STUDIES IN TURBINE GEOMETRY—III*
THE NON-ORIENTED LINE ELEMENT IN TWO-DIMENSIONAL MOBIUS GEOMETRY

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1. SOPHUS LIE obtained two representations of line elements in a plane by points in 3-space which have been shown[1] to be essentially representations of oriented line elements in a Euclidean plane, and of non-oriented elements in a minimal plane in such a manner that circles on the planes correspond to the lines of a linear complex. Representations of the manifold of line elements of a plane as binary-ternary,[2] double ternary[3] (in contragredient variables) and of oriented line elements of a sphere as triple binary[4] domains, and their representation in 3-space by projection from their associated Segre Manifolds have appeared in recent years. A fair idea of the work done in this field may be had from the " Geschitlische Entwicklung der Lehre von der Geraden-Kugel Transformation " by Prof. E. A. Weiss, whose own contributions to this region are not inconsiderable.

In this paper the main interest is not any scheme of analytic representation but the geometry of line elements under the extended point transformations of the inversive group. A representation free from singularities of the line element of the Mobius plane on a V_3^4 in a projective S_5, is obtained

* The first two papers of this series are:

(I) " The Concepts of Turbine Geometry," *Jour. Ind. Math. Soc.*, New Series, **3**, 96–108.

(II) " On the Sub-geometries of Lie which belong to the Mobius Laguerre Pencil," *Proc. Ind. Acad. Sci.*, 1938, **8**, 179–86.

These will be referred to briefly as T.G. I and T.G. II.

[1] E. A. Weiss, " Die Geschictliche Entwicklung der Lehre von der Geraden-Kugel Transformation," *Deutsche Mathematik*, Jahrg. 3 S. 19. I have to thank Prof. Weiss for a set of reprints of his recent papers.

[2] E. A. Weiss, " Das Linienelement als singulare Punktreihe," *Jour. f.d. reine u, ange. Mathematik*, Bd. 117 (1937). This paper has several points of contact with the present one.

[3] Schake, " Le geometrie degli elementi lineari etc.," *Atti Congr. Bologna*, **4**, 45–50. *Vide* also Beck, "Uber die Lieschen Abbildungen der Linienelemente auf Raumpunkte, " *Math. Zeit.*, Bd. 42.

[4] E. A. Weiss, "Die orientierten Linienelemente einer Kugel als dreifach binares Gebeit," *Deutsche Mathematik*, Jahrg. 3.

and then transferred to a semi-metric 3-space by projection from a generating line. Concurrent line elements are represented by the lines of an elliptic congruence, while circles and turbines correspond to conics. It is found that the transformation group in this representative space is Cubo-cubic. In the last paragraph, it is shown that the analogue to line-element geometry on the Mobius plane is the geometry of "rods" (point pairs at a constant distance apart) under the group of G_k inversions.

2. The invariant concepts of Sphere-Geometry enable us to associate with each prime element in a Euclidean S_n a pencil of spheres in mutual contact and *vice versa*. By means of the representations discussed in T.G. I § 4, these spheres are represented by the points of a tangent line to a quadric Ω in a projective S_{n+1}, or of a generator of a quadric $\overline{\Omega}$ in S_{n+2} according as the elements and hence also the spheres are not, or are oriented. Thus

> The transformations of prime elements in the extended point transformation of Mobius Geometry (Inversive Geometry) in a Euclidean S_n are isomorphic with the transformations of tangent lines to a fixed quadric in a projective S_{n+1} under the automorphic group of the quadric. Hence the inversive geometry of prime elements in a Mobius S_n is essentially identical in structure with the geometry of isotropic lines in a non-Euclidean S_{n+1}. (2·1)

Similarly

> The geometry of oriented prime elements in a Euclidean S_n under the Lie Group of Sphere transformations is essentially identical with the geometry of generating lines of a quadric variety in a projective S_{n+2} under the automorphic group of the quadric. (2·2)

The generators of a quadric $\overline{\Omega}$ in S_{n+2} may be projected into the tangent lines of a quadric Ω in S_{n+1} by means of a vertex of projection not on $\overline{\Omega}$. We thus establish a (2, 1) correspondence between oriented and non-oriented prime elements, which may be identified with the two single orientations of a doubly oriented [hence non-oriented by T.G. I (1·1)] prime element, if we take for vertex, of projection the point m which is mentioned in T.G. I (3·4) and (4·7).

