) a proof that at each of the points of multiple incidence in the configuration, the concurrent circles cut at the same angles as at any other of these points.

2. *The Projective View-point.*

Let p be the projective plane with a degenerate Caylean Absolute consisting of two points I, J. The line IJ is the "line at infinity" and any conic through I and J is a "circle".

Let (1) and (2) denote two circles through a point O which will also be denoted by the symbol ( ). Their other intersection (12)[6] is the Miquel Point of (1) and (2). With three circles (1), (2), (3) we have three such points (12), (23), (31) which lie on the Clifford Circle (123). If however, (3) is a singular circle through O consisting of the line OI and a line l_j through J, the intersections (31) and (32) (other than O, I, J) are on l_j so that the circle (123) breaks up into l_j and the line joining the Miquel Point (12) to I.

With four circles through O, one of which, say (4), consists of OI and l_j we have one proper Clifford Circle (123) and three degenerate Clifford Circles each of which breaks up into l_j and another line. The Miquel Point (1234) common to all these circles is thus the intersection of l_j with (123), other than J. Thus, if we take five circles of which (5) breaks up into OI and l_j, four of the Miquel-points lie on l_j. Hence the Clifford Circle of the five concurrent circles breaks up into and l_j the join of (1234) with I. It is easy to see that these considerations may be indefinitely extended and apply to either circular point. Hence,

> Given an even number of circles (1), (2), (3),$\cdots\cdots(n)$ through O they have a Miquel Point M $\equiv$ $(12\cdots\cdots n)$. With every additional circle (x) through O may be associated the Clifford Circle $(123\cdots\cdot nx)$ through M. When (x) breaks up into OI and a line l_j through J, the associated circle $(12\cdots\cdot nx)$ breaks up into l_j and the line MI. .. (2 · 1)

It is thus seen that point-circles through O correspond to point-circles through M. Among the circles through both O and M, there are two which are

[6] Both here and throughout this paper the order of the letters in a bracket is a matter of indifference. The notation is that of my paper on *Cremona Transformations*, etc.

point-circles. These points will be called the "cross-pair" of O and M and denoted by C and C′. They stand respectively for the line pairs OI, MJ and OJ, MI. We shall now prove that

if the additional circle (x) through O be either of the point-circles C or C′ (the cross-pair of O and M), the associated Clifford Circle $(12 \cdot \cdot \cdot \cdot nx)$ is the point-circle at M. .. .. (2 · 2)

This follows from (2·1) since C is the line pair OI, MJ so that the associated Clifford Circle is MI, MJ. Similarly for C′.

3. *The Inversive View-point.*

It is known that the geometry of the plane under the inversion group is isomorphic with the projective geometry of a 3-space S_3 with an invariant quadric Ω whose points represent the point-circles on the plane. The point-circles O, M on p are represented by points o and m on Ω while the two groups of circles $(1), (2) \cdot \cdot \cdot \cdot \cdot (n)$ and $(23 \cdot \cdot \cdot \cdot \cdot n)$, $(13 \cdot \cdot \cdot \cdot \cdot n)$, $\cdot \cdot \cdot \cdot \cdot$ $(12 \cdot \cdot \cdot \cdot \cdot n-1)$ which pass respectively through O and M are represented by points on the tangent planes π_0 and π_m to Ω at o and m. The intersection of the two tangent planes represents the circles in p through O and M, and the two points c, c' where the line cuts Ω correspond obviously to the cross-pair C, C′ of O, M. We shall use the bracket symbols to represent both the circle on the plane p and the representative point in S_3.

By (2·1) we have for every point (x) on π_0, an associated point $(12 \cdot \cdot nx)$ on π_m. In my paper on *Cremona Transformations* cited earlier, I have discussed the transformation $(x) \longrightarrow (12 \cdot \cdot \cdot \cdot \cdot nx)$ and shown that

(i) it is an involutoric transformation of order $t+1$ (where $n = 2t$) of the De Jonquiere type .. .. (3 · 1)

(ii) its F-elements of unit multiplicity in the two planes are the points $(1)\ (2) \cdot \cdot \cdot \cdot (n)$ and $(23 \cdot \cdot \cdot \cdot n), (13 \cdot \cdot \cdot \cdot n), \cdot \cdot \cdot \cdot \cdot (12 \cdot \cdot \cdot \cdot n-1)$ and its elements of multiplicity t are $O = (\quad)$ and $M = (12 \cdot \cdot \cdot \cdot n)$; .. .. (3 · 2)

so that

(iii) the P-curve corresponding to (r) in π_0 is the line joining M to $(12 \cdot \cdot \cdot r-1\ r+1 \cdot \cdot \cdot \cdot \cdot n)$ on π_m .. .. (3 · 3)

(iv) while the P-curves of O and M are curves of order t on π_m and π_0 having m and o for singular points of order $t-1$ .. (3 · 4)

