STUDIES IN TURBINE GEOMETRY—IV
THE TOPOLOGY OF ORIENTED AND NON-ORIENTED LINE ELEMENTS IN THE INVERSIVE PLANE

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THE inversive plane may be defined as the totality of all real points on the circumference of a circle and in its interior together with the inverses of the latter and an ideal " point at infinity " which is adjoined to correspond to the centre of the circle. Such a point set is carried over in a biuniform bicontinuous manner into itself by every operation of the inversive group, that is by inversion in a circle or by a succession of inversions.

The inversive plane ω differs from the Euclidean plane in being a closed point set, and from the projective plane in closure being effected by an ideal point and not by the adjunction of an ideal line. The totality of points in the interior of a circle gives a topological image of the Euclidean plane, while if the circumference be also included and diametrically opposite points considered identical, we have a set homeomorphic with the projective plane. To get a topological image of the inversive plane we shall have to consider all points on the circumference as identical.

By continuous deformation and coincidence of points considered identical, this may be transformed into the points on the surface of a sphere in 3-space, *i.e.*, a 2-sphere Ω. A homeomorphism may be established between the 2-sphere Ω and the inversive plane ω by stereographic projection, the centre of projection corresponding to the " point at infinity ".

As an arithmetical continuum the points of the Euclidean and projective planes may be defined as

(1) real number pairs x, y subject to $x^2 + y^2 < a^2$ (Euclidean plane).

(2) ratios $x : y : z$ of real number triads not all 0 (projective plane).

The inversive plane may be specified in a similar manner by

(3) the ratios $x^2 + y^2 : xz : yz : z^2$ where x, y, z are real numbers not all zero.

By making the points defined by $x : y : z$ in (2) and (3) correspond, we have a correspondence between the projective and inversive planes which is (1–1) except for points on the line $z = 0$ of the former all of which correspond to (1, 0, 0, 0) of the latter.

2. Stereographic projection provides more than a mere mode of establishing a correspondence between ω and Ω. The transformations of the inversive group are carried over into projective transformations carrying Ω into itself and hence the inverse group into the automorphic collineation group of Ω in the projective space S_3. This is a mixed 6 parameter group G'_6 consisting of two continuous systems of transformations G_6, H_6. One system G_6 corresponds to an even number of inversions and carries each oriented circle into another circle oriented the same way, while the transformations of H_6 correspond to an odd number of inversions and reverse the orientation. They correspond respectively to the homeographies and antigraphies of the inversive plane, each general homeography being expressible as the product of 4 inversions and each antigraphy as the product of 3 inversions.[1] The homeographic transformations G_6 constitute a sub-group of G'_6.

3. *Oriented Line Elements of the Inversive Plane*

The totality of line elements of the inversive plane constitute a 3-dimensional aggregate whose topological character differs according as they are considered oriented one way or oriented both ways, *i.e.* (non-oriented).[2] In either case each line element of the inversive plane corresponds by stereographic projection to a line element and hence to a tangent line to the sphere Ω.

Locally, the line elements in a small region constitute a product manifold—that of a circle and a 2-cell. The outstanding characteristic of the totality of all line elements in the large is that *it is not a topological product.* This statement is equivalent to the result that it is not possible to associate with each point of the sphere a line element, oriented or otherwise, so that they vary in a continuous manner.[3] Were such a covering of the sphere

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[1] *Vide* Morley : *Inversive Geometry*, p. 64.

For a discussion of the invariant theory of the inversion group by quaternary and double binary methods reference may be made to an important paper on the subject by E. Kasner : *Transactions of the American Mathematical Society*, Vol. I, pp. 430–98.

[2] "Studies in Turbine Geometry—I," *Jour. Ind. Math. Soc.*, pp. 97, (1.1).