Two-dimensional circle geometry is somewhat peculiar in that the turbine splits up into two semi-turbines according to my terminology (Kasner's "Turbines"), and requires a separate treatment. In this paper we deal with two-dimensional Mobius Geometry as a geometry of non-oriented line elements.

3. *The Structure of* V_3^4

We have seen that every non-oriented line element in the inversive plane ω (the plane of Mobius Geometry) may be represented by a tangent line to a quadric Ω in S_3. This is equivalent to the transfer of each line element from the plane ω to the sphere Ω by stereographic projection. Again, the lines of a projective S_3 may be represented by points on a Plücker quadric Σ in S_5, the tangent lines Ω being given by the intersection V_3^4 of Σ with another quadric Σ' the special quadratic complex of tangents to Ω. Since each of these representations is free from singular elements, we see that

> the line elements on the Mobius plane may be represented in a (1, 1) manner by the points of a V_3^4 in S_5. The inversive group of the plane induces projective transformations in S_5 which carry the above variety into itself. (3·1)

The lines of the two regulii on the sphere Ω correspond to points on two conics Γ_1, Γ_2 whose planes are conjugate with respect to Σ. Since every tangent to Ω belongs to a pencil formed by two generators one from each regulus, the variety V_3^4 is traced by the lines joining in every way the points on the two conics. The points on each of these "generating lines" (lines in S_5 intersecting Γ_1, and Γ_2) correspond to line-elements through the same point which form what may be called a "star".

The quadric Σ has two systems of generating planes which correspond to lines in the same plane and to lines through a common point in S_3. The sections of Σ' by these two systems of planes[5] give us two systems of conics on V_3^4 whose points correspond respectively to line elements of the same circle, and these line elements turned each through one right angle (a $\pi/2$ turbine). From the known relations between the points in a plane and circles or $\frac{\pi}{2}$ turbines, we infer the following properties of the two systems of "generating conics" of V_3^4. The first of them, for example, follows from the fact that two circles have only two points in common, and hence have common line elements with only two stars.

> Two conics belonging to the same or to different systems are met in general by two generating lines. (3·2)

> Three generating lines are met by one conic of the first system and by one conic of the second system. (3·3)

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[5] Σ is well defined but not Σ' which may be any number of the pencil $\Sigma + k\,\Sigma'$. However, all these cut a generating plane of Σ in the same conic section.

The totality of all generating conics of either system passing through a given point P on V_3^4 generate the same quartic variety P_2^4, called the *turbinoid*[6] *of P*. The surface P_2^4 contains also the generator through P and is formed by the section of V_3^4 by the tangent prime to Σ at P. (3·4)

The points on the turbinoid correspond to the totality of all line elements on the Mobius plane which are concyclic with a given line element, namely the one represented by P. The corresponding tangent lines to Ω meet a fixed tangent line to Ω and thus belong to a special linear complex, whence the result (3·4).

More generally, the tangent lines of Ω which belong to any linear complex, correspond to a prime section of V_3^4. This section contains four generating lines corresponding to four stars at the vertices of a twisted quadrilateral of complex lines lying on Ω. All the four stars coincide in the case of a turbinoid.

If all the line elements of a circle on the sphere Ω be turned in the same sense through the same angle θ, we get a non-oriented semi-turbine. For brevity we shall refer to such a configuration as a θ-turbine. Since the tangents of the sphere corresponding to such line elements belong to a regulus, the representative points in S_5 lie on a conic. The line elements of the turbine associated with the same circle but with the angle $-\theta$ form lines of the complementary regulus, and correspond to a conic on another plane in S_5, the two planes being conjugate with respect to Σ. By varying θ but keeping the circle fixed we get an outer pencil of semi-turbines which are incident with the same " stars ". Interpreting this and the constancy of the angle we see that

there are ∞^4 generating conics on V_3^4 whose planes are the generating planes of the ∞^1 quadrics of the system $\Sigma + k\,\Sigma'$, and these correspond to the θ-turbines (including circles which are 0-turbines) on the Mobius plane ω. An outer pencil of turbines (obtained by varying θ) correspond to conics which meet the same set of generators. Any two of these conics along with Γ_1 and Γ_2 determine on each generator which they all meet, ranges which have the same cross ratio. (3·5)

It is to be noted that the planes of the four conics like those mentioned above are not in general position since there are only 3 lines which meet four arbitrary planes in S_5.[7] It is not difficult to verify that

[6] The use of the word " turbinoid " in this sense is due to Dr. Weiss.

[7] R. Vaidyanathaswamy, " On the number of lines which meet 4 regions in hyperspace," *Proc. Camb. Phil. Soc.*, **22**, p. 51.

