Upstream propagating curved shock in a steady transonic flow*

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Abstract

We derive a new set of equations to give successive positions of slowly moving and slowly turning curved shock front in a transonic flow. The equations are in conservation form - two of these are kinematical conservation laws.

keywords Curved shock propagation, shock dynamics, ray theory, transonic flow.

1 Introduction

It was initially proposed that, due to trapping of upstream propagating waves at different points of a sonic surface and increase in their amplitude at decelerating sonic points, a continuous mixed supersonic and subsonic flow is unstable (Kuo 1951). But much later, theoretical and experimental investigators at NLR (see Spee, 1971 for details) showed that two-dimensional continuously decelerating transonic flows were not unstable in the strict sense. Based on linear geometrical acoustics, Spee argued that the upstream moving waves were not really trapped at the sonic point due to turning effect but penetrate through the decelerating part of the flow and thus move away from the sonic point. A nonlinear analysis of interaction of trapping of waves and simultaneous turning due to gradient of the *solution* along the wavefront forms a challenging mathematical problem. The word *solution* in italics is significant, which we shall explain later. It was pointed by Prasad (1973) that a necessary condition for waves to get trapped is the vanishing of all components of the bicharacteristic (or ray) velocity. He showed that this condition is satisfied at all points of a sonic surface for upstream propagating wavefronts orthogonal to the fluid velocity at these points but only waves whose fronts are convex to the flow direction are likely to get trapped for a longer time. Such waves tend to become plane. Based on these results, Prasad derived a general theory without turning effect which he used to discuss various cases of the trapped transonic pulses.

*Based on the invited lecture at the Symposium Transsonicum IV, September 2-6, 2002, Goettingen, Germany

The weakly nonlinear ray theory (WNLRT) developed by Prasad since 1975 (see Prasad, 2001) shows that a wavefront in such a wave turns due to gradients along the wavefront of (i) the steady flow and (ii) the amplitude of the wave. The significance of the word *solution* in italics in the first paragraph is now clear. It emphasizes both effects (i) and (ii). The first effect was taken into account by Prasad and Krishnan (1977) and equations globally valid in a transonic flow under a thin aerofoil theory, were derived but only local analysis was presented in the neighbourhood of a given sonic point. Prasad (2001) extended this result to include equally important second effect (ii) (see case 2, section 5.3). **In what follows the section numbers refers to sections in the book: Prasad (2001).** Though, it is theoretically possible to fit in a shock (when it appears) in a curved pulse, it would become extremely complex procedure even numerically.

In this paper we present a full set of equations for the propagation of a upward propagating curved weak *shock front* with normal direction making a small angle to the stream lines in the steady transonic flow taking into account both effects (i) and (ii). Shock fronts with convex geometry to the on coming transonic flow are continuously produced by disturbances at the rear end of the aerofoil and hence the model equations which we have derived here are extremely valuable to trace the full history of these curved shocks in the entire transonic region.

2 Equations of WNLRT for upstream propagating waves in an steady flow

Weakly nonlinear ray theory (WNLRT) for nonlinear waves in multi-dimensions have been developed for a general system of hyperbolic equations over a number of years (sections 4 - 4.3.3). Consider a particular case of this theory for Euler's equations governing unsteady two dimensional motion of a polytropic gas in space-time $(\mathbf{x}, t), \mathbf{x} = (x, y)$. Let ρ denote mass density, $\mathbf{q}$ fluid velocity, p gas pressure and $a \left(= \sqrt{\frac{\gamma p}{\rho}}\right)$ the local sound velocity, γ being the ratio of specific heats. We denote the solution in a steady flow by $(\rho_0, \mathbf{q}_0, p_0)$ and the sound velocity by a_0. Consider now a small amplitude upward propagating pulse in high frequency approximation so that a one parameter family of wavefronts (with normal direction $\mathbf{n} = (\cos\theta, \sin\theta)$) generating the pulse is defined. Let the amplitude of the pulse on a particular wavefront Ω_t be denoted by w which is assumed to be small. Then the perturbation due to the wave on the steady flow is given by

$$\rho - \rho_0 = -\frac{\rho_0}{a_0} w, \;\; q_\alpha - q_{\alpha_0} = n_\alpha w, \;\; \alpha = 1, 2, 3 \text{ and } p - p_0 = -\rho_0 a_0 w \tag{2.1}$$

