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# RAY THEORIES FOR HYPERBOLIC WAVES, KINEMATICAL CONSERVATION LAWS (KCL) AND APPLICATIONS

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Ray theory, for the construction of the successive positions of a wavefront governed by linear hyperbolic equations, is a method which had its origin from the work of Fermat (and is related to Huygen's method). However, for a nonlinear wavefront governed by a hyperbolic system of quasilinear equations, the ray equations are coupled to a transport equation for an amplitude of the intensity of the wave on the wavefront and some progress has been made by us in its derivation and use. We have also derived some purely differential geometric results on a moving curve in a plane (surface  $\in I\!R^3$ ), these kinematical conservation laws are intimately related to the ray theory. In this article, we review these recent results, derive same new results and highlight their applications, specially to a challenging problem: sonic boom produced by a maneuvering aerofoil.

Key Words: Ray theory; kinematical conservation laws; nonlinear waves; conservation laws; shock propagation; curved shock; bicharacteristic lemma; shock dynamics; sonic boom; Cauchy problem; hyperbolic and elliptic systems; Fermat's principle

1. RAY EQUATIONS AND FERMAT'S PRINCIPLE

Waves involve transfer of energy from one part of a medium to another part, usually without transfer of material particles [5]. When we take such a general definition of waves, we may not be in position to identify some special propagating surfaces which we shall like to call wavefronts (propagating with finite speeds). Identification of a wavefront requires an approximation: there is a more rapid change in the state of the medium as we cross the wavefront transversely compared to more gradual changes in the state, which is already present prior to the onset of the wave, or when we move

along the wave front (PP - $\S3.21$ )<sup>1</sup>. Thus, when we encounter a wave we can see a short wavelength variation in the state of the system at a given time or a high frequency variation with respect to time at a fixed point. Let us denote a wavefront at a given time t by  $\Omega_t$ , then as time t changes  $\Omega_t$  occupies different positions in (x)-space. Identification of a wavefront  $\Omega_t$  implies finite speed C of propagation of  $\Omega_t$ . Let  $\Omega_t$  be represented by

$$\Omega_t: \phi(\mathbf{x}, t) = 0, \ \mathbf{x} \in \mathbb{R}^m, t \in \mathbb{R}$$
(1.1)

then

$$C = -\phi_t / | \nabla \phi | . \tag{1.2}$$

Evolution of  $\Omega_t$  is given with the help of a ray velocity  $\chi$  and this ray velocity can be obtained only when the nature of the curve  $\Omega_t$  is known i.e., by the dynamics of the curve. For example when  $\Omega_t$  is a crest line of a curved solitary wave on shallow water,  $\chi$  is given by water wave equations and boundary conditions on the surface of the water and the bottom surface [1].

Position of the surface  $\Omega_t$  can be obtained from  $\Omega_{t_0}$   $(t > t_0)$  as the locus of the tips P of rays starting from points  $P_0$  on  $\Omega_{t_0}$  and moving with the ray velocity  $\chi$  i.e.,  $\frac{d\mathbf{x}}{dt} = \chi$  (see equation (1.6a) below). The ray velocity  $\chi$  at any point  $\mathbf{x}$  of  $\Omega_t$  depends also on the unit normal  $\mathbf{n}$  of  $\Omega_t$  at  $\mathbf{x}$ . Thus  $\chi = \chi(\mathbf{x}, t, \mathbf{n})$ . The velocity C of  $\Omega_t$  is the normal component of  $\chi$  i.e.,

$$C = \langle \mathbf{n}, \boldsymbol{\chi} \rangle, \quad \mathbf{n} = \nabla \phi / | \nabla \phi |.$$
 (1.3)

Using (1.2) and (1.3) we get the eikonal equation

$$\phi_t + \langle \boldsymbol{\chi}, \nabla \phi \rangle = 0 \tag{1.4}$$

which is a first order nonlinear partial differential equation giving successive positions of  $\Omega_t$  as time evolves.

**Theorem 1** — In order that the vector  $\boldsymbol{\chi}(\mathbf{x}, t, \mathbf{n}) = (\chi_1, \chi_2, \dots, \chi_m)$  qualifies to be a ray velocity, it must satisfy a consistency condition [22]

$$n_{\beta}n_{\gamma}\left(n_{\beta}\frac{\partial}{\partial n_{\alpha}} - n_{\alpha}\frac{\partial}{\partial n_{\beta}}\right)\chi_{\gamma} = 0, \text{ for each } \alpha = 1, 2, \cdots, m$$
(1.5)

*Note:* A repeated Greek index implies sum over the range  $(1,2,\dots,m)$ . Later we shall encounter repeated subscripts i, j, k for which the range of summation will be  $(1,2,\dots,n)$ .

PROOF: Derivation of this condition is simple when we note that the left hand side of (1.5) appears as an additional term on the right hand side of (1.6a) below. This additional term is inconsistent with the statement that  $\chi$  is the ray velocity and hence must vanish. More explicitly the eikonal equation (1.4) is a first order nonlinear equation:  $\varphi_t + \langle \chi(\mathbf{x}, t, \nabla \varphi | | \nabla \varphi |), \nabla \varphi \rangle = 0$  for

<sup>&</sup>lt;sup>1</sup>We shall frequently refer to various sections of the book [21] by PP followed by a hyphen and the section number.

the function  $\phi(\mathbf{x}, t)$ . This equation is a Hamilton-Jacobi equation. The Charpit equations of this first order equation reduce to Hamilton's canonical equations, which when written for  $\mathbf{x}$  and  $\mathbf{n}$  instead of  $\mathbf{x}$  and  $\nabla \varphi$  give (1.6a,b) with the additional term on the right hand side of (1.6a). The theorem is proved.

Thus when condition (1.5) is satisfied, we have also derived the ray equations from the eikonal (1.4) in the form

$$\frac{dx_{\alpha}}{dt} = \chi_{\alpha},\tag{1.6a}$$

$$\frac{dn_{\alpha}}{dt} = -n_{\beta}n_{\gamma} \left(\frac{\partial}{\partial \eta_{\beta}^{\alpha}}\right) \ \chi_{\gamma} \equiv \psi_{\alpha}, \text{ say}$$
(1.6b)

where

$$\frac{\partial}{\partial \eta_{\beta}^{\alpha}} = n_{\beta} \frac{\partial}{\partial x_{\alpha}} - n_{\alpha} \frac{\partial}{\partial x_{\beta}}.$$
(1.7)

The derivatives  $\frac{\partial}{\partial \eta_{\beta}^{\alpha}}$  represent tangential derivatives on the surface  $\Omega_t$ . We may be tempted to ask a question: does a ray defined by (16a,b) starting from one point  $P_0 \in \mathbb{R}^m$  to another point  $P_1 \in \mathbb{R}^m$  chooses a path such that the time of transit is stationary with respect to small variations in the path? What we are asking is equivalence of rays given by (16a,b) to the rays satisfying Fermat's Principle. It is easy to see this equivalence (PP- § 3.2.5) for an isotropic propagation of a wavefront  $\Omega_t$ , in which case

$$\boldsymbol{\chi} = \mathbf{n} \ C \tag{1.8}$$

where C is independent of n. In this case (1.5) is automatically satisfied. It will be interesting to prove this equivalence for a general ray velocity  $\chi$  satisfying (1.5).

An important class of waves, called hyperbolic waves, appear in a medium governed by a hyperbolic system of partial differential equations. In this case every signal in the medium propagates with finite speed in a very strict sense: if the state of the system is perturbed at any time in a closed bounded domain, then the effect of the perturbation at any later time is not felt outside another closed bounded domain. The signal in such a domain may travel with a shock ray velocity  $\chi$  when the governing hyperbolic system is derived from a system of conservation laws. In the case of a curved solitary wave mentioned above, the crest line does represent a wave front propagating with a finite speed but the solitary wave is not a hyperbolic wave [5]. It is governed by the KdV equation.