The sections of V_3^4 by the ∞^3 generating planes of the same system of a particular quadric $\Sigma + k\Sigma'$ correspond to the totality of turbines with the same angle θ. The sections by the generating planes of the other system of the same quadric represent turbines whose angle is $\theta + \frac{\pi}{2}$. (3·6)

Besides generating lines and conics, the variety V_3^4 contains ∞^6 twisted cubics each of which is its residual section with any S_3 passing through a generating line. We shall call the corresponding system of line elements on the Mobius plane ω a *cubic series*. These are the contact elements determined by pairs of circles belonging one to each of two coaxal system, one of the limiting points of one system coinciding with one of the fixed points of the other, so that a star separates out as part of the locus. If this common point be identified with the " point circle at infinity " of the Mobius plane, we see that

> A set of line elements whose points lie on a circle and whose lines pass when produced through a fixed point on the circle constitute a cubic series. The general cubic series may be obtained from such particular cases by the operations of the inversive group. (3·7)

A cubic series of the particular type mentioned above (which is the residual locus of a star at ∞ on ω) is defined by a circle with a point marked out on its circumference. The circle is then said to be " cut " at that point. Hence the cubic series defined by a " cut circle " consists of line elements whose points lie on the circle and which are directed towards the point where it is cut.

4. *Projection of* V_3^4 *on a 3- space R*

We now project the V_3^4 on a 3-space R taking as the region of vertices a generating line cutting the conics Γ_1 and Γ_2 at, say, J_1 and J_2 respectively, and thus obtain a representation of the line-elements of ω by points in R. A plane (two-space) containing the line $J_1 J_2$ meets V_3^4 in 4 points of which 3 are absorbed by $J_1 J_2$ and one point Q stands out distinct. If the same plane meets R in a point q, we regard q as the projection of Q.

The conics Γ_1 and Γ_2 project into two lines γ_1 and γ_2 which are thus in projective correspondence with the two conics. To the points J_1 and J_2 thus correspond two points j_1 and j_2 which are the intersections of γ_1 and γ_2 with the planes through $J_1 J_2$ and tangents to the conics at J_1 and J_2. A generating line meeting Γ_1 and Γ_2 at P_1 and P_2 projects into a line meeting γ_1, γ_2 at p_1, p_2. When P_1 and P_2 tend to J_1 and J_2, the projection approaches $j_1 j_2$ as a limit. However, the correspondence between the points on $J_1 J_2$

and $j_1 j_2$ will depend on the manner of approach of P_1 and P_2 so that all we can assert is that

> the projection of a point J on $J_1 J_2$ is a point j on $j_1 j_2$ whose position depends on the manner in which J is approached. (4·1)

The (1, 1) character of the correspondence between V_3^4 and R breaks down also for points on γ_1 and γ_2. For if p_1 on γ_1 be the projection of P_1 on Γ_1, the plane $J_1 J_2 p_1$ cuts V_3^4 in $J_1 J_2$ and in the line $J_2 P_1$.

Thus

> Any point p_1 on γ_1 is the projection not only of a point P_1 on Γ_1 but also of all points on the line $J_2 P_1$. Similarly any point p_2 on γ_2 is the projection of all points on a line $J_1 P_2$. (4·2)

The ∞^4 generating conics of V_3^4 project into ∞^4 conics in R. Now each conic determines a (1, 1) correspondence between Γ_1 and Γ_2 by means of the generating lines which meet all the three conics. Let J_1, J_2 correspond to P_2 and P_1 in this correspondence. In R we have now a (1, 1) correspondence between γ_1 and γ_2 and hence the joins generate a quadric regulus containing, in particular, the lines $j_1 p_2$ and $j_2 p_1$. Also since the conic in S_5 meets $J_1 P_2$ and $J_2 P_1$, its projection in R passes through p_2 and p_1 by (4·2). Thus $j_1 p_1 = \gamma_1$, $j_2 p_2 = \gamma_2$, $j_1 p_2$, and $j_2 p_1$ form a twisted quadrilateral on the quadric, so that $p_1 p_2$ is the polar line of $j_1 j_2$. The conic in R is thus a section of the quadric by a plane through $p_1 p_2$, that is, by the polar plane of a point on $j_1 j_2$.

Hence

> The ∞^4 generating conics of V_3^4 representing the turbines (and in particular, the circles) of ω project into ∞^4 conics which are the sections of ∞^3 quadrics through γ_1, γ_2 by their polar planes w. r. t. the ∞^1 points on $j_1 j_2$. (4·3)

Now the points or stars, associated with the line elements of a turbine correspond to the generators of the quadric, and they are the same for all turbines of an outer pencil.