We introduce a ray coordinate system (ξ', t) such that ξ' = constant give rays and t = constant give successive positions of Ω_t. Let $g' d\xi'$ be the element of distance along Ω_t. The equations of the WNLRT giving the successive positions of Ω_t and amplitude distribution on it consists of ray equations (section 4.3.2)

$$x_t = q_{10} - n_1 a_0 + \frac{\gamma + 1}{2} n_1 w = \chi_1, \text{ say} \tag{2.2}$$

$$y_t = q_{20} - n_2 a_0 + \frac{\gamma+1}{2} n_2 w = \chi_2, \text{ say} \tag{2.3}$$

$$\theta_t = (1/g')\left(a_{0\xi'} - \cos\theta\, q_{10\xi'} - \sin\theta\, q_{20\xi'} - \frac{\gamma+1}{2} w'_{\xi'}\right) \tag{2.4}$$

and a transport equation for the amplitude w

$$\frac{dw}{dt} = (K - a_0\Omega)w\ , \ \ \Omega = -\frac{1}{2}\langle \nabla, \mathbf{n}\rangle \tag{2.5}$$

where Ω is the mean curvature of Ω_t and

$$K = -\frac{1}{2\rho_0 a_0}\langle \mathbf{q}_0 - a_0\mathbf{n}, \nabla\rangle(\rho_0 a_0) - \frac{1}{2}\left\{\gamma\langle\nabla, \mathbf{q}_0\rangle + \langle \mathbf{n}, \langle \mathbf{n}, \nabla\rangle \mathbf{q}_0\rangle\right\} \tag{2.6}$$

To these equations we need to add an equation for $g = \left(x_\xi^2 + y_\xi^2\right)^{1/2}$, see equation (9) below.

The system of 5 equations in (2) to (5) and (9) below for 2 components of $\mathbf{x}, \theta, g$ and w form a coupled system. The term containing w in the equations in (2) and (3) represents stretching of rays and that containing $w_{\xi'}$ in (4) represents nonlinear diffraction or turning of rays. For WNLRT not only the amplitude w is small but also the pulse containing Ω_t is localized in a neighbourhood ϵ of the surface Ω_t, where ϵ is small compared to a characteristic length H in the steady solution over which $(\rho_0, \mathbf{q}_0, p_0)$ varies and a length R which is smaller of the radii of curvature of Ω_t.

Discussion of the section 6.1 (with suitable modification done here for upward propagating waves and nonconstant known steady state ($\rho_0(\mathbf{x})$, $q_0(\mathbf{x})$ and $p_0(\mathbf{x})$) is applicable here. We note that
(i) Given the initial position Ω_0 of the nonlinear wavefront (so that the unit nomal $\mathbf{n}$ of Ω_0 and value of g_0 can be calculated) and amplitude distribution w_0 on it, we can set up an initial value problem for the system (2) - (5) and (9) to find the successive positions of Ω_t as t varies.
(ii) The nonlinear wavefront Ω_t is self-propagating (section 3.3.1), which means that a nonlinear wavefront is determined by the information only on it at any previous time and is not influenced by the wavefronts which follow or precede it. The equations which we shall derive show that a shock front is not self-propagating (section 9.4).

The nonlinear wavefront Ω_t (or a shock front) usually develop kinks, which can be studied with kinematical conservation laws (KCL), see section 3.3. The KCL for Ω_t governed by the above equations, are

$$(g\sin\theta)_t + \left(q_{10} - a_0\cos\theta + \frac{\gamma+1}{2} w\cos\theta\right)_\xi = 0 \tag{2.7}$$

$$(g\cos\theta)_t - \left(q_{20} - a_0\sin\theta + \frac{\gamma+1}{2} w\sin\theta\right)_\xi = 0 \tag{2.8}$$

For a smooth solution, these conservation laws give (4) and

$$g_t = -q_{10\xi'}\sin\theta + q_{20\xi'}\cos\theta + \left(-a_0 + \frac{\gamma+1}{2} w\right)\theta_\xi \tag{2.9}$$

The last equation can also be derived from the relation $g^2 = x_\xi^2 + y_\xi^2$ and (2) and (3).

3 Approximate equations for waves in a transonic flow produced by a thin aerofoil

Consider a two-dimensional steady transonic flow produced by a thin aerofoil with τ as the camber i.e., the ratio of its maximum thickness to its length l. Then τ is a small nondimensional quantity. Take a free undisturbed flow with high subsonic speed in the direction of the aerofoil, which is chosen as the x-direction. The y direction is perpendicular to the aerofoil. Let a^* is the sound speed at a sonic point. We assume the steady transonic flow with an embedded supersonic flow around the thin aerofoil to be represented by (Guderley (1942))

$$q_{10} = a^* + \tau a^* \bar{q}_{10} \ , \quad q_{20} = \tau^{3/2} a^* \bar{q}_{20} \ , \quad q_{30} = 0 \tag{3.10}$$

where $\bar{q}_{10}$ and $\bar{q}_{20}$ are quantities of order one and are functions of nondimensional variables

$$\bar{x} = x/l \ , \quad \bar{y} = \tau^{1/2} y/l \ , \quad \bar{t} = t a^* \tau / l \ , \quad \bar{\xi} = \tau^{1/2} \xi' / l \tag{3.11}$$