Consider a hyperbolic system of n first order partial differential equations in m + 1 independent variables  $(\mathbf{x}, t)$ :

$$A(\mathbf{u}, \mathbf{x}, t)\mathbf{u}_t + B^{(\alpha)}(\mathbf{u}, \mathbf{x}, t)\mathbf{u}_{x_{\alpha}} + \mathbf{C}(\mathbf{u}, \mathbf{x}, t) = 0$$
(1.9)

where  $\mathbf{u} \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B^{(\alpha)} \in \mathbb{R}^{n \times n}$  and  $\mathbf{C} \in \mathbb{R}^n$ . Then the velocity C of a wavefront  $\Omega_t$ , across which  $\mathbf{u}$  is continuous, is equal to an eigenvalue c of (1.9) and  $\Omega_t$  is the projection on **x**-space of a section of the characteristic surface  $\Omega$  by a t = constant plane. Note that  $\Omega$  is a surface

in space time i.e  $(\mathbf{x}, t)$  - space. This follows from the high frequency approximation (PP- §4.2, §4.3) without the small amplitude assumption. Thus, we substitute  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t, \frac{\phi(\mathbf{x}, t)}{\epsilon})$ , where  $\epsilon$  is a small quantity, in the equation (1.9) and equate the leading order terms i.e. other terms of the order  $1/\epsilon$  equal to zero. This gives the characteristics partial differential equation of (1.9) as the eikonal equation and the result follows. The ray velocity components  $\chi_{\alpha}$  corresponding to the eigenvalue c are given by the lemma on bicharacteristics [7]

$$\chi_{\alpha} = \frac{\underline{IB}^{(\alpha)}\mathbf{r}}{\underline{IA}\mathbf{r}}$$
(1.10)

where l and r are left and right null vectors satisfying

$$\mathbf{l}(n_{\alpha}B^{(\alpha)} - cA) = 0, \quad (n_{\alpha}B^{(\alpha)} - cA)\mathbf{r} = 0$$
(1.11)

and

$$C = c = n_{\alpha} \chi_{\alpha} = \frac{\mathbf{l}(n_{\alpha} B^{(\alpha)}) \mathbf{r}}{\mathbf{l} A \mathbf{r}}$$
(1.12)

**Theorem 2** — If  $\chi$  be a given by (1.10) the compatibility condition (1.5) is satisfied and the rays are given by

$$\frac{dx_{\alpha}}{dt} = (\mathbf{l}B^{(\alpha)}\mathbf{r})/(\mathbf{l}A\mathbf{r}) = \chi_{\alpha}$$
(1.13)

$$\frac{dn_{\alpha}}{dt} = -\frac{1}{\mathbf{l}A\mathbf{r}} \mathbf{l} \left\{ n_{\beta} \left( n_{\gamma} \frac{\partial B^{(\gamma)}}{\partial \eta_{\beta}^{\alpha}} - c \frac{\partial A}{\partial \eta_{\beta}^{\alpha}} \right) \right\} \mathbf{r} = \psi_{\alpha}, \quad say.$$
(1.14)

Further, the system (1.9) implies a compatibility condition on a characteristic surface  $\Omega$  of hyperbolic system in the form

$$\mathbf{l}A\frac{d\mathbf{u}}{dt} + \mathbf{l}(B^{(\alpha)} - \chi_{\alpha}A)\frac{\partial\mathbf{u}}{\partial x_{\alpha}} + \mathbf{l}C = 0.$$
(1.15)

*Note:* The theorem in this form was first stated in [17], see also (PP-  $\S$  2.4). We give here a complete proof.

PROOF: Post (pre) multiplying the first (second) result in (1.11) by  $\mathbf{r}$  (l) and using  $c = -\phi_t / | \nabla \phi |$ and  $\mathbf{n} = \nabla \phi / | \nabla \phi |$ , we get the eikonal equation (1.4), where  $\boldsymbol{\chi}$  is given by (1.10).

We note that **l** and **r** (and hence  $\chi$ ) depend on **n** but A and  $B^{(\alpha)}$  do not. Hence

$$n_{\gamma} \frac{\partial \chi_{\gamma}}{\partial n_{\alpha}} = n_{\gamma} \frac{\partial}{\partial n_{\alpha}} \left( \frac{\mathbf{l}B^{(\gamma)} \mathbf{r}}{\mathbf{l}A\mathbf{r}} \right)$$

$$= \frac{1}{(\mathbf{l}A\mathbf{r})^{2}} \left[ n_{\gamma} \left( \frac{\partial}{\partial n_{\alpha}} (\mathbf{l}) \right) \left\{ \left( B^{(\gamma)} \mathbf{r} \right) (\mathbf{l}A\mathbf{r}) - (A\mathbf{r}) \left( \mathbf{l}B^{(\gamma)} \mathbf{r} \right) \right\} \right]$$

$$+ \frac{n_{\gamma}}{(\mathbf{l}A\mathbf{r})^{2}} \left[ \left\{ \left( \mathbf{l}B^{(\gamma)} \right) (\mathbf{l}A\mathbf{r}) - (\mathbf{l}A) \left( \mathbf{l}B^{(\gamma)} \mathbf{r} \right) \right\} \left( \frac{\partial}{\partial n_{\alpha}} (\mathbf{r}) \right) \right]$$

$$= \frac{1}{(\mathbf{l}A\mathbf{r})} \left[ \left( \frac{\partial}{\partial n_{\alpha}} (\mathbf{l}) \right) \left( n_{\gamma} B^{(\gamma)} - cA \right) \mathbf{r} + \mathbf{l} \left( n_{\gamma} B^{(\gamma)} - cA \right) \left( \frac{\partial}{\partial n_{\alpha}} (\mathbf{r}) \right) \right] = 0.$$
(1.16)

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Replacing  $\alpha$  by  $\beta$ , we also get  $n_{\gamma} \frac{\partial}{\partial n_{\beta}} \chi_{\gamma} = 0$ . Therefore, the condition (1.5) is satisfied and (1.13) is proved.

In the equation (1.6b), the derivatives  $\frac{\partial}{\partial \eta_{\beta}^{\alpha}}$  and hence  $\frac{\partial}{\partial x_{\alpha}}$  operate on  $\chi_{\alpha}$ , which is a very complicated expression in x and n. In the original Charpit equations from which (1.6b) has been derived involves differentiation with respect to  $x_{\alpha}$  keeping  $\phi_t$  and  $\nabla \phi$  fixed i.e., n fixed in (1.6b). We use this information to simplify the right hand of (1.6b) in such a way that derivtives  $\frac{\partial}{\partial x_{\alpha}}$  appear only on A and  $B^{(\gamma)}$ . To do this we follow the producer of differentiation in (1.16), now with respect to  $x_{\alpha}$  instead of  $n_{\alpha}$  and finally we get

$$n_{\gamma} \frac{\partial \chi_{\gamma}}{\partial x_{\alpha}} = \frac{1}{\mathbf{l}A\mathbf{r}} \mathbf{l} \left[ n_{\gamma} \frac{\partial B^{(\gamma)}}{\partial x_{\alpha}} - c \frac{\partial A}{\partial x_{\alpha}} \right] \mathbf{r}.$$
 (1.17)

This leads to (1.14). A proof of the third part i.e., (1.15) of the theorem is available in PP- $\S$ 2.4.

The derivatives in the compatibility condition (1.15) on a characteristics surface  $\Omega$  are so grouped that each group represents a tangential derivative on  $\Omega$ . The derivative  $\frac{d}{dt} = \frac{\partial}{\partial t} + \chi_{\alpha} \frac{\partial}{\partial x_{\alpha}}$  along a ray is the time derivative along a bicharacteristics i.e., tangential to  $\Omega$ . It can also be shown (PP-§ 2.4) that each of the derivatives  $\tilde{\partial}_j = \ell_i \left( B_{ij}^{(\alpha)} - \chi_{\alpha} A_{ij} \right) \frac{\partial}{\partial x_{\alpha}} (j = 1, 2, \dots, n)$  operating on  $u_j$  in the second term represents tangential derivatives not only on  $\Omega$  but also on  $\Omega_t$ . We note that  $|n_{\alpha}| = 0$  so that only m - 1 components of **n** are independent and we can show that only m - 1equations in (1.14) are independent. For a linear hyperbolic system, the ray equations (1.13) and (1.14) decouple from (1.15) and hence can be solved to give rays. For a quasilinear system (1.9), the matrices A and  $B^{(\alpha)}$  depend on **u** and hence the terms on the right sides of (1.13) and (1.14), when evaluated, would contain **u** and  $\frac{\partial \mathbf{u}}{\partial \eta_{\beta}^{\alpha}}$ . In this case the system (1.13)-(1.15) in 2m + n quantities  $\mathbf{x}$ , **n**, **u** is an under-determined unless n = 1. Therefore, this system is of limited use for n > 1 unless high frequency approximation is made leading to the weakly nonlinear ray theory (WNLRT) or the shock ray theory discussed in the next two sections. However, one important use is in development of numerical methods, namely characteristic Galerkin method [12], a topic of very active research today.

## 2. A WEAKLY NONLINEAR RAY THEORY (WNLRT)

The high frequency approximation, mentioned in the end of the last section, when applied to a hyperbolic system of partial differential equations, gives (PP - Chapter 4) a representation of the solution in terms of a wave amplitude w and a phase function  $\phi$ . The high frequency approximation further implies that

(i) the function  $\phi$  satisfies an eikonal equation, with the help of which we can define rays,

(*ii*) all components of the state variables **u** of the system are expressed in terms of the an amplitude w and the unit normal **n** of the wavefront  $\Omega_t : \phi(\mathbf{x}, t) = \text{constant}$ , and in addition

(*iii*) when the expression for **u** in terms of w is substituted in (1.15) we get a transport equation for the amplitude w along the rays.