Hence

> The section of the same quadric through γ_1, γ_2 by the polar planes of different points on $j_1 j_2$ correspond to an outer pencil of turbines. (4·4)

Complementary to the above is the following result which we shall presently establish:

> The section of different quadrics through γ_1, γ_2 by their polar planes w.r.t. the same point j (θ) on $j_1 j_2$ are turbines with the same angle θ.

Hence

The planes of all such conics cut $j_1 j_2$ at a fixed point $j' (\theta)$ the harmonic conjugate of $j (\theta)$ w.r.t. j_1 and j_2. (4·5)

A turbine is determined by its outer circle and the angel θ ; we thus have the striking result that the quadric determines the former and the plane of section, the latter.

A star is a turbine whose outer circle is of zero radius, and whose angle θ is indeterminate. Hence among the system of generating planes of $\Sigma + k\Sigma'$ where $k = k (\theta)$ all of which cut V_3^4 in θ turbines [*vide* (3·6)], there will be one through each generating line of V_3^4. These planes have the property that all their 4 intersections with V_3^4 are absorbed by the generating line through which they pass, unlike other planes which have one residual intersection. For, the 4 intersections of a general plane in S_5 with V_3^4 represent the 4 generators of an arbitrary regulus which touch the sphere Ω. When the regulus degenerates into two plane pencils, one of which is a tangent pencil to Ω at say A, and the other has vertex B, the plane in S_5 passes through a generator corresponding to the star at A, and the residual intersection represents the tangent in the pencil with vertex B other than BA. Now the line elements of a turbine on Ω define a regulus of a quadric having ring contact with Ω and so a turbine will contain a star only when this ring contact quadric breaks up into two planes. But in such a case the planes are coincident and a residual line not belonging to the star does not exist. Hence the property stated above.

Let p be an arbitrary generating plane of $\Sigma + k\,\Sigma'$ of that system whose section by V_3^4 gives θ turbines, and p_j the particular generating plane which passes through $J_1\, J_2$. Since p_j has no residual intersection with V_3^4 outside $J_1\, J_2$ it follows by (4·1) that it meets $j_1\, j_2$ in a point $j' (\theta)$. Also p and p_j meet since they are generating planes of the same system of a quadric in S_5, and the projection of this point is j' since all points on p_j project into j'. It follows, therefore, that the projecion of p is a plane through $j' (\theta)$ which proves (4·5).

The range of points $j (\theta)$ or $j' (\theta)$ are in projective correspondence with the quadrics $\Sigma + k\,\Sigma'$. The points j_1 and j_2 correspond to the turbines formed by the generators of Ω, *i.e.*, the turbines whose ring contact quadric coincides with Ω.

I shall use the word " T-conics " to denote those conics in the representative 3-space R which correspond to turbines on the sphere Ω or the inversive plane ω. The particular T-conics which represent circles will be called " C-conics ".

On any particular plane in R there are ∞^1 T-conics which have double contact with one another at the points where the plane meets γ_1 and γ_2. The pole of this chord of contact lies on $j_1 j_2$. (4·6)

For, if the plane meets $j_1 j_2$ in j', these conics are by (4·3) the sections of quadrics through γ_1, γ_2 by their polar planes w. r. t. j, the harmonic conjugate of j' w.r.t. j_1 and j_2.

For using the notation of (4·5), if the given plane cuts γ_1, γ_2 in p_1, p_2 it will be the polar plane of the point j only when the quadric passes through $j_1 p_2$ and $j_2 p_1$, in addition to $j_1 p_1 = \gamma_1$ and $j_2 p_2 = \gamma_2$. The pole of $p_1 p_2$ for all such sections is j'.

5. By selecting a particular generator $J_1 J_2$ as the axis of projection we have singled out a particular point J on Ω and hence also a point on ω for special notice. We identify this with the " point circle at infinity " of the Mobius plane.

A straight line in general position in R is the projecion of the section of V_3^4 by a 3-space through the line and $J_1 J_2$, *i.e.*, of a twisted cubic which is the residual of a " star at infinity ".

Hence

A straight line in general position in R corresponds to the line elements of a cut circle. If the line meets $j_1 j_2$, at j' (θ) the line elements cut a straight line at a constant angle θ. (5·1)

We have already seen that any quadric through $\gamma_1 \gamma_2$ defines a circle on ω. (5·1) enables us to interpret the two systems of generating lines of such a quadric.