[3] This result is due to Poincare. Bronwer extended it to all n-spheres for which n is even and showed that it fails when n is odd. Thus on the 3-sphere $x_0^2+x_1^2+x_2^2+x_3^2 = 1$ a continuous vector distribution is given by means of the components (x_1, x_0, x_3, x_2) where x is a point on the sphere. Alexandroff : "Topologie I," p. 481.

with line elements possible, we should be able by continuous rotation of all the elements to represent all line elements on the sphere as the topological product of the 2-sphere and a circle. In fact in the case of no two dimensional variety except the anchor ring, which is itself a topological product, do the totality of line elements constitute a topological product.[4]

If the line elements be oriented, we may fix one of them l as reference initial element. Every other element m may be obtained from l by a unique rotation of the sphere about a suitably chosen diameter through an angle θ not exceeding two right angles. This rotation may be represented by a vector of length θ drawn from the centre O along that diameter one way or the other according to the sense of the rotation. We associate with the line element l itself the centre O of the sphere. We thus have a representation of all oriented line elements of the sphere by points in the interior and on a sphere of radius π. Diametrically opposite points of this sphere must be considered identical since a rotation about a diameter through $+\pi$ or $-\pi$ carries l into the same element. Hence the totality of all oriented line elements of Ω is homeomorphic with a projective manifold of 3-dimensions. An actual representation of the totality of oriented line elements of the inversive plane on a projective 3-space, or what is its equivalent, a representation of the rotations of a sphere about its centre by means of 4-homogeneous parameters may be obtained by means of the quaternion formula :

$$x' = a^{-1} x a$$

where x, x' are vectorial quaternions representing the old and new positions of the point and $a = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ the quaternion which determines the rotation. All the a's are real numbers and it will be seen that the transformation $x \rightarrow x'$ is determined by the ratios $a_0 : a_1 : a_2 : a_3$, *vide* E. A. Weiss : *Einfuhrung in die Liniengeometrie und Kinematik*, p. 96.

There are two types of fundamental closed curves in projective space—the straight line and the circle. In the first case closure is dependent on the identification of diametrically opposite points on the bounding sphere, while in the second it is independent of such identification. The former corresponds to a continuous succession of line elements in which the total rotation of the element about its middle point is 2π such as a star or a turbine. The latter corresponds to a succession of elements in which the total rotation is zero.

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[4] *Vide* W. Threlfall: "Raume aus Linienelemente," *Jahresbericht der Deutschen Mathematiker Vereinigung*, 1933, **42**, p. 108.

4. *Non-Oriented Line Elements in the Inversive Plane*

The totality of non-oriented line elements of the inversive plane or of the 2-sphere are represented twice over on the oriented line elements of Ω. These latter correspond, as we have seen, to points on a projective 3-space which again is covered twice over by points on a 3-sphere. Since the fundamental group of a 3-sphere is the identity, we may take it to be the universal covering manifold (Universelle Überlagerungsmannigfaltigkeit). The fundamental group of M is thus of order 4. From the finiteness of the fundamental group it follows that the closed manifold M is orientable.[5]

Each non-oriented line element of the sphere Ω in a projective 3-space S_3 determines a tangent line to the sphere. By means of Plücker's line coordinates p_{ij} we represent all lines in S_3 by points in a quadric Q of signiture zero in a projective S_5 and, in particular, the tangent lines of Ω by the section of Q with another quadric Q′ corresponding to a special quadratic complex. The intersection variety V_3^4 in S_5 is a topological image of the manifold M.

5. *Fibred Spaces*

Both the non-oriented and oriented line elements of a 2-sphere constitute varieties known as fibred spaces, the fibres corresponding to the line elements with the same point (stars). For we have here closed 3-dimensional varieties whose points group themselves into closed curves—the fibres. Through each point passes a unique fibre and each fibre possesses a neighbourhood of fibres, *i.e.*, a subset of fibres which may be represented as the topological product of a 2-cell and a circle. The variety of decomposition or base variety of the fibration, *i.e.*, the variety obtained by taking each fibre as an element is the 2-sphere Ω itself.[6]