Thus, the high frequency approximation reduces the problems of finding the successive positions of a wavefront and the amplitude distribution on it to the integration of a closed system of equations consisting of the ray equations and a transport equation. The ray equations and the transport equations are decoupled for a linear hyperbolic system and coupled for a quasi-linear system. When the amplitude of the wave is a small perturbation over a known state  $\mathbf{u}_0$ , the relation between  $\mathbf{u} - \mathbf{u}_0$  and w is linear. This leads to a simple weekly nonlinear ray theory (WNLRT) in which the equations can be integrated numerically, at least in theory. We review these important results below.

The high frequency approximation was first applied in 1911 [26] to the wave equation

$$u_{tt} - a_0^2 \Delta u = 0, \ \Delta = \sum_{\alpha=1}^m \frac{\partial^2}{\partial x_\alpha^2}$$
(2.1)

where  $a_0$  is the constant sound velocity in the uniform medium. We shall briefly review the results of high frequency approximation for the most general case as given in [PP - chapter 4], but in order that the theory is clear we shall first present here the simplest case i.e. the case of the wave equation (2.1). Small amplitude high frequency approximation of a perturbation of a constant basic state u = 0 consists of assuming a solution of (2.1) in the form

$$u = \epsilon u_1(\mathbf{x}, t, \theta) + \epsilon^2 u_2(\mathbf{x}, t, \theta) + \cdots$$
(2.2)

where  $\epsilon$  is small,

$$\theta = \frac{1}{\epsilon} \phi(\mathbf{x}, t) \tag{2.3}$$

and  $\phi$  is the phase function. Substituting (2.2) in (2.1) and equating the first two order terms in  $\epsilon$  on both sides, we get

$$(\phi_t^2 - a_0^2 | \nabla \phi |^2) u_{1\theta\theta} = 0$$
(2.4)

and

$$(\phi_t^2 - a_0^2 | \bigtriangledown \phi |^2) u_{2\theta\theta} + 2(\phi_t u_{1t\theta} - a_0^2 \phi_{x_\alpha} u_{1x_\alpha\theta}) + (\phi_{tt} - a_0^2 \phi_{x_\alpha x_\alpha}) u_{1\theta} = 0$$
(2.5)

For a nonzero perturbation of order  $\epsilon$ ,  $u_{1\theta\theta} \neq 0$  so that (2.4) gives the eikonal equation

$$\phi_t^2 - a_0^2 |\nabla \phi|^2 = 0 \tag{2.6}$$

and consequently the ray equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{n}a_0, \ \frac{d\mathbf{n}}{dt} = 0 \tag{2.7}$$

(2.7) shows that the set of all rays of the wave equation from a point  $\mathbf{x}^*$  starting at time  $t_0$  are straight lines emanating from  $\mathbf{x}^*$ :

$$\mathbf{x} = \mathbf{x}^* + \mathbf{n}(t - t_0), \ |\mathbf{n}| = 1$$
 (2.8)

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where n is constant. Time rate of change along a bicharecteristic or ray is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx_{\alpha}}{dt} \frac{\partial}{\partial x_{\alpha}} = \frac{\partial}{\partial t} + a_0 n_{\alpha} \frac{\partial}{\partial x_{\alpha}}$$
$$= -\frac{1}{a_0 |\nabla \phi|} \left( \phi_t \frac{\partial}{\partial t} - a_0^2 \phi_x \frac{\partial}{\partial x} \right)$$
(2.9)

using  $\phi_t = -a_0 | \nabla \phi |$  for a wave moving in the direction of **n**. There are two important quantities, which frequently appear in the propagation of a curved wavefront  $\Omega_t$ . They are the mean curvature  $\Omega$  (not to be confused with the characteristic surface  $\Omega$ , which is not in italics) and the ray tube area A (PP-§ 2.2.3) related to the divergence of the unit normal **n** of  $\Omega_t$  and the derivatives of  $\phi$  by

$$\Omega = -\frac{1}{2a_0A}\frac{dA}{dt} = -\frac{1}{2}div(\mathbf{n})$$
$$= \frac{1}{2a_0^2|\nabla\phi|}\left(\phi_{tt} - a_0\nabla^2\phi\right).$$
(2.10)

Using (2.6), (2.9) and (2.10), we deduce from (2.5) the transport equation for  $u_{1\theta}$ 

$$\frac{du_{1\theta}}{dt} = a_0 \Omega u_{1\theta} = -\frac{1}{2A} \frac{dA}{dt} u_{1\theta}.$$
(2.11)

We integrate this equation with respect to the fast variable  $\theta$  and assuming that the amplitude is non-zero only in an  $\epsilon$  neighbourhood of  $\phi = 0$  (i.e.  $\theta = 0$ ), so that in this integration we may treat  $\Omega$  and A to be constant, we get

$$\frac{du_1}{dt} = a_0 \Omega u_1 = -\frac{1}{2A} \frac{dA}{dt} u_1 \tag{2.12}$$

showing that the amplitude  $u_1$  satisfies

$$u_1 = u_{10}/A^{1/2}, \ u_{10} = \text{constant.}$$
 (2.13)

This is an example of a linear theory of wave propagation where the high frequency approximation (geometrical optics) gives a value of the leading term  $u_1$  of the amplitude which tends to infinity as  $A \rightarrow 0$  i.e., as a focus or a caustic is approached. The small amplitude approximation implied in the expansion (2.2) breaks down as  $A \rightarrow 0$ . In reality, the amplitude of the wave when evaluated more accurately does remain finite [6, 11]. We shall see later that the genuine nonlinearity present in the mode of propagation in consideration would prevent even the leading term  $u_1$  of perturbation to remain finite and of the same order as the order of its initial value. This is because the nonlinearity avoids formation of a caustic, which is replaced by a new type of singularity called kink (PP-§ 3.3.3). The WNLRT remains valid in the caustic region of the linear theory.

The basis of the weekly nonlinear ray theory applied to the system (1.9) for a wave in a mode having genuine nonlinearity is to capture the wave amplitude  $\tilde{w}$  (=  $u_1$  of (2.2)) in the eikonal equation itself. One way to achieve this is to modify the expansion (2.3) by (PP-§4.4 or [20]).

$$\mathbf{u}(\mathbf{x}, t, \epsilon) = \mathbf{u}_0 + \epsilon \tilde{\mathbf{v}}(\mathbf{x}, t, \frac{\phi(\mathbf{x}, t, \epsilon)}{\epsilon}, \epsilon)$$
(2.14)

$$\tilde{\mathbf{v}}(\mathbf{x}, t, \theta, \epsilon) = \tilde{\mathbf{v}}_0(\mathbf{x}, t, \theta, \epsilon) + \epsilon \tilde{\mathbf{v}}(\mathbf{x}, t, \theta, \epsilon), \quad \theta = \frac{\phi}{\epsilon}$$
(2.15)

for a perturbation of a basic solution  $\mathbf{u} = \mathbf{u}_0$  of (1.9). It is possible to derive the WNLRT for the general hyperbolic system (1.9), but in order to keep the derivation simpler, we consider the reducible system (PP - p77)

$$A(\mathbf{u})\mathbf{u}_t + B^{(\alpha)}(\mathbf{u})\mathbf{u}_{x_\alpha} = 0.$$
(2.16)

Now we substitute (2.14) and (2.15) in it and go through a long mathematical steps (PP - § 4.4) carefully collecting terms up to order  $\epsilon^2$ . This leads to a derivation of equations of WNLRT with long expressions of terms in the ray and transport equations. We write these equations for the Euler equations of a polytropic gas

$$\rho_t + \langle \mathbf{q}, \nabla \rangle \rho + \rho \langle \nabla, \mathbf{q} \rangle = 0 \tag{2.17}$$

$$\mathbf{q}_t + \langle \mathbf{q}, \nabla \rangle \mathbf{q} + \frac{1}{\rho} \nabla p = 0 \tag{2.18}$$

$$p_t + \langle \mathbf{q}, \nabla \rangle p + \rho a^2 \langle \nabla, \mathbf{q} \rangle = 0$$
(2.19)

where  $\mathbf{u} = (\rho, \mathbf{q} = (q_1, q_2, q_3), p)^T$  is a vector whose components are mass density  $\rho$ , fluid velocity  $\mathbf{q}$  and gas pressure p, and a is the local sound velocity

$$a^2 = \gamma p/\varrho. \tag{2.20}$$

We take the basic constant solution to be a constant equilibrium state  $\mathbf{u}_0 = (\rho_0, \mathbf{q} = \mathbf{0}, p_0)$ , then the leading order perturbation  $\tilde{\mathbf{v}}_0$  is given by,

$$\tilde{v}_{10} = \rho - \rho_0 = \epsilon \frac{\rho_0}{a_0} \tilde{w}, \quad (\tilde{v}_{20}, \tilde{v}_{30}, \tilde{v}_{40}) = \epsilon \mathbf{n} \tilde{w}, \quad \tilde{v}_{50} = p - p_0 = \epsilon \rho_0 a_0 \tilde{w} \quad .$$
(2.21)