One system of generating lines of a quadric surface through γ_1 and γ_2 correspond to stars incident with a particular circle. The other system of lines represent cubic series obtained by the same circle cut at any of its points. (5·2)

Again consider a conic in R which represents the line elements of a circle on ω. The plane of this C-conic passes through j' (0). The line joining any two points on the conic corresponds to the line elements of a cubic series in ω determined by any two line elements of the circle. They belong to a cut circle passing through the centre C of the given circle and cut at the diametrically opposite point. The line element at any point P will be found to be perpendicular to CP. By forming all cubic series by combining pairs of elements of the circle, we get all line elements corresponding to points on the plane.

Hence

The line elements on ω corresponding to points on a plane in R meeting $j_1 j_2$ in j' (0), belong to a system of concentric circles. The conics of (4·6) correspond to the several circles, and the congruence line $p_1\ p_2$ to the star at the common centre. (5·3)

It can be similarly proved that

If all the line elements of a system of concentric circles on be turned through the same angle θ they correspond to points on a plane cutting $j_1\ j_2$ in $j'\ (\theta)$. (5·4)

If, however, a plane in R passes through $j_1\ j_2$, it is the section of R by the turbinoid of a point on $J_1\ J_2$. On Ω it represents line elements concyclic with a particular line element through the "point at infinity".

Hence,

A plane through $j_1\ j_2$ represents a field of parallel line elements. In particular, the planes through $j_1\ j_2\ \gamma_1$ and $j_1\ j_2\ \gamma_2$ corresponding to isotropic line elements. (5·5)

6. *Real Representations in a Semi Metric R*

We may, without loss of generality, choose a system of homogeneous co-ordinates ξ_r $(r = 0, 1, 2, \cdots 6)$ in S_5 so that points on the conic Γ_1 (corresponding to one system of generators of Ω) may be represented by $(\rho_1\lambda^2,\ \rho_1\lambda,\ \rho_1,\ 0,\ 0,\ 0)$ and points on the conic Γ_2 corresponding to the other system by $(0,\ 0,\ 0,\ \rho_2,\ \rho_2\mu,\ \rho_2\mu^2)$. The quadrics $\Sigma + k\ \Sigma'$ are thus given by $k_1\ (\xi_0\xi_2 - \xi_1^2) + k_2\ (\xi_3\ \xi_5 - \xi_4^2) = 0$, and we may identify Σ with the quadric $k_1 = k_2$, Σ' corresponding to an unspecified member of the system.

Any line element on Ω through the point (λ, μ) is specified by the ratio $d\lambda/d\mu$. Hence the variety V_3^4 is given by $(\rho_1\lambda^2 d\ \mu,\ \rho_1\lambda d\ \mu,\ \rho_1 d\ \mu,\ \rho_2 d\lambda,\ \rho_2\mu\ d\lambda,\ \rho_2\mu^2 d\ \lambda)$. On every generating line the points corresponding to the values 0 and ∞ for $d\lambda/d\mu$ are known. The projective metric on the line will be completely fixed by giving say, the point for which $d\lambda/d\mu = 1$. We take this to be the point where the prime $\xi_2 = \xi_3$ cuts it, so that we have $\rho_1 = \rho_2$. This is equivalent to taking a parallel field of elements on ω, the turbinoid of the point $(1,\ 0,\ 0,\ 0,\ 0,\ -1)$.

The points on Γ_1 and Γ_2 corresponding to infinite values of the generator co-ordinates λ and μ are $(1,\ 0,\ 0,\ 0,\ 0,\ 0)$ and $(0,\ 0,\ 0,\ 0,\ 0,\ 1)$ and these we identify with J_1 and J_2. We project the point on V_3^4

$$(\xi_0,\ \xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \xi_5) = (\lambda^2 d\mu,\ \lambda d\mu,\ d\mu,\ d\lambda,\ \mu d\lambda,\ \mu^2 d\lambda) \qquad (6\cdot1)$$

from $J_1\ J_2$ as axis on the 3-space $\xi_0 = 0$, $\xi_5 = 0$ which we identify with R, and obtain the point

$$(\xi_1,\ \xi_2,\ \xi_3,\ \xi_4) = (\lambda d\mu,\ d\mu,\ d\lambda,\ \mu d\lambda) \qquad (6\cdot2)$$