The problem of determining all the fibred spaces of n-dimension whose variety of decomposition of dimension d is known does not appear to have been solved in general. However, when $n = 3$ and $d = 2$ which is our special case the problem can be resolved completely. We have an infinity of fibred spaces with a 2-sphere for the variety of decomposition each of which is completely characterised by an integer $b > 0$. For $b = 0$ it is the topological product of the 2-sphere and the circle. For $b > 0$ we have[7] a

[5] Threlfall : "Aufgabe 135," *Jahresbericht d. Deutschen Matth. Vereinigung.*

[6] For discussion of fibred spaces in which the fibres are of dimensions higher than unity reference may be made to H. Whitney, "Sphere Spaces," *Proc. of the National Academy of Sciences of the U.S.A.*, Vol. 21, No. 7.

[7] Threlfall : "La Topologie der varieties," *L Enseignement Mathematique*, 1936, p. 251.

lenticular space whose coefficient of torsion is b, and whose fundamental group is cyclic and of order b.

Such a lenticular space[8] is bounded by two equal spherical caps intersecting in a circular edge C. The circumference of C is divided into p equal segments. Each point on either of the bounding caps is identified with the point of the other obtained by a reflection in the plane of C and a rotation about the axis of C through an angle $2\pi q/p$ where $0 \leqslant q \leqslant p/2$ is an integer prime to p. The p vertices, edges and faces are all identical for the resulting polyhedron so that its Euler characteristic is 0. For $p = 1$ and $p = 2$ we obtain respectively a 3-sphere and a projective 3-space. The lenticular space obtained when $p = 4$, $q = 1$ may be identified with the totality of all non-oriented line elements of the inversive plane. Its torsion coefficient is 4. Its Betti numbers p^i are connected by the relation

$$-p^0 + p^1 - p^2 + p^3 = \text{Euler-Poincare Characteristic} = 0,$$

$p^0 = 1$ for a connected manifold, while $p^3 = 1$ since the manifold is orientable. Hence $p^1 = p^2$ both being zero in this case.

6. The actual representation of the non-oriented line elements of Ω on such a lenticular space may be effected as follows : We take a fixed oriented reference line element and associate with every other line element of Ω a quaternion

$$x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$$

as in para 3 but of unit norm., so that

$$\Sigma : \ x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$$

diametrically opposite points of Σ being considered identical. The reference element corresponds to $\pm e_0$. A rotation through π (*i.e.*, a reflection in the axis of rotation) is represented by a quaternion whose scalar component x_0 vanishes. The rotation which reverses the direction of the reference element is of this type and may be taken to be $\pm e_1$. The product $e_1 x$ corresponds to a reversal of the orientation of the line element and then a rotation x. Hence it corresponds to the same element as x but with its orientation reversed. Since our line elements are not oriented we identifiy x and $e_1 x$. Since e_1 is cyclic and of order 4 we have tetrads of identical elements x, $e_1 x$, $-x$, $-e_1 x$.

Now $$e_1 x = -x_1 e_0 + x_0 e_1 - x_3 e_2 + x_2 e_3$$

so that the transformation $x \to e_1 x$ is equivalent to a simultaneous rotation of the $x_0 x_1$ plane and of the $x_2 x_3$ plane about the origin through one right

[8] Siefert-Threlfall : *Lehrbuch. der Topologie*, p. 210.

angle. We project Σ stereographically on the 3-space $x_1 = 0$ from the vertex (1, 0, 0, 0). The projection of the circle on the $x_0 x_1$ plane is the x_1 axis which we now endow with an elliptic metric in which the separation of two points is measured by the angle between the two corresponding points on the $x_0 x_1$ circle. A simultaneous rotation in the $x_0 x_1$ and $x_2 x_3$ planes is now projected into a kind of helicoidal motion in the 3-space. The lenticular region is one whose base surface is carried over into the other by this movement.[9]

[9] Trelfall : *loc. cit.*, p. 245—7.