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Canonical form of a quasilinear hyperbolic system of first order equations

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Dedicated to the memory of Professor Krishna M. Das

ABSTRACT

A canonical form of the compatability condition along a characteristic surface for a quasilinear hyperbolic system of first order equations in $m+1$ independent variables is derived. This form of the compatibility condition is distinguished by the fact that special emphasis is given to the interior derivative in the bicharacteristic direction, which alone contains derivatives of the type $\partial/\partial t$, whereas all other interior derivatives present are spatial in nature.

1. Introduction.

The bicharacteristic method for numerical solution of hyperbolic equations (Reddy, Tikekar and Prasad, 1982) is based on the existence of a compatibility condition along the characteristic surface with special emphasis on the bicharacteristic direction. There does not seem to be a general method to derive this compatibility condition. In fact, Butler (1960) goes through very complicated geometrical considerations to derive it for the gas dynamic equations. In this short paper, we have derived the compatibility condition for a general quasilinear hyperbolic system of first order equations. We believe that the equation derived here will not only be useful in numerical solution of the hyperbolic equations but will also lead to simpler proofs of many important theoretical results.

In what follows, the suffixes i, j, k take values $1, 2, \ldots, n$ and α, β, γ the values $1, 2, \ldots, m$; where n and m are positive integers. A repeated suffix (except when the suffix is M) implies sum over the range of the suffix.

2. The canonical form of the compatibility condition

Consider a quasilinear system of first order equations

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$$A \frac{\partial U}{\partial t} + B^{(\alpha)} \frac{\partial U}{\partial x_\alpha} + C = 0, \tag{2.1}$$

where U is a column vector with n components ; x_α, t ($\alpha = 1, 2, \ldots, m$) are the $m+1$ independent variables ; A and $B^{(\alpha)}$ are $n \times n$ matrices, A being nonsingular and C a column vector with n components. A, $B^{(\alpha)}$, C are functions of U, x_α and t. We assume the system to be hyperbolic [Prasad and Ravindran, 1984] with t as a time-like variable, i. e., for an arbitrary set of real numbers $\{n_\alpha\}$, there are n real characteristic roots c_i (not necessarily distinct) of the characteristic equation and that there exist n linearly independent left eigenvectors $l^{(k)}$ and n linearly independent right eigenvectors $r^{(k)}$ satisfying

$$l^{(M)}\, n_\alpha\, B^{(\alpha)} = c_M\, l^{(M)}\, A,\; n_\alpha\, B^{(\alpha)}\, r^{(M)} = c_M\, A r^{(M)},\; M = 1, 2, \ldots, n. \tag{2.2}$$

Let L and R be $n \times n$ matrices with ith row of L as $l^{(i)}$ and ith column of R as $r^{(i)}$. Let $S = [s_{ij}]$ be the inverse of R. In passing we remark that it is possible to normalise L and R satisfying

$$L\, A\, R = I \text{ (identity matrix)}. \tag{2.3}$$

Then

$$S \equiv [s_{ik}] = L\, A = [l_j^{(i)}\, A_{jk}]\, . \tag{2.4}$$

However, it is not necessary for us to use (2.3) and (2.4), which makes the algebra very complicated in practical problems.

Multiplying (2.1) by the left eigenvector $l^{(M)}$ of the Mth characteristic field, we get a system equivalent to (2.1) :

$$l^{(M)}\, A \frac{\partial U}{\partial t} + l^{(M)} B^{(\alpha)} \frac{\partial U}{\partial x_\alpha} + l^{(M)}\, C = 0,\; M = 1, 2, \ldots, n\, . \tag{2.5}$$

Consider the equation (2.5) for a given value of M. If $\phi^{(M)}(x_\alpha, t)$ = constant, represents a one parameter family of characteristic manifolds of the Mth field, we define a set of $m+1$ functions

$$\phi^{(M)}(x_\alpha, t),\, \eta_1^{(M)}(x_\alpha, t),\, \eta_2^{(M)}(x_\alpha, t),\, \ldots,\, \eta_m^{(M)}(x_\alpha, t) \tag{2.6}$$

such that

$$\frac{\partial(\phi^{(M)}, \eta_1^{(M)}, \ldots, \eta_m^{(M)})}{\partial(t, x_1, \ldots, x_m)} \neq 0\, .$$

Here $(-c_M, n_\alpha) \propto (\phi_t^{(M)}, \phi_{x_\alpha}^{(M)})$ and the equation (2.5) reduces to

$$l^{(M)} \left(A\, \eta_{\beta t}{}^{(M)} + B^{(\alpha)}\, \eta_{\beta x_\alpha}{}^{(M)}\right) \frac{\partial U}{\partial \eta_\beta{}^{(M)}} + l^{(M)} C = 0. \tag{2.7}$$