and these we take to be the homogeneous co-ordinates of the representative point in R. We may identify the generator co-ordinates λ, μ on Ω with the isotropic co-ordinates $y - ix$ and $y + ix$ on ω thus converting the latter into a double binary domain. To get a representation of real line elements by real points in R, we perform a transformation of co-ordinates from ξ_r to X_r thus getting

$$\left.\begin{aligned} X_1 &= (\xi_1 + \xi_4)/2 = (\lambda d\mu + \mu d\lambda)/2 = x\,dx + y\,dy \\ X_2 &= (\xi_1 - \xi_4)/2i = (\lambda d\mu - \mu d\lambda)/2i = ydx - xdy \\ X_3 &= (\xi_2 + \xi_3)/2 = (d\mu + d\lambda)/2 = dy \\ X_4 &= (\xi_2 - \xi_3)/2i = (d\mu - d\lambda)/2i = dx \end{aligned}\right\} \tag{6·3}$$

as the homogeneous co-ordinates of the line element determined by the point (x, y) and the direction dy/dx on the Cartesian plane.

If we prefer to use non-homogeneous co-ordinates we may take

$$X = x + yp,\ Y = y - xp,\ Z = p, \text{ where } p = dy/dx. \tag{6·4}$$

These give

$$x = \frac{X - YZ}{1 + Z^2};\ y = \frac{XZ + Y}{1 + Z^2};\ p = Z. \tag{6·5}$$

The points j_1, j_2 are found to be given by $(X_1, X_2. X_3, X_4) = (1, -i, 0, 0)$ and $(1, i, 0, 0)$ respectively so that $j_1\ j_2$ is the line $X_3 = X_4 = 0$ (the " line at infinity " on $Z = 0$). The lines γ_1, γ_2 are given by $X_2 + i\,X_1 = 0$, $X_3 - iX_4 = 0$, and $X_2 - iX_1 = 0$, $X_3 + iX_4 = 0$; or by $Y = \mp iX$, $Z = \pm i$.

We have in R three significant lines $j_1\ j_2, \gamma_1, \gamma_2$. We take the first of these as a " line at infinity " and j_1 and j_2 as the absolute points on it. Thus any plane $Z =$ const. is Euclidean. However, the choice of $X_4 = 0$ as the " plane at infinity " in the X, Y, Z co-ordinates (6·4) is arbitrary, and there is no Absolute conic in this plane, unless we select a particular quadric through γ_1, γ_2 and hence also a particular circle on ω for special notice. Thus our space is semi-metric in the sense that metrical ideas apply only to the planes $Z =$ constant.

It will be shown later that the inversive group of ω induces a cubo-cubic Cremona space transformation of R into itself so that projective concepts in R have no invariant significance. However, since the two systems of generating lines on Ω are transformed into themselves or into each other, it follows that

> The inversive group of ω induces projective transformations of $\gamma_1\ \gamma_2$ into themselves or each into the other. Any transversal of $\gamma_1\ \gamma_2$ is transformed into another transversal, and the two are in projective correspondence. (6·6)

The last statement merely expresses that stars are inverted into stars. Other straight lines in R will, in general, be transformed into twisted cubics.

The equations (6·3) and (6·4) enable us to verify the truths of the results already proved. Thus the line elements of the concentric system of circles given by $(x-a)+(y-b)p=0$ are represented by points on $X-bZ-a=0$, a plane parallel to the Y-axis. Hence the point at infinity on the Y-axis (which we shall denote by Y_∞) is the point j' (0), and the point at infinity on the X-axis (X_∞) is j (0). A line through j (0) is given by $Y=c$, $Z=m$, *i.e.*, by $y=xm+c$, $p=m$ which are the line elements of a straight line. Equations (5·4) show that the line of the congruence corresponding to the star at (x, y) is given by

$$\frac{X-x}{y}=\frac{Y-y}{-x}=Z$$

which cuts $Z=0$ in $(x, y, 0)$.

Hence,

Let us identify the plane ω with $Z=0$ representing a point on it by (a, b) or $(a, b, 0)$ according as we conceive of it as belonging to the former or the latter. The star at any point P (x, y) is then represented by the congruence line through P $(x, y, 0)$. A particular line element is located by the additional relation $Z=p$. (6·7)

The spatial co-ordinates of a line element may be obtained geometrically thus: its X co-ordinate is the intercept of the normal to the element on the X-axis, its Y-co-ordinate is the intercept which the line of the element makes on the Y-axis, and its Z is p.