From Bernoulli's equation, we get the following approximate expression for the sound speed in terms of $\bar{q}_{10}$,

$$a_0 = a^* \{1 - \frac{1}{2}(\gamma - 1)\tau \bar{q}_{10} - \frac{1}{8}(\gamma - 1)(\gamma + 1)\tau^2 \bar{q}_{10}^2\} + O(\tau^3) \tag{3.12}$$

Using the above scaling it is possible to derive a simpler form of the equations for WNLRT for the upstream propagating small amplitude waves assuming the amplitude parameter ϵ to be same as τ (see Prasad and Krishnan (1977) for an introduction). Then $H = O(l)$ in x-direction and $O(l/\tau^{1/2})$ in y-direction. What is important theoretically is that the scaling, introduced in (10) and (11), gives a flow field in which (i) waves, which are almost perpendicular to the x-axis, are trapped in such a way that the angle θ, which the normal to the wavefront makes with the x-axis, remains small and will have to be appropriately scaled; (ii) the first two terms in (2) and (3)i.e., $\mathbf{L}a_0$ and $n_\beta \mathbf{L} q_{\beta 0}$ become small and of the same order as the third term $\mathbf{L}w$ so that the wavefront turns slowly and remains trapped in the transonic region for a time interval of the order $l/(a^*\tau)$; and (iii) we are able to follow the complete history of the nonlinear wavefront as it transverses the entire transonic region. In this case, K and Ω on the right hand side of (5) are also of the same order as that of w. We introduce a scaled angle $\bar{\theta}$ and scaled amplitude $\bar{w}$ by

$$\bar{\theta} = \theta/\tau^{1/2}, \ \ \bar{w} = w/(\tau a^*) \tag{3.13}$$

We note that $n_1 = \cos\theta$ and $n_2 = \sin\theta$.

The approximate equations which can be derived, following the procedure in Prasad and Krishnan (1977), are

$$\bar{x}_t = \frac{1}{2}(\gamma + 1)\bar{q}_{10} + \frac{1}{2}\bar{\theta}^2 + \frac{1}{2}(\gamma + 1)\bar{w}, \quad \bar{y}_t = -\bar{\theta} \tag{3.14}$$

$$\bar{\theta}_t = -\frac{\gamma+1}{2}\left\{\frac{1}{g}\frac{\partial}{\partial\bar{\xi}}(\bar{q}_{10}+\bar{w})\right\} \tag{3.15}$$

$$g_t = -\theta_{\bar{\xi}} \tag{3.16}$$

and

$$w_t \equiv \left[\frac{\partial}{\partial t} + \left\{\frac{1}{2}(\gamma+1)\bar{q}_{10} + \frac{1}{2}\bar{\theta}^2 + \frac{1}{2}(\gamma+1)\bar{w}\right\}\frac{\partial}{\partial x} - \bar{\theta}\frac{\partial}{\partial y}\right] w$$
$$= (\bar{K} - \bar{\Omega})\bar{w} \tag{3.17}$$

where

$$\bar{\Omega} = -\frac{1}{2g}\frac{\partial\bar{\theta}}{\partial\bar{\xi}}, \quad \bar{K} = -\frac{\gamma+1}{2}\frac{\partial\bar{q}_{10}}{\partial\bar{x}} \tag{3.18}$$

We proceed now to derive from these equations a new set of equations of the shock ray theory (SRT) valid for a shock front heading the pulse containing Ω_t, so that in the state ahead of the shock we have $w = 0$.

4 Equations of SRT governing the motion of a shock

Shock ray theory (SRT) consists of shock ray equations and an infinite system of compatibility conditions along a shock ray for the shock strength and successive normal derivatives of a physical variable at the shock (sections 9 - 9.5). Derivation of the equations of SRT for a shock of arbitrary strength is extremely cumbersome. However, SRT for a *weak* shock can be derived from the equations of WNLRT by using a theorem (section 9.2, see also Monica and Prasad, 2001). A weak shock front is followed by one parameter family of nonlinear waves belonging to the same characteristic field. They catch up with the shock, interact with it and then disappear. A nonlinear wavefront, while interacting with the shock will be instantaneously coincident with it. The above mentioned theorem states that the shock ray velocity components and the rate of turning of a shock ray are respectively means of the bicharacteristic velocity components and rates of turnings of the instantaneously coincident nonlinear wavefronts ahead and behind the shock. We also note that on the length scale ϵ ($= \tau$ by our choice in the last section) over which the nonlinear pulse (containing Ω_t) extends in its normal direction, the steady solution as well as $\mathbf{n}$ (i.e. θ) may be treated as constant. Since the normal direction of Ω_t and the direction of the x-axis are almost parallel, it follows that the steady solution and $\mathbf{n}$ may be treated as constant when x varies over the distance ϵ. This helps in differentiating the transport equation (17) with respect to x treating $\bar{q}_{10}$ and $\bar{\theta}$ to be constant.