The ray equations and the transport equation of the WNLRT are given by

$$\frac{d\mathbf{x}}{dt} = \left(a_0 + \epsilon \frac{\gamma + 1}{2}\tilde{w}\right)\mathbf{n}, \quad \frac{d\mathbf{n}}{dt} = -\epsilon \frac{\gamma + 1}{2}\mathbf{L}\,\tilde{w} \tag{2.22}$$

and

$$\frac{d\tilde{w}}{dt} \equiv \left\{\frac{\partial}{\partial t} + (a_0 + \epsilon \frac{\gamma + 1}{2}\tilde{w})\langle \mathbf{n}, \nabla \rangle\right\}\tilde{w} = \Omega a_0 \tilde{w}$$
(2.23)

where

$$\mathbf{L} = \nabla - \mathbf{n} \langle \mathbf{n}, \nabla \rangle \tag{2.24}$$

and  $\Omega$  is now the mean curvature of the nonlinear wavefront  $\Omega_t$ . Note that the symbols  $\Omega_t$  and  $\Omega$  represents respectively a surface in  $\mathbb{R}^m$  (here m = 3) and mean curvature of  $\Omega_t$ . The components of

the operator **L** represents tangential derivatives on the surface  $\Omega_t$ .  $L_{\alpha}$  can be expressed as a linear combination of the operators  $\frac{\partial}{\partial \eta_{\beta}^{\alpha}}$  (PP - p73).

The equation (2.23) is a true generalization of the one dimensional Burgers' equation  $u_{t'} + uu_{\xi'} = 0$  to multidimensions. This can be seen from the fact that in one dimension  $\mathbf{n} = (1, 0, 0)$  so that  $\Omega = 0$ ,  $\frac{d}{dt} = \frac{\partial}{\partial t} + (a_0 + \epsilon \frac{\gamma + 1}{2}\tilde{w})\frac{\partial}{\partial x}$  and then (3.23) reduces to the Burger's equation in  $(\xi', t')$  coordinates  $(\xi' = x - a_0t, t' = t)$  for the variable  $u = \epsilon \frac{\gamma + 1}{2}\tilde{w}$ . The coupled system of 6 equations (2.22) and (2.23) (note that only two component of  $\mathbf{n}$  are independent) describe completely a new property (not observed before WNLRT was formulated): interaction of three independent of the wave amplitude on the wavefront and (iii) geometric amplification due to convergence of rays (PP-pages 206-7).

# 3. SHOCK RAY THEORY (SRT)

When the moving surface  $\Omega_t$  is a shockfront, special care is required in the definition of a shock ray [16]. The shock ray velocity in a gas is unambiguously defined by

$$\chi = \mathbf{q}_{\mathbf{r}} + \mathbf{N} A \tag{3.1}$$

where  $\mathbf{q}_{\mathbf{r}}$  is the fluid velocity ahead of the shock, N the unit normal to the shock front and A is the normal speed of the shock relative to the gas ahead of it. If we represent the shock surface  $\Omega_t$  at any time by  $s(\mathbf{x}, t) = 0$ , then the eikonal equation or what we call a shock manifold partial differential equation (SME), appropriately interpreted in [16], is

$$s_t + \langle \mathbf{q}_r, \nabla \rangle s + A |\nabla s| = 0. \tag{3.2}$$

When we try to derive (3.2) from the jump relations or Rankine-Hugoniot (RH) conditions, we run into difficulty. There are many other jump relations, which can be derived from the RH conditions and each one of them would lead to an SME. For example, the well known Prandtl relation for a curved shock, when expressed in terms of *s*, is

$$\{s_t + \langle \mathbf{q}_l, \nabla \rangle s\} \{s_t + \langle \mathbf{q}_r, \nabla \rangle s\} - a_*^2 |\nabla s|^2 = 0$$
(3.3)

where  $a_*$  is the common critical speed on the two sides of the shock

$$a_*^2 = (p_r - p_l) / (\rho_r - \rho_l).$$
(3.4)

The ray velocity obtained from (3.3) gives a different expression for the shock ray velocity  $\chi$ . The question arises: "are the shock ray velocities (or more precisely, shock ray equations) obtained from different SMEs the same?"

The concept of a SME and their equivalence in the above sense was first discussed in [16]. Making further use of the RH conditions, it was shown that the shock ray equations given by the two SMEs (3.2) and (3.3) are equivalent. This result was generalized in [24] for almost all SMEs.

Since the high frequency approximation is satisfied exactly on a shock front, the derivation of the eikonal equation and definition of a shock ray does not require any approximation. The next step i.e., the derivation of the compatibility condition along a shock ray was done in 1978 independently by Grinfeld [8] and Maslov [13]. The compatibility conditions form an infinite system and their derivation involves extremely difficult mathematical steps. Hence we first write down these compatibility conditions in the case of one space dimensional problem and that too for a conservation law from Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{3.5}$$

Consider a shock along a curve  $\Omega : x = X(t)$  in (x, t)-plane, then

$$\frac{dX(t)}{dt} = \frac{1}{2}(u_l + u_r) = C, say$$
(3.6)

where  $u_l$  and  $u_r$  are states on the left and right of the shock. The state on x < X(t) satisfies  $u_t + uu_x = 0$  which we write as,

$$u_t + \frac{1}{2} (u + u_r) \ u_x = -\frac{1}{2} (u - u_r) u_x.$$
(3.7)

Taking the limit  $x \to X(t)$ -, we get

$$\frac{du_l}{dt} = -\frac{1}{2} \left( u_l - u_r \right) \left( u_x \right)_l, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{2} \left( u_l + u_r \right) \frac{\partial}{\partial x}.$$
(3.8)

Differentiating  $u_t + uu_x = 0$  with respect to x we get,

$$(u_x)_t + u (u_x)_x = -u_x^2$$

which we write as

$$(u_x)_t + \frac{1}{2} (u + u_r) (u_x)_x = -\frac{1}{2} (u - u_r) (u_x)_x - u_x^2$$
(3.9)

and taking the limit as  $x \to X(t)$ , we get

$$\frac{d(u_x)_l}{dt} = -\frac{1}{2} \left( u_l - u_r \right) \left( u_{xx} \right)_l - \left( (u_x)_l \right)^2$$
(3.10)

where again

$$\frac{d}{dt} = \frac{\partial}{\partial t} + C \frac{\partial}{\partial x}$$
(3.11)

represents the rate of change as we move with the shock. This way, an infinite system of compatibility conditions can be derived.

We define the limiting value of the ith derivative of u on the left of the shock divided by i! by  $v_i(t)$  i.e.,

$$v_i(t) = \frac{1}{i} \lim_{x \to X(t) -} \frac{\partial^i u}{\partial x^i}, \ i = 1, 2, 3, \cdots$$
 (3.12)

and introduce  $v_0(t) = u_l(t)$ . The equation (3.6) and the infinite system of compatibility conditions along  $\Omega$  are (PP-§ 7.1)

$$\frac{dX}{dt} = \frac{1}{2}(u_0 + u_r)$$
(3.13)

$$\frac{du_0}{dt} = -\frac{1}{2} \left( u_0 - u_r \right) v_1 \tag{3.14}$$

$$\frac{dv_i}{dt} = -\frac{i+1}{2} \left( u_0 - u_r \right) v_{i+1} - \frac{i+1}{2} \sum_{j=1}^i v_j v_{i-j+1}, \quad i = 1, 2, 3, \cdots$$
(3.15)

Given an initial value  $u(x, 0) = \phi(x)$  for (3.5) with a single shock at  $X_0$ , we can derive initial values of X(t),  $u_0(t)$ ,  $v_i(t)$ ,  $i = 1, 2, 3, \cdots$  in the form

$$X(0) = X_0, \quad u_0(0) = u_{00}, v_i(0) = v_{i0}, \quad i = 1, 2, 3, \cdots$$
(3.16)

However, we face new problems (PP -  $\S7.2$ ):

- The solution of an infinite dimensional problem (3.13) (3.16) is more difficult than the Cauchy problem for the original conservation law (3.5).
- For an analytic initial value  $\phi(x)$  for (3.5) in a neighbourhood of  $x = X_0$ , the analytic solution of the problem (3.13)–(3.16) is unique and give a function u(x, t) by

$$u(x,t) = u_0(t) + \sum_{i=1}^{\infty} v_i(t)(x - X(t))^i$$
(3.17)

which tends in a neighbourhood of x = X(t) to the analytic solution of the initial value problem for (3.5) for small time.