7. *The Hart-Circle and Miquel-Clifford Configurations in Line Element Geometry*

The considerations of the preceding paragraphs provide us with a principle of transference by which any configuration in the inversive plane may be associated with a corresponding configuration in R. It is only necessary to remember that to a point corresponds a "congruence line" meeting γ_1 and γ_2, the angle between two line elements corresponding to the "separation" of the two points with reference to the intersections with γ_1 and γ_2 as absolute points. Also that to a circle corresponds a C-conic and that

a C-conic is one which meets γ_1 and γ_2 and has its tangents at these points parallel to the Y-axis. (7·1)

If we apply the Principle of Transference to a Hart System we have the result :—

It is possible to find in R two tetrads of C-conics such that every conic of one tetrad intersects all the conics of the other tetrad.

In the same manner, the Miquel-Clifford Configuration regarded as one in the inversive plane gives rise to the following :

For every positive integral value of n it is possible to find systems of 2^n C-conics and 2^n congruence lines with the property

(i) every congruence line is met by exactly $n+1$ C-conics;

(ii) every conic is met by exactly $n+1$ congruence lines; also,

(iii) the range of points determined on any congruence line by the conics which meet it, is projective with the range on any other congruence line.[8] (7·2)

8. *The Cubo-Cubic Transformations in R induced by the Inversive Group on the Plane*

The effect of inversion with respect to a given circle is to replace every star on ω by a star at the inverse point. Corresponding line elements are those which lie on the same circle. Hence the effect of the inversion on R is firstly to replace a congruence line by another congruence line, such that their intersections with $Z=0$ are inverse points. A C-conic which meets both the congruence lines cuts out corresponding points on them.

Let us examine how planes in R are transformed. A plane in R is the projection from $J_1 J_2$ of the section of V_3^4 by the 4-space through the plane and J_1 and J_2. Since inversion is equivalent to a projective transformation of V_3^4 into itself, the section is carried over into another section of V_3^4 by a 4-space through another generator $Q_1 Q_2$. This section is a V_2^4 which meets all generators and, in particular, $J_1 J_2$. Hence its projection from $J_1 J_2$—the transform of the original plane—is a cubic surface through $q_1 q_2$. This last line represents the transform of the star at the "point at infinity" *i.e.*, the star at the centre of inversion. If the centre of inversion be $(0, 0)$ the line $q_1 q_2$ is the Z axis.

Geometrical considerations lead to the conclusion that the cubic surfaces pass through the lines γ_1, γ_2. For, equations (6·4) show that all line elements of the same isotropic line on the projective plane ω correspond to the same point in R, while line elements of different isotropic lines correspond to the several points on γ_1 or γ_2. Now, among the line elements of a concentric system of circles there are only two which are isotropic, since all such circles touch at the circular points. These two correspond, in fact, to the two intersections of the representative plane in R with γ_1 and γ_2. On inversion, we get the line elements of a co-axial system of circles and these include line elements belonging to all the isotropic lines in the plane. Hence the result. We shall

[8] *Vide* "The Miquel-Clifford Configuration in the Geometries of Mobius and Lagurre", *Annamalai Univ. Jour.*, 7, 6–12.

see later that the cubic surfaces touch along γ_1 and γ_2 so that these count each as two lines.[9]

A line in R is the projection of a section of V_3^4 by the 3-space through it and $J_1 J_2$, that is of a twisted cubic on V_3^4 meeting $J_1 J_2$ twice. This is carried over into another twisted cubic which meets $Q_1 Q_2$ —the transform of $J_1 J_2$— but which will not, in general meet, $J_1 J_2$. Hence a straight line transforms into a twisted cubic meeting $q_1 q_2$. The transformations of R induced by the inversive group of ω is thus cubo-cubic.

Analytically, the effect of inversion in say, $x^2 + y^2 = k^2$ is to replace the line element x, y, p by $k^2 x/(x^2 + y^2)$, $k^2 y/(x^2 + y^2)$, p'

where
$$p' = \frac{(x^2 + y^2)\,p - 2y\,(x + yp)}{(x^2 + y^2) - 2x\,(x + yp)}\,.$$

Hence if X_1', X_2', X_3', X_4' are the co-ordinates of the transform of $X_1\ X_2\ X_3\ X_4$, we have,

$$\left.\begin{aligned} \rho\, X_1' &= k^2 X_1\,(X_3^2 + X_4^2) \\ \rho\, X_2' &= -\,k^2 X_2\,(X_3^2 + X_4^2) \\ \rho\, X_3' &= [\,(X_1^2 - X_2^2)\,X_3 + 2X_1\,X_2\,X_4] \\ \rho\, X_4' &= X_1^2 - X_2^2 - 2X_1\,X_2\,X_3. \end{aligned}\right\} \qquad (8\cdot 1)$$

It will be found that the homoloidal family of cubic surfaces pass through the lines $X_1 = X_2 = 0$ and through $X_3 = X_4 = 0$ and touch the planes $X_3 = \pm\, iX_4$ along the lines γ_2 and γ_1 respectively. The F-element is thus a sextic curve which breaks up into 6 lines, of which two coincide with γ_1 and two with γ_2. The four points where these lines meet are obviously singular points for the cubic surfaces.