Following the procedure given in the section 10.1 (see also Monica and Prasad, 2001), we deduce from the equations (15) - (17) the equations of the SRT. We first introduce notations

$$\left.\begin{array}{l}(\bar{x},\bar{y})|_{\text{shock}} = (X,Y),\ \ \bar{\theta}|_{\text{shock}} = \Theta,\ \ g|_{\text{shock}} = G,\ \ \bar{\xi}|_{\text{shock}} = \xi \\ \bar{w}|_{\text{shock}} = \mu\ ,\quad \frac{\partial \bar{w}}{\partial \bar{x}}|_{\text{shock}} = \mu_1 \quad \text{and} \quad \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}|_{\text{shock}} = \mu_2\end{array}\right\} \tag{4.19}$$

The shock ray equations (with two compatibility conditions) are

$$X_T = \frac{1}{2}(\gamma+1)\bar{q}_{10} + \frac{1}{2}\Theta^2 + \frac{\gamma+1}{4}\mu,\quad Y_T = -\Theta \tag{4.20}$$

$$\Theta_T = -\frac{\gamma+1}{2G}\left\{\left(\bar{q}_{10} + \frac{1}{2}\mu\right)_\xi\right\} \tag{4.21}$$

$$G_T = -\Theta_\xi \tag{4.22}$$

$$\mu_T = (K_s - \Omega_s)\mu - \frac{\gamma+1}{4}\mu\mu_1 \tag{4.23}$$

and

$$\mu_{1T} = (K_s - \Omega_s)\mu_1 - \frac{\gamma+1}{2}\mu_1^2 - \frac{\gamma+1}{4}\mu\mu_2 \tag{4.24}$$

where

$$\Omega_s = -\frac{1}{2G}\frac{\partial\Theta}{\partial\xi},\quad K_s = -\frac{\gamma+1}{2}\frac{\partial\bar{q}_{10}}{\partial\bar{x}}|_{\text{shock}} \tag{4.25}$$

$$\frac{d}{dT} = \frac{\partial}{\partial t} + \left\{\frac{1}{2}(\gamma+1)\bar{q}_{10} + \frac{1}{2}\Theta^2 + \frac{\gamma+1}{4}\mu\right\}\frac{\partial}{\partial X} - \Theta\frac{\partial}{\partial Y} \tag{4.26}$$

The above system of equations are not closed due to the presence of μ_2 in (24). As suggested in section 7.3 for the new theory of shock dynamics (NTSD), we drop the last term in (24). The final set of equations to study the full history of a upstream propagating weak shock as it traverses the transonic region are (20) - (24) with $\{(\gamma+1)/4\}\,\mu\mu_2$ omitted in (24).

To deal with kinks which may develop on the shock front, we may write these equations in an equivalent system of physically realistic conservation forms. Conservation laws equivalent to (21) and (22) can be derived directly from (7) and (8) by making substitution of (10), (11), (13) and (19), retaining the leading terms, and using the theorem mention in the beginning of this section:

$$(G\Theta)_T + \left(\frac{1}{2}\Theta^2 + \frac{\gamma+1}{2}\bar{q}_{10} + \frac{\gamma+1}{4}\mu\right)_\xi = 0 \tag{4.27}$$

$$G_T + \Theta_\xi = 0 \tag{4.28}$$

Eliminating Θ_ξ between (22) and (23), and again between (22) and (24) we get two more equations in conservation form

$$(\mu^2 G)_T = \mu^2 G\left\{2K_s - \frac{\gamma+1}{2}\mu_1\right\} \tag{4.29}$$

and

$$(\mu_1^2 G)_T = \mu_1^2 G\left\{2K_s - (\gamma+1)\mu_1\right\} \tag{4.30}$$

The four equations (27) - (30) for four quantities μ, μ_1, Θ and G are coupled to the two equations (20) for X and Y due to the presence of the term $K_s = -\frac{\gamma+1}{2}\frac{\bar{q}_1}{\partial \bar{x}}|_{\text{shock}}$ and these six equations are to be solved simultaneously (unlike the problem dealt by Monica and Prasad (2001)). Given initial position of the shock (from where Θ may be calculated), the initial shock strength μ and initial value of μ_1, we can numerically solve these equations. Procedure for setting up initial value is same as that in section 10.3 (see also Monica and Prasad, 2001).

Acknowledgement: We thank AR&DB, Ministry of Defence, Govt. of India for financial support through the project "Nonlinear Hyperbolic Waves in Multi-Dimensions with Special Reference to Sonic Booms"(No. DARO/08/1031199/M/I).

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