• For a more general non-analytic initial data, the solution of initial value problem (3.13)–(3.16) is non-unique.

Ravindran and Prasad [24] proposed a new theory of shock dynamics by setting  $v_{n+1} = 0$  in the nth equation in (3.15) so that the first n + 1 equations in (3.14) and (3.15) form a closed system. Then the first (n+2) equations in (3.13)-(3.15) form a system of ordinary differential equations, which can be easily integrated numerically with initial data (the first n + 2 in (3.16)). The new theory of shock dynamic gives excellent results for the case when  $\phi'(x) > 0$  behind the shock at  $x = X_0$ . There are many open mathematical questions to be answered but this method has been very successful for many practical problems in multi-dimensions - we shall mention these later on.

Derivation of the compatibility conditions from Euler's equations (2.17)-(2.19) along a shock ray given by the shock ray velocity (3.1) requires extremely complex calculations (in fact it is complex even in one-dimension) (PP -§ 8.2). Hence we present a derivation for a weak shock from the equations (2.22)–(2.23) of the WNLRT (PP - §10.1 or [14]) - this was first proposed in 1993 [17]. For such a shock we have a theorem, first stated in [16] in 1982 (Theorem 9.2.1, PP - p267), which states

**Theorem 3**—For a weak shock, the shock ray velocity components are equal to the mean of the bicharacteristic velocity components just ahead and just behind the the shock, provided we take the wavefronts generating the characteristic surface ahead and behind to be instantaneously coincident with the shock surface. Similarly, the rate of turning of the shock front is also equal to the mean of the rates of turning of such wavefronts just ahead and just behind the shock.

Consider a shock front propagating into a polytropic gas at rest ahead of it. Then the shock will be followed by a one parameter family of nonlinear wavefronts belonging to the same characteristic field (or mode). Each one of these wavefronts will catch up with the shock, interact with it and then disappear. A nonlinear wavefront, while interacting with the shock will be instantaneously coincident with it due to the short wave assumption. The ray equations of the WNLRT in threespace-dimensions for a particular nonlinear wavefront are (2.22). We denote the unit normal to the shock front by **N**. For the linear wavefront just ahead of the shock and instantaneously coincident with it (this is actually a linear wavefront moving with the ray velocity: **N** multiplied by the local sound velocity  $a_0$ ),  $\tilde{w} = 0$  and the bicharacteristic velocity is **N** $a_0$ . For the nonlinear wavefront just behind the shock and instantaneously coincident with it, we denote the amplitude  $\tilde{w}$  by  $a_o\mu$ . Then  $\epsilon\mu$  is a non-dimensional shock amplitude of the weak shock under consideration. Using the above theorem and the results (2.22) with  $\mathbf{n} = \mathbf{N}$ , we find that a point **X** on the shock ray satisfies

$$\frac{d\mathbf{X}}{dT} = \frac{1}{2} \left\{ a_0 \mathbf{N} + \mathbf{N} a_0 \left( 1 + \epsilon \frac{\gamma + 1}{2} \mu \right) \right\} = \mathbf{N} a_0 \left( 1 + \epsilon \frac{\gamma + 1}{4} \mu \right)$$
(3.18)

$$\frac{d\mathbf{N}}{dT} = -\frac{1}{2} \left\{ 0 + \epsilon \frac{\gamma + 1}{2} a_0 \mathbf{L} \mu \right\} = -\epsilon \frac{\gamma + 1}{4} a_0 \mathbf{L} \mu$$
(3.19)

where T is the time measured while moving along a shock ray. We take  $\tilde{w} = a_o \mu$  and  $\mathbf{n} = \mathbf{N}$  in (2.23) and write it as

$$\frac{d\mu}{dT} \equiv \left\{ \frac{\partial}{\partial t} + a_0 \left( 1 + \epsilon \frac{\gamma + 1}{4} \mu \right) \langle \mathbf{N}, \nabla \rangle \right\} \mu$$

$$= -\frac{1}{2} a_0 \langle \nabla, \mathbf{N} \rangle \mu - \epsilon \frac{\gamma + 1}{4} \mu \langle \mathbf{N}, \nabla \rangle \tilde{w}$$
(3.20)

where we note that since  $\mu$  is defined only on the shock front (and also on the instantaneously coincident nonlinear wavefront behind it but not on the other members of the one parameter family of wavefronts following it), the normal derivative  $\langle \mathbf{N}, \nabla \rangle \mu$  dose not make sense. We introduce new

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variables, defined on the shock front:

$$V = \epsilon \left\{ \langle \mathbf{N}, \nabla \rangle \tilde{w} \right\} |_{\text{shock front}}, \quad \mu_2 = \epsilon^2 \left\{ \langle \mathbf{N}, \nabla \rangle^2 \tilde{w} \right\} |_{\text{shock front}}$$
(3.21)

where powers of  $\epsilon$  appears to make both V and  $\mu_2$  of O(1) since variation of  $\tilde{w}$  with respect to the fast variable  $\theta$  (introduced in (2.15)) is of the order of 1 and hence  $\langle N, \nabla \rangle \tilde{w} = O\left(\frac{1}{\epsilon}\right)$  and  $\langle N, \nabla \rangle^2 \tilde{w} = O\left(\frac{1}{\epsilon^2}\right)$ . The equation (3.20) leads to the first compatibility condition along a shock ray

$$\frac{d\mu}{dT} = a_0 \Omega_s \mu - \frac{\gamma + 1}{4} \mu V \tag{3.22}$$

where

$$\Omega_s = -\frac{1}{2} \langle \nabla, N \rangle \tag{3.23}$$

is the value of the mean curvature of the shock. To find the second compatibility condition along a shock, we differentiate (2.23) in the direction of  $\mathbf{n}$  but on the length scale over which  $\theta$  varies. On this length scale,  $\mathbf{n}$  is constant and we get, after rearranging some terms,

$$\left\{\frac{\partial}{\partial t} + \left(a_0 + \epsilon \frac{\gamma + 1}{4}\tilde{w}\right) \langle \mathbf{n}, \nabla \rangle \right\} \langle \mathbf{n}, \nabla \rangle \tilde{w} = -\frac{1}{2}a_0 \langle \nabla, \mathbf{n} \rangle \langle \mathbf{n}, \nabla \rangle \tilde{w}$$
$$-\epsilon \frac{\gamma + 1}{4} \left\{ \langle \mathbf{n}, \nabla \rangle \tilde{w} \right\}^2 - \epsilon \frac{\gamma + 1}{4} \tilde{w} \langle \mathbf{n}, \nabla \rangle^2 \tilde{w}.$$
(3.24)

Writing this equation on the wavefront instantaneously coincident with the shock, multiplying it by  $\epsilon$  we get

$$\frac{dV}{dT} = a_0 \Omega_s V - \frac{\gamma + 1}{4} V^2 - \frac{\gamma + 1}{4} \mu \mu_2$$
(3.25)

which is the second compatibility condition along shock rays given by (3.18) and (3.19). Similarly, higher order compatibility conditions can be derived.

Thus, for the Euler's equations, we have derived the infinite system of compatibility conditions for a weak shock just from the dominant terms of WNLRT. As we have already mentioned, the shock ray theory is an exact theory (weak shock assumption is another independent assumption) but since there are infinite number of compatibility conditions on it, it is impossible to use it for computing shock propagation. We now use the new theory of shock dynamics (NTSD) according to which the system of equations (3.18), (3.19), (3.22) and (3.25) can be closed by dropping the term containing  $\mu_2$  in the equation (3.25). This step is justified in the case V > 0, which occurs very frequently in a applications such as a blast wave. But for multidimensional problems, dropping  $\mu\mu_2$  gives excellent results in all cases [3]. When we consider propagation of even stronger shocks in gas dynamics, the results of [PP - chapter 8] shows that neglecting the term  $\mu\mu_2$  in the second compatibility condition gives good results for one space dimensions not only when V > 0 but also when V < 0 (as exemplified by the accelerating piston problem after an initial push of the piston).