It follows from (3 · 2) that a line through O in π_0 say the join of O and (x) is transformed into a line through M in π_m namely, the join of M and $(12 \cdot \cdot \cdot nx)$; when (x) approaches any point on the line O (r) so that O (x) tends to O (r), the corresponding line through M should by (3 · 3) approach the join of M and ($12 \cdots r-1 \;\; r+1 \cdots n$). Hence

> The pencil of lines through o in the plane π_0 is in one-one correspondence with the pencil of lines through m in π_m which are their transforms. To the line joining o to (r) corresponds the join of m with the associated Miquel Point ($12 \cdots r-1 \;\; r+1 \cdots n$). The generators oc, oc' correspond respectively to the generators mc' and mc. (3 · 5)

The last statement follows from the following considerations: Since c represents the circle OI, MJ and o the circle OI, OJ the points on the line oc correspond to the circles of the pencil OI, l_j where l_j is any line through J. Similarly points on mc' represent the pencil of circles MI, l_j. From (2·1) it follows that points on oc correspond to points on mc'. A similar proof holds for oc' and mc.

From (2·2) it follows that the P-curves of o and m pass through both c and c'. Let q and q' be points on the planes π_0 and π_m corresponding each to the other in the involutoric Cremona Transformation. As q' approaches any point on the P-curve of o, q approaches o in a particular direction. When q' approaches c, Mq' tends to Mc and hence by (3·5) oq tends to oc'. Also by (2·2), q tends to the limit o. Hence

> The P-curves of o and m pass through both c and c'. The transform of a curve in one of the planes, say π_m which passes through c (or c') is a curve in the other plane π_0 having a singular point at o, one of the tangents at o being the generator oc' (or oc). .. .. (3 · 6)

4. *Applications.*

Consider now the transform of the line cc', the intersection of π_0 and π_m, which represents the circles in p through O and M. Regarded as a line in π_m, it is the transform of, or transforms into, (since the transformation is involutoric) a curve Γ_0 of degree $t+1$ in π_0 having by (3·2), a singularity of order t at o with mc and mc' as two of the nodal tangents (by 3·6), and passing through the points (1) (2) $\cdots$ (n). If we take o to be the point on Ω representing the "point-circle at infinity," the points on Γ_0 correspond to straight lines on p which envelop a curve of class $t+1$ having the line at infinity (which corresponds to o) as a multiple tangent of order t, two of the points of contact being the circular points I and J, and also touching the lines (1) (2) $\cdots$ (n). When another tangent (x) to such a curve is taken, their Clifford Circle ($12 \cdots nx$) being the transform of (x) corresponds to a point on cc', that is, it is a straight line through M. Thus the Clifford Circle of any $n+1 = 2t+1$ tangents to such a curve is a straight line, and this is precisely Ramaswami Aiyar's Theorem mentioned in the introduction.

More generally, if the Clifford Circle $(12 \cdot \cdot \cdot \cdot nx)$ is to cut a given circle orthogonally, (x) must lie on a curve in π_0 which is the transform of a straight line on π_m. Such a curve is of class $t + 1$ with a node of order t at o and passes through $(1) \cdot \cdot \cdot \cdot (n)$, but it will not have oc and oc' for nodal tangents. Interpreting in terms of lines in plane p with o corresponding to the line at infinity, we have the following result :—

> Let Γ be an t-fold parabola (a curve of class $t + 1$ touching the line at infinity t times) and (1) (2) $\cdot \cdot \cdot \cdot \cdot (2t)$ denote $2t$ tangents to it. If (x) be any variable tangent to Γ, the Clifford Circles $(123 \cdot \cdot \cdot \cdot 2tx)$ will for all positions of (x) be orthogonal to a fixed circle. (4 · 1).

Finally, let us determine the relation between $2t + 1$ lines so that their Clifford Circles may be of given radius.

Now circles of constant radius on p correspond to points on a quadric Ω_1, touching Ω at the "point-circle at infinity" which we take to be o. Since Ω_1 belongs to the linear system containing Ω and the squared tangent plane π_0, its section by π_m belongs to the linear system containing the line pair mc, mc' and the squared line cc'. Hence the section of Ω_1 by π_m is a conic touching mc and mc' at c and c'. Different conics of this linear system correspond to different values of the radius.

In order that the Clifford Circle $(12 \cdot \cdot \cdot \cdot nx)$ may be of fixed radius, (x) must lie on the locus γ in π_0, which is the transform of a conic touching mc and mc' at c and c'. It is a curve of degree $2t + 2$ having o for a singular point of order $2t$ and the points (1) $(2) \ldots . (n)$ for double points. Since the conic passes through c and c', its transform γ has by (3·6) the generators oc and oc' for two of the tangents at the node. By (3·5) the other intersections of the conic with mc and mc' go over into points on oc' and oc, and as these are also located at c and c', it follows that oc and oc' are inflexional tangents at o to two of the branches through it.