The equations of transformation take a simpler form when referred to the imaginary co-ordinates

$$\xi_1' : \xi_2' : \xi_3' : \xi_4' = k^2\,\xi_2\,\xi_3\,\xi_4 : \xi_1\,\xi_4^2 : \xi_1^2\,\xi_4 : k^2\,\xi_1\,\xi_2\,\xi_3 \qquad (8\cdot 2)$$

A pair of points in R have an absolute invariant under the Cremona Group, and hence there is a sort of a metric though not of the usual type. For two points on the same congruence line, it is the " separation " mentioned in the beginning of Para 6 and corresponds to the angle between the line elements of the same star. If the points are not on the same congruence line, the invariant in question is the " separation " of either point from the intersection of the turbinoid of the other with the congruence line through it. On ω it is the angle between either line element and the circle through its point and the other line element. It may also be interpreted as twice

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9 A line along which the surfaces of a homoloidal web touch a plane counts as two F-lines. *Vide* Hudson, *Cremona Transformations* (1927), p. 248.

the angle of the unique turbine which contains both the line elements. It will be obvious that

(i) the locus of line elements on ω with a zero "separation" from a given line element is the totality of elements concyclic with it—the turbinoid of that element;

(ii) while the "separation" of line elements of different stars is unique even when approached as a limit, the "separation" of elements of the same star becomes indeterminate as a limit.

9. *Analogous Results in the Geometry G_k*

In my paper (T.G. II), I have worked out the details of a generalisation of Mobius Geometry in which non-oriented circles of a fixed radius k (called k-circles) play the same role as do point circles in the former. The totality of k-circles may be represented by points on a quadric Ω' which touches Ω at its point at infinity J. By stereographic projection from J we may establish a one-one correspondence between the Mobius plane ω and the plane ω' of G_k geometry (supposed superposed) in which each point P on ω corresponds to the circle with centre P and radius k on ω'. To a line element of ω correspond two neighbouring k-circles and these cut in 2 points at a constant distance $2k$. We shall refer to such a configuration of two points as a "rod". Thus

In the correspondence between Mobius- and G_k-geometries, a line element of the former corresponds to a "rod" perpendicular to the element and whose centre coincides with the point of the element. (9·1)

Just as tangent lines to Ω define line elements by means of co-axal systems of circles which all touch, so also a tangent line to Ω' defines a rod by means of a co-axal system of circles whose common points are the extremities of the rod. Thus rod-geometry in G_k is isomorphic with the geometry of tangent lines to Ω' under the projective group. We may thus reinterpret all our results taking V_3^4 as the representative space of line elements on Ω' instead of on Ω. I shall content myself with giving only the results, since they follow readily from the results already established in this paper and the results in T.G. II.

No.	Representative 3-space with an invariant quadric Q (Ω or Ω')	Mobius Plane	G_k Plane
1	Point P on Q	Point circle P	k-circle P
2	Point on tangent plane at P	Proper circle ($r \neq 0$) through P	Proper circle ($r \neq k$) bisecting P
3	Line element on Q	Line element	" Rod "
4	Angle between line elements of the same star (invariant)	Angle between line elements at a point (invariant)	Angle between two rods with the same centre (invariant)
5	Corresponding tangents in an automorphic involution of Q meet	A line element and its inverse are concyclic	The extremities of a rod and its G_k inverse are concyclic
6	Turbine on Q	Turbine	Rods with centres on a circle and cutting it at a constant angle
7	Line elements belonging to a linear complex	Line elements of a co-axal system of circles	Rods touching the circles of the orthogonal co-axal system at their middle points
8	Special linear complex of lines meeting a given tangent line	Turbinoid of a given element	Rods whose extremities are concyclic with those of a given rod

In conclusion it may be added that by fixing the order in which the two points which define a rod are to be taken we have a geometry of oriented rods analogous to the geometry of oriented line elements on a plane or sphere.