## 4. KINEMATICAL CONSERVATION LAWS IN TWO SPACE DIMENSIONS

Kinematical conservation laws (KCL) are equations of evolution of a moving (m-1)-dimensional surface  $\Omega_t$  in  $\mathbb{R}^m$ . These equations are in conservation form in a special coordinate system, namely the ray coordinate system  $(\xi_1, \xi_2, ..., \xi_{m-1}, t)$ . Since we have been able to make many important applications of KCL only in two space dimensions, we restrict our discussion only to two space dimensions. We denote the spatial coordinates by x and y. The unit normal  $(n_1, n_2)$  is expressed in terms of  $\theta$  by  $n_1 = \cos \theta$ ,  $n_2 = \sin \theta$ . Let us assume that the motion of  $\Omega_t$  is governed by its points moving along rays according to

$$\frac{dx}{dt} = \chi_1, \quad \frac{dy}{dt} = \chi_2 \tag{4.1}$$

where the ray velocity field  $\chi$  depends on the nature of  $\Omega_t$ . We define a ray coordinate system $(\xi, t)$  such that  $\xi$  = constant is a ray and t= constant is the wavefront  $\Omega_t$  at time t. Let g be the metric associated with  $\xi$  i.e.,  $gd\xi$  is an element of length along  $\Omega_t$ , then

$$g = \sqrt{x_{\xi}^2 + y_{\xi}^2}$$
 and  $\frac{\partial}{g\partial\xi} = \left(-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\right).$  (4.2)

The normal and tangential components of  $\chi$ , denoted by C and T respectively, are

$$C = n_1 \chi_1 + n_2 \chi_2, \quad T = -n_2 \chi_1 + n_1 \chi_2. \tag{4.3}$$

If P(x, y) be a point on  $\Omega_t$  and Q(x + dx, y + dy) be an arbitrary point on  $\Omega_{t+dt}$ , then we can reach Q from P by moving along the ray (Cdt along the normal to  $\Omega_t$  from P and Tdt along the tangent to  $\Omega_{t+dt}$ ) and then moving along  $\Omega_{t+dt}$  by  $gd \xi$  (see [19] and PP - §3.3.2). This gives the differential relation between (dx, dy) and ( $d\xi, dt$ ) from which we can derive,

$$\begin{pmatrix} x_{\xi} & x_t \\ y_{\xi} & y_t \end{pmatrix} = \begin{pmatrix} -g\sin\theta & C\cos\theta - T\sin\theta \\ g\cos\theta & C\sin\theta + T\cos\theta \end{pmatrix}.$$
(4.4)

For a smooth curve  $\Omega_t$  and smooth ray velocity  $\chi$ , the results  $x_{\xi t} = x_{t\xi}$  and  $y_{\xi t} = y_{t\xi}$  give the required pair of kinematical conservation laws:

$$(gsin\theta)_t + (C\cos\theta - T\sin\theta)_{\xi} = 0 \tag{4.5}$$

$$(g\cos\theta)_t - (C\sin\theta + T\cos\theta)_{\xi} = 0. \tag{4.6}$$

In two dimensions, the ray equations (1.6) reduce to

$$\frac{dx}{dt} = \chi_1, \quad \frac{dy}{dt} = \chi_2, \quad \frac{d\theta}{dt} = -\frac{1}{g} \left( n_1 \frac{\partial \chi_1}{\partial \xi} + n_2 \frac{\partial \chi_2}{\partial \xi} \right). \tag{4.7}$$

**Theorem 4** — Let  $\chi$  be a smooth function of x, y, t and  $\mathbf{n}$ , and satisfy (1.5). Then the ray equations (4.7) for the propagation of a smooth curve  $\Omega_t$  are equivalent to the KCL (4.5) and (4.6).

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PROOF: The proof that the ray equations imply KCL is too simple. Given  $\chi$  as a function of x, y, t and  $\theta$ , and an arbitrarily prescribed  $\Omega_0$ , we can construct the rays and the family of curves  $\Omega_t$ . Then we choose a variable  $\xi$  and construct the ray coordinate system  $(\xi, t)$ . g is given by (4.2). As long as a singularity does not appear on  $\Omega_t$ , the mapping from (x, y) plane to  $(\xi, t)$  - plane for a given  $\Omega_t$  is well defined and one to one. Now we can derive KCL in just few steps (as shown above). Alternately, we shall show that it is simple to deduce the differential from of KCL

$$\theta_t = -\frac{1}{g}C_{\xi} + \frac{1}{g}T\theta_{\xi}, \quad g_t = C\theta_{\xi} + T_{\xi}$$
(4.8)

from the ray equations (4.7). Using  $\chi_1 = n_1C - n_2T$  and  $\chi_2 = n_2C + n_1T$  and noting that d/dt becomes  $\partial/\partial t$  in  $(\xi, t)$ -plane, we find that the third equation in (4.7) reduces to the first equation in (4.8). We now differentiate the relation  $g^2 = x_{\xi}^2 + y_{\xi}^2$  with respect to t and use  $n_1 = y_{\xi}/g$ ,  $n_2 = -x_{\xi}/g$  and also use  $x_t = \chi_1$ ,  $y_t = \chi_2$  to get the second equation in (4.8).

To prove the converse, we note that (4.5) and (4.6) imply existence of two functions  $x(\xi, t), y(\xi, t)$  satisfying (4.4). The mapping from  $(\xi, t)$  to (x, y)-plane is one to one as long as the Jacobian

$$\frac{\partial(x,y)}{\partial(\xi,t)} = -gC \tag{4.9}$$

does not vanish and remain finite. Image of a line t = constant in  $(\xi, t)$ -plane is a curve, let us denote it by  $\Omega_t$ , along which  $\xi$ -varies. The first column of (4.4) gives  $x_{\xi} = -g \sin \theta$ ,  $y_{\xi} = g \cos \theta$ , which show that g is the metric associated with  $\xi$  and the normal to  $\Omega_t$  makes an angle  $\theta$  with the x-axis. Propagation of the curve  $\Omega_t$  in (x, y)-plane is governed according to the second column of (4.4) with a ray velocity  $\chi = (\chi_1, \chi_2) := (C \cos \theta - T \sin \theta, C \sin \theta + T \cos \theta)$ . This shows that C and T satisfy (4.3) and so they are the normal and tangential components of the ray velocity  $\chi$ . Using this relation between  $(\chi_1, \chi_2)$  and (C, T) we get the third equation in (4.7) from the first equation in (4.8). Thus, we have derived the ray equations from KCL. However, the quantities C and T appearing in KCL must satisfy the consistency condition (1.5) through  $\chi_1$  and  $\chi_2$ . This completes the proof of the theorem.

We note that KCL is equivalent to the ray equations only for smooth fronts  $\Omega_t$ . However, when we numerically integrate the equations of WNLRT,  $\Omega_t$  ceases to be smooth after some  $t_c > o$  even though  $\Omega_o$  is smooth. After the time  $t_c$ , the differential equations of the WNLRT in § 2 cease to be valid. It was this situation, when we were forced to look for conservation forms of the ray equations and we discovered KCL for the case when  $\chi$  is orthogonal to  $\Omega_t$  [15]. The KCL gives a new type of singularity on  $\Omega_t$ , namely kinks (PP-§ 3.33). It turned out KCL represent conservation of distance in two independent directions for non-smooth solutions and hence are physically realistic.

# 5. Conservation Forms of Two Ray Theories: Weakly Nonlinear Ray Theory (WNLRT) and Shock Ray Theory (SRT) in a Polytropic Gas

KCL, being only two equations in four quantities  $g, \theta, C$  and T, is an under determined system. This is expected as KCL is a purely mathematical result and the dynamics of a particular moving

curve has not been taken into account in their derivation. We describe here two sets of closure equations. Both of these belong to the case of an isotropic wave, where T = 0 i.e., the rays are normal to the front. When a small amplitude curved wave front (across which the physical variables are continuous) or a shock front propagates into a medium at rest and in equilibrium with density  $\rho = \rho_0$ , fluid velocity  $\mathbf{q} = 0$  and gas pressure  $p = p_0$ , the perturbation on the wavefront or behind the shock front is given by (2.21), where  $\tilde{w}$  has dimension of velocity.

*Non-Dimesionalization* — In this and the subsequent sections we shall use non-dimensional form of space and time coordinates:  $\bar{\mathbf{x}} = \mathbf{x}/L$ , and  $t = a_0 t/L$ , where L is an appropriate length scale. After making the non-dimensionalization, we remove bar from  $\bar{\mathbf{x}}$  and  $\bar{t}$  so that the non-dimensional variables are denoted by  $\mathbf{x}$  and t. The Mach numbers m of a weakly nonlinear wavefront and M of a shock front are given by

$$m = 1 + \frac{\gamma + 1}{2} \frac{\epsilon \tilde{w}}{a_0}, \ M = 1 + \frac{\gamma + 1}{4} \frac{\epsilon \tilde{w}|_s}{a_0}$$
(5.1)

where  $\tilde{w}|_s$  is the value of  $\tilde{w}$  on a suitable side of the shock (behind a shock for a shock propagating in to the constant state ( $\rho_0$ ,  $\mathbf{q} = \mathbf{0}$ ,  $p_0$ ) and ahead of a shock which joins a constant state behind it to the disturbed state ahead of it). The non-dimensional value of C in (4.3) is m or M as the case may be.