When the points of γ are interpreted as lines in p, o corresponds to the line at infinity and the intersections of γ with the generator oc (or oc') to the tangents from the circular points I (or J) to the envelope of the lines. The inflexional tangents oc, oc' thus imply that three of the tangents from I and J are coincident with the line at infinity, *i.e.*, I and J are cusps with the line at infinity for the cuspidal tangent in each case. Hence we have the result :—

> Given a curve of class $2t + 2$ having the $2t$ lines (1) $(2) \cdot \cdot \cdot \cdot \cdot \cdot (2t)$ for bitangents and the line at infinity as a singular line of order $2t$, being the cuspidal tangent at the circular points I and J, if (x) be any variable tangent to the curve, then the Clifford Circle $(12 \cdot \cdot \cdot \cdot 2mx)$ is of constant radius. .. .. (4 · 2)

5. *Other Applications. Angle Properties.*

We have seen (3·5) that the lines of the pencil vertex o in π_0 are in projective correspondence with the lines of the pencil vertex m in π_m.

If

we make the convention that any figure repeated twice in a bracket symbol destroys itself and may be omitted altogether, .. (5 · 1)

so that we have $(23 \cdots n) \equiv (123 \cdots n\ 1)$, the correspondence is concisely expressed by the statement that

for all values of x ($x \leqslant n$ and $x > n$) the line in π_0 through (x) corresponds to the line in π_m through $(12 \cdots nx)$ .. .. (5 · 2)

Now a line through o corresponds in p to a pencil of circles touching one another at O and so determines a line element at O. We thus have a correspondence in p between the line elements at O and M. With the isotropic line element along OI is associated the pencil of circles OI, l_j and we have seen in (2·1) that the corresponding circles through M belong to the pencil MI, l_j. Hence the isotropic directions OI, OJ correspond to MI, MJ and it follows that

the angle between (r) and (s) is equal to the angle between $(12 \cdots nr)$ and $(12 \cdots ns)$. .. .. (5 · 3)

The equality is a direct equality and not an inverse equality (since OI corresponds to MI and OJ to MJ), so that a rotation through a suitable angle $\theta_{12 \cdots n}$ will carry the line elements at O into those at $M = (12 \cdots n)$.

The equality of the angles formed by the line elements at the various points of multiple incidence holds not only for extreme points like O and M, but also in the case of any two of them. Thus take the point (123456). This is the Miquel Point of (1) (2) (3) (4) (5) (6) through ( ), and hence by (5·3), the angle betweeen (r) and (s) is the angle between $(12 \cdots r-1\ r+1 \cdots 6) = (12 \cdots 6r)$ and $(12 \cdots s-1\ s+1 \cdots 6) = (12 \cdots 6s)$ provided $r, s \leqslant 6$. If $r > 6$, (r) is no longer a singular element of the Cremona Transformation between the tangent planes π_0 and π_t to Ω at o and $t = (12 \cdots 6)$, but (r) is carried over into $(12 \cdots 6r)$ and hence by (3·5) the line elements correspond. Thus with the convention (5·1) we may assert that there is a single projective correspondence in which the line element of (r) through O corresponds to the line element of $(12 \cdots 6r)$ through t, r taking all the values $1, 2 \cdots n$. Hence the result.[7]

[7] *Vide* A. Narasinga Rao, "The Miquel-Clifford Configuration in the Geometries of Mobius and Laguerre," *Annamalai University Jour.*, VII, 1937.

The angles formed by the line elements of the n circles which meet at any point of multiple of incidence is the same for all such points. The line elements at any such point can be carried over into the corresponding line elements at any other point by a pure rotation. .. (5·4)

This may be regarded as a generalisation of the well-known result that the angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the opposite segment. For, when $n = 2$ and we start with two straight lines (1) (2) meeting at (12), the transform of a third straight line (3) is the circumcircle (123). Now (5·4) asserts that the angle between say (2) and (3) at the point-circle at infinity of the tetracyclic plane, [whose magnitude may be taken to be the angle between the lines (2) and (3) since two circles cut at the same angle at both intersections] is equal to the angle between their transforms (122) = 1 and (123).

Summary.

In this paper the Miquel-Clifford configuration is studied both from the standpoint of the projective geometry of the plane and of Mobius Geometry —the inversive geometry of the plane. By combining both view-points conditions are obtained for the Clifford Circle of $2t+1$ lines to be (i) a straight line, (ii) a circle of given radius. It is also shown that at each of the points of multiple intersection of the configuration the concurrent circles cut at the same angles as at any other of these points.

2154-37 Printed at The Bangalore Press, Bangalore City, by G. Srinivasa Rao, Superintendent, and Published by The Indian Academy of Sciences, Bangalore.