Equation (2.7) contains only the m independent interior derivatives on a characteristic manifold $\phi^{(M)} =$ constant and hence represents a compatibility condition on it. Theoretically we can choose one of the interior derivatives in the direction of the bicharacteristic curve leading to the result we are looking for. Following the same idea Butler (1960) goes through very complicated geometrical considerations even for the gas dynamic equations. Prasad (1975) carried out this procedure for a general hyperbolic system and rearranged (2.7) to include the derivative along the bicharacteristic curve of the variable w defined by $U = Rw$. However, in this formulation, non-spatial derivatives of w also occurred in the equation $\left(\text{e. g , the term } \frac{\partial}{\partial x_\alpha} r^{(i)} \text{ in equation (2.23) in his paper}\right)$ which had to be carefully gathered together, before any numerical integration could be performed. We shall derive here a canonical form of the compatibility condition (2.5), which separates out the differentiation in the bicharacteristic direction from other spatial interior derivatives in the characteristic manifold. This is also the form which lends itself to easy numerical computations. Since $RS = I$, we can rewrite (2.5) as :

$$l^{(M)}\, A\, R\, S\, \frac{\partial U}{\partial t} + l^{(M)}\, B^{(\alpha)}\, RS \frac{\partial U}{\partial x_\alpha} + l^{(M)}\, C = 0 \tag{2.8}$$

or using summation convention except for M we get

$$l^{(M)}\, A\, r^{(i)}\, s_{ij} \frac{\partial u_j}{\partial t} + l^{(M)}\, B^{(\alpha)}\, r^{(i)}\, s_{ij} \frac{\partial u_j}{\partial x_\alpha} + l^{(M)}\, C = 0$$

Since $l^{(i)}\, A\, r^{(j)} = 0$ when $i \neq j$, this relation gives

$$s_{Mj} \left[\left(l^{(M)}\, A\, r^{(M)}\right) \frac{\partial u_j}{\partial t} + \left(l^{(M)}\, B^{(\alpha)}\, r^{(M)}\right) \frac{\partial u_j}{\partial x_\alpha} \right] + \sum_{i \neq M} l^{(M)}\, B^{(\alpha)}\, r^{(i)}\, s_{ij} \frac{\partial u_j}{\partial x_\alpha} + l^{(M)}\, C = 0. \tag{2.9}$$

Let the equation of the bicharacteristic curve in space-time be given by $x = x_\alpha(\sigma_M)$, $t = \sigma_M$, then from the Lemma on bicharacteristic direction [Prasad and Ravindran (1984), 82, Chapter 3], we get

$$\frac{dx_*}{d\sigma_M} = (l^{(M)} B^{(*)} r^{(M)}) / (l^{(M)} A r^{(M)}), \frac{dt}{d\sigma_M} = 1, \tag{2.10}$$

The equation (2.9) now becomes

$$s_{Mj} (l^{(M)} A r^{(M)} \frac{du_j}{d\sigma_M} + \sum_{i \neq M} l^{(M)} B^{(*)} r^{(i)} s_{ij} \frac{\partial u_j}{\partial x_*} + l^{(M)} C = 0, \tag{2.11}$$

no sum over M.

The derivative $d/d\sigma_M$ represents the time rate of change of a quantity along the bicharacteristic curves of the Mth characteristic family. The compatibility condition (2.11) on the characteristic manifold is of a very special type, the derivative along the bicharacteristic direction has a special status in that it is the only one which contains the time derivative $\partial/\partial t$, the other interior derivatives $l^{(M)} B^{(*)} r^{(i)} s_{ij} \frac{\partial}{\partial x_*}$ on the characteristic manifold contain only spatial derivatives.

References

1. D S Bulter (1960) The numerical solution of hyperbolic systems of partial differential equations in three independent variables *Proc. Roy. Soc. London Series A* 225 232-252.
2. P Prasad (1975) Approximation of the perturbation equations of a quasilinear hyperbolic system in the neighbourhood of a bicharacteristic *J. Math. Anal. Appl.* 50 470-482.
3. P Prasad and R Ravindran (1984) *Partial Differential Equations* Wiley Eastern New Delhi.
4. A S Reddy, V G Tikekar and P Prasad (1982) Numerical computation of hyperbolic equations by bicharacteristic method *J. Mathematical and Physical Sciences* 16 575-603

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