The KCL of  $\Omega_t$  when it is a weakly nonlinear wavefront, are

$$(g\sin\theta)_t + (m\cos\theta)_{\xi} = 0, \ (g\cos\theta)_t - (m\sin\theta)_{\xi} = 0.$$
(5.2)

The closure equation of this under-determined system is obtained from (2.23), where we note that the rate of change of  $\frac{d}{dt}$  along a ray becomes the partial derivative  $\frac{\partial}{\partial t}$  in ray coordinates  $(\xi, t)$ . An expression for the non-dimensional mean curvature  $\Omega$  in 2-D is  $\Omega = \frac{1}{2} (\theta_x \sin \theta - \theta_y \cos \theta) = -(1/2g)\theta_{\xi}$ . Now it is simple to deduce (following the derivation of the equation (6.19) in PP - §6.1) the following conservation form of the equation (2.23)

$$\left\{g(m-1)^2 \ e^{2(m-1)}\right\}_t = 0 \tag{5.3}$$

• A general procedure for the derivation of a conservation form of a transport equation is available in [3]. (5.2) and (5.3) form the equations of the weakly nonlinear ray theory (WNLRT). The mapping from  $(\xi, t)$ -plane to (x, y)-plane is given by the equations (4.4), say by  $x_t = m \cos \theta$ ,  $y_t = m \sin \theta$ .

Next we choose  $\Omega_t$  to be a shock front. By taking the shock to be weak and by truncating the infinite system (i.e., by dropping the last term on the right of (3.25)) we can construct an approximate shock ray theory, which forms an efficient system of equations for calculation of successive positions of a curved shock front in two space dimensions [PP - §10.2]. We represent the unit normal to the shock front  $\Omega_t$  in 2-D as  $\mathbf{N} = (\cos \Theta, \sin \Theta)$ . A system of conservation form of the equations for a weak shock  $\Omega_t$ , are two KCL and two additional closure equations [3]:

$$(G\sin\Theta)_t + (M\cos\Theta)_{\xi} = 0, \ (G\cos\Theta)_t - (M\sin\Theta)_{\xi} = 0, \tag{5.4}$$

$$(G(M-1)^2 e^{2(M-1)})_t + 2M(M-1)^2 e^{2(M-1)} GV = 0,$$
(5.5)

$$(GV^2e^{2(M-1)})_t + GV^3(M+1)e^{2(M-1)} = 0, (5.6)$$

where G is the metric associated with the variable  $\xi$  and  $\frac{d}{dT}$ , defined by (3.20), becomes the partial derivative  $\frac{\partial}{\partial t}$  in the ray coordinate system  $(\xi t)$ . The normal derivative  $\langle \mathbf{N}, \nabla \rangle \tilde{w}$  in (3.20) is first obtained in the region behind the shock if the shock is moving into the undisturbed region and then the limit is taken as we approach the shock. The mapping from  $(\xi, t)$ -plane to (x, y)-plane can be obtained by integrating the first part (3.18) of the shock ray equations i.e.,  $x_t = M \cos \Theta$ ,  $y_t = M \sin \Theta$ .

(5.4)–(5.6) form the equations of SRT, which is ideally suited in dealing with many practical problems involving propagation of a curved shock since (i) it has been shown that it gives results which agree well with known exact solutions and experimental results, [10], (ii) it gives sharp geometry of the shock and many finer details of geometrical features of the shock ([10], [14] and PP - § 10), (iii) results obtained by it agree well with those obtained by numerical solutions of full Euler's equations, [2] and [10], (iv) it takes considerably less computational time (say less than 10%) compared to the Euler's numerical solution and (v) for a problem like sonic boom [4], where it is difficult to get information in a long narrow region away from the aircraft by Euler's numerical solution, and SRT and WNLRT are most suited.

We mention two important results obtained by WNLRT and SRT: (i) the genuine nonlinearity in the original system causes a strong nonlinear diffraction of the rays and does not allow rays from a converging wavefront to form a caustic so that the caustic is resolved ([3], [14], [23], and PP -  $\S6$  and  $\S10$ ) and (ii) again the genuine nonlinearity significantly accelerates a non-circular shock to evolve into a circular shock [3].

This article will remain incomplete unless we present a nontrivial application of WNLRT and SRT. Therefore, in the next section we take up their application to a new formulation of a very interesting physical problem: finding geometry and signature of a sonic boom, which is a very difficult problem for a maneuvering aircraft.

# 6. FORMULATION OF THE PROBLEM OF SONIC BOOM BY A MANEUVERING AEROFOIL AS A ONE PARAMETER FAMILY OF CAUCHY PROBLEMS

For details of the results in this section, please refer to [4]. Consider a two dimensional unsteady flow produced by a thin maneuvering aerofoil moving with a supersonic velocity along a curved path. The aerofoil moves in a uniform polytropic gas at rest. We are interested in calculating the sonic boom produced by the aerofoil, the point of observation being far away, say at a distance L, from the aerofoil. We use coordinates x, y and time t nondimensionalized by L and the sound

velocity  $a_0$  in the ambient medium. In a local rectangular coordinate system (x', y') with origin O' at the nose of the aerofoil and O'x' axis tangential to the path of the nose, which moves along a curve  $(X_0(t), Y_0(t))$ , let the upper and lower surfaces of the aerofoil be given by

$$\left(x' = \zeta, y' = b_u(\zeta)\right) \text{ and } \left(x' = \zeta, y' = b_l(\zeta)\right), \ -d < \zeta < 0 \tag{6.1}$$

respectively. Here d is the non-dimensional camber length. We assume that  $b'_u(-d) > 0$ ,  $b'_u(0) < 0$ ,  $b'_l(-d) < 0$  and  $b'_l(0) > 0$ , so that the nose and the tail of the aerofoil are not blunt. We further assume that

$$d = \frac{\bar{d}}{L} = O(\epsilon), O\left\{\frac{\max_{-d < \zeta < 0} b_u(\zeta)}{d}\right\}$$
$$= O\left\{\frac{\max_{-d < \zeta < 0} (-b_l(\zeta))}{d}\right\} = O(\epsilon)$$
(6.2)

where  $\epsilon$  is a small positive number. Then the nondimensional amplitude  $w = \epsilon \tilde{w}/a_0$  of the perturbation in the sonic boom also satisfies  $w = O(\epsilon)$ .

In Fig. 1, we show the geometry of the aerofoil and the sonic boom produced by it at a time t. The sonic boom produced either by the upper or lower surface consists of a leading shock LS:  $\Omega_t^{(0)}$  and a trailing shock TS:  $\Omega_t^{(-d)}$  and (since high frequency approximation is satisfied by the flow between the two shocks) a one parameter family of nonlinear wavefronts  $\Omega_t^{(\zeta)}(-d < \zeta < 0, \zeta \neq G)$  originating from the points  $P_{\zeta}$  on the aerofoil in between the two shocks. The nonlinear wavefronts produced from points on the front part of the aerofoil start interacting with the LS  $\Omega_t^{(0)}$  and those from the points near the trailing edge do so with the TS  $\Omega_t^{(-d)}$ , and after the interaction they keep on disappearing continuously from the flow. These two sets, one interacting with LS and another interacting with TS are separated by a linear wavefront  $\Omega_t^{(G)}$ , which originates from a point  $P_G$  where the function  $b_u(\xi)(b_l(\xi))$  is maximum (minimum). Fig. 2 shows an enlarged version of the upper part of the Fig. 1 near the aerofoil.

Let us introduce a ray coordinate system  $(\xi, t)$  for  $\Omega_t^{(\zeta)}$ . The front  $\Omega_t^{(\zeta)}$  at a given time t can be obtained as locus of the tip of the rays (at time t ) in (x,y)-plane starting from all positions  $P_{\zeta|\eta}$  of  $P_{\zeta}$  at times  $\eta < t$  shown in Fig. 3. Therefore, a value of  $\eta$ ,  $\eta \leq t$  identifies a ray and we choose

$$\xi = -\eta, \ \eta \le t \tag{6.3}$$

for  $\Omega_t^{(\zeta)}$  from the upper surface (for lower surface we shall choose  $\xi = \eta, \ \eta \leq t$ ).

When  $\xi \equiv -\eta = -t$ , the points A, B and C in the Fig. 3 coincide. Hence the base point  $P_{\zeta}$  of  $\Omega_t^{(\zeta)}$ , which lies on the upper surface of the aerofoil, corresponds to a point on the line  $\xi + t = 0$  in the  $(\xi, t)$ -plane.

The nonlinear wavefront  $\Omega_t^{(\zeta)}$ ,  $(-d < \zeta < 0, \zeta \neq G)$  satisfies the system (5.2)–(5.3). The Cauchy data on  $\xi + t = 0$  for our sonic boom problem can be determined from the inviscid flow condition



FIG. 1: Sonic boom produced by the upper and lower surfaces:  $y' = b_u(x')$  and  $y' = b_l(x')$  respectively. The boom produced by either surface consists of a one parameter family of nonlinear wavefronts bounded by leading and trailing shocks.

on the surface of the aerofoil. Retaining only the leading order terms, this is [4]

$$m(\xi, -\xi) = m_0(\xi) := 1 - \frac{(\gamma + 1)(\dot{X}_0^2 + \dot{Y}_0^2)b'_u(\xi)}{2(\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{\frac{1}{2}}}$$
(6.4)

$$g(\xi, -\xi) = g_0(\xi) := (\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{\frac{1}{2}}$$
(6.5)

$$\theta(\xi, -\xi) = \theta_0(\xi) := \frac{\pi}{2} + \psi - \sin^{-1}\{1/(\dot{X}_0^2 + \dot{Y}_0^2)^{\frac{1}{2}}\}$$
(6.6)

where  $\psi = \tan^{-1} \left\{ \dot{Y}_0 / \dot{X}_0 \right\}$ . Since  $b'_u(\zeta) < 0$  for  $G < \zeta \le 0$  and  $b'_u(\zeta) > 0$  for  $-d \le \zeta < G$ ,  $m_0 > 1$  on  $P_{\zeta}$  for  $G < \zeta \le 0$  and  $m_0 < 1$  on  $P_{\zeta}$  for  $-d \le \zeta < G$ . This can be used to argue that

$$m > 1 \text{ on } \Omega^{(\zeta)}, \text{ for } G < \zeta \le 0 \text{ and } m < 1 \text{ on } \Omega^{(\zeta)}, \text{ for } -d \le \zeta < G.$$
 (6.7)



FIG. 2: An enlarged version of the upper part of the Figure 1 near the aerofoil.



FIG. 3: A formulation of the ray coordinate system  $(\xi, t)$  for  $\Omega_t^{(\zeta)}$ . AB represents the path of a fixed point  $P_{\zeta}$  on the aerofoil. A and B are the positions of  $P_{\zeta}$  at times  $\eta$  and t respectively,  $\eta < t$ .

Since the eigenvalues of the system (5.2)–(5.3) are

$$\lambda_1 = -\sqrt{(m-1)/(2g^2)}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{(m-1)/(2g^2)}$$
 (6.8)

we get a Cauchy problem for a hyperbolic system (5.2)–(5.3) for  $\Omega_t^{(\zeta)}$  for each  $\zeta$  satisfying  $G < \zeta < 0$  and an elliptic system for  $\Omega_t^{(\zeta)}$  for each  $\zeta$  satisfying  $-d < \zeta < G$  (we call it elliptic even

though  $\lambda_2 = 0$  is real).

The derivation of the Cauchy data on  $\xi + t = 0$  for the system (5.4)–(5.6) governing the evolution of the leading and trailing shock fronts is far more complex. We quote from [4] the leading order terms in this Cauchy data for  $\Omega_t^{(-d)}$ 

$$M(\xi, -\xi) = M_0(\xi) := 1 - \frac{(\gamma + 1)(X_0^2 + Y_0^2)b'_u(\xi)}{4(\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{\frac{1}{2}}}$$
(6.9)

$$G(\xi, -\xi) = G_0(\xi) := (\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{\frac{1}{2}}$$
(6.10)

$$\Theta(\xi, -\xi) = \Theta_0(\xi) := \frac{\pi}{2} + \psi - \sin^{-1}\{1/(\dot{X}_0^2 + \dot{Y}_0^2)^{\frac{1}{2}}\}$$
(6.11)

$$V(\xi, -\xi) = V_0(\xi) := \frac{\gamma + 1}{4} \{ \Omega_{p_{(-d)}} w_0(\xi) - \mathcal{F}(-d, t) \}$$
(6.12)

where  $w_0(\xi)$  is related to  $M_0(\xi)$  by  $M_0(\xi) = 1 + \frac{\gamma+1}{4}w_0(\xi)$ ,

$$\begin{split} \Omega_{P_{(-d)}} &= \frac{(\dot{X}_{0}\ddot{X}_{0} + \dot{Y}_{0}\ddot{Y}_{0})}{2g_{0}(\dot{X}_{0}^{2} + \dot{Y}_{0}^{2})(\dot{X}_{0}^{2} + \dot{Y}_{0}^{2} - 1)^{1/2}} + \frac{\dot{X}_{0}\ddot{Y}_{0} - \dot{Y}_{0}\ddot{X}_{0}}{g_{0}\ddot{X}_{0}^{2}} \\ \mathcal{F}(\zeta, t) &= \frac{(\dot{X}_{0}^{2} + \dot{Y}_{0}^{2})b_{u}''(\zeta)}{(\dot{X}_{0}^{2} + \dot{Y}_{0}^{2} - 1)^{1/2}} \{\dot{\mathcal{X}}_{0}(t)\} - \frac{(\dot{X}_{0}^{2} + \dot{Y}_{0}^{2} - 2)(\dot{X}_{0}\ddot{X}_{0} + \dot{Y}\ddot{Y}_{0})}{(\dot{X}_{0}^{2} + \dot{Y}_{0}^{2} - 1)^{3/2}}b_{u}'(\zeta) \end{split}$$

$$\mathcal{X} = X_0 \cos \psi + Y_0 \sin \psi.$$

Again M > 1 on  $\Omega_t^{(0)}$  and M < 1 on  $\Omega_t^{(-d)}$  so that the two eigenvalues  $\Lambda_1 = \sqrt{(M-1)/2G^2}$ ,  $\Lambda_2 = -\sqrt{(M-1)/2G^2}$  of (5.4)–(5.6) are real for  $\Omega_t^{(0)}$  and purely imaginary for  $\Omega_t^{(-d)}$ . The other two eigenvalues are  $\Lambda_{11} = 0$ ,  $\Lambda_{12} = 0$ . Thus, for the LS we get a Cauchy problem for a system which is hyperbolic and for the TS we get it for a system which has elliptic nature.

We have numerically solved the system (5.2)–(5.3) for a nonlinear wavefront with Cauchy data (6.4)-(6.6) for  $\zeta = 0$  for two cases. This nonlinear wavefront from the leading edge is immediately annihilated by the shock  $\Omega_t^{(0)}$ . The first case is for an accelerating aerofoil in a straight path and the second one is for an aerofoil moving with a constant speed but on a curved path. We have also solved the system (5.4)–(5.6) with data (6.9)–(6.12) for the same paths and same geometry of the aerofoil. We present some results in Fig. 4 and Fig. 5.

For an accelerating aerofoil along a straight line we note that the linear wavefront from the nose develops fold in the caustic region but the nonlinear wavefront does not fold and instead has a pair of kinks. For a supersonic aerofoil moving on a highly curved path (curved downwards), the nonlinear wavefront from the upper surface is smooth but from the lower surface has a pair of kinks. The most interesting result seen from our new formulation of the sonic boom problem is the elliptic nature of the equations governing  $\Omega_t^{(-d)}$ . This implies that whatever may be the flight path and acceleration



FIG. 4: Sonic boom wavefront at t = 2 from the leading edge of an accelerating aerofoil moving in a straight path. Kinks on the nonlinear wavefront are shown by dots. The initial Mach number is 1.8 and the acceleration is 10 in the time interval (0,1/2) and  $b'_u(0) = -0.01$ .



FIG. 5: Successive position of the nonlinear wavefront from the leading edge of an aerofoil moving with a constant Mach number 5 along a path concave downwards with  $b'_u(0) = -0.01$ 

of the aerofoil, the trailing shock  $\Omega_t^{(-d)}$  must be free from kinks. All these features, which we obtain from our theory are seen in the Euler's numerical solution in [9], discussed also in [4].

We also have some results for the leading shock LS from the SRT (see a paper by Baskar and Prasad in the Proc. International Symposium on Nonlinear Acoustics including International Sonic Boom Forum, Pennsylvania State University, 18-22 July, 2005). We find that though the geometric shape of nonlinear wavefront is not only topologically same as that of  $\Omega_t^{(0)}$ , it is very close to it. Hence the nonlinear wavefront from the leading edge gives valuable information about the shape and the position of  $\Omega_t^{(0)}$